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THE FIRST RETURN TIME PROPERTIES OF AN IRRATIONAL ROTATION

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Abstract. If an ergodic system has positive entropy, then the Shannon-McMillan-Breiman theorem provides a relationship between the entropy and the size of an atom of the iterated partition. The system also has Ornstein-Weiss' first return time property, which offers a method of computing the entropy via an orbit. We consider irrational rotations which are the simplest model of zero entropy. We prove that almost every irrational rotation has the analogous properties if properly normalized. However there are some irrational rotations that exhibit different behavior.

1. Introduction

Let μ be a probability measure on X and $T : X \to X$ be a μ-preserving transformation. For a measurable subset $E \subset X$, the first return time R_E is defined by

$$R_E(x) = \min\left\{j \geq 1 : T^j x \in E\right\}.$$

Let $\mathcal{P}$ be a finite partition of X. We call $(x_0, x_1, \ldots, x_{n-1})$ the $\mathcal{P}_n$ name of x if $T^i x \in P_{x_i}$ for $i = 0, \ldots, n-1$. Let $\{\mathcal{P}_n\}$ be the sequence of partitions of X according to $\mathcal{P}_n$ names; that is, $\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}$, where $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. The Shannon-McMillan-Breiman theorem, which is called the equipartition property, states that if the entropy with respect to a partition $\mathcal{P}$, $h(T, \mathcal{P})$ is positive and T is ergodic, then we have

$$\lim_{n\to\infty} \frac{-\log \mu(P_n(x))}{n} = h(T, \mathcal{P}) \quad \text{a.e.},$$

where $P_n(x)$ denotes the element in $\mathcal{P}_n$ containing x. The asymptotic behavior between the measure of P_n and $R_{\mathcal{P}_n}$ has been studied since Wyner and Ziv's work [25] for ergodic processes. Ornstein and Weiss [19] showed the return time property which says that if T is ergodic, then

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{n} = h(T, \mathcal{P}) \quad \text{a.e.}$$

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We would like to study the analogous properties for irrational rotations. Our motivation for the investigation comes from the study of the properties of entropy zero systems which recently become more important in the analysis of physical models as well as of general group actions (groups other than $\mathbb{Z}$ or $\mathbb{R}$). General group actions of entropy zero unlike positive entropy have interesting subdynamics ([5], [18], [20]). These examples lead us to a more thorough investigation of entropy zero systems. To measure the complexity of entropy zero dynamical systems, we define the entropy dimension following the idea of J. Milnor [10]. Roughly speaking, we say $(X, \mathcal{P})$ has entropy dimension α with respect to the partition $\mathcal{P}$ if

$$\alpha(\mathcal{P}) = \inf\{\beta : \lim_{n\to\infty} \frac{1}{n^\beta} H(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}) = 0\},$$

where $H(\mathcal{P})$ is the entropy of the partition $\mathcal{P}$. Note that the limit may not exist for some of the systems. Also, if a system has positive entropy with respect to $\mathcal{P}$, then $\alpha(\mathcal{P}) = 1$. Since this entropy dimension depends on $\mathcal{P}$, it is necessary to implement the definition so that it is intrinsic to the dynamical system (see [10] and [12] for more details). It is known [10] that for each $0 \le \alpha \le 1$, there exists a system of entropy dimension α.

In this paper we study the Shannon-McMillan-Breiman theorem and the Ornstein-Weiss theorem for the least 'complex' examples of irrational rotations. Clearly an irrational rotation has entropy dimension zero. It is still wide open as to how analogous theorems hold for entropy zero systems of positive entropy dimension. Let T be the rotation by an irrational θ on $[0, 1)$ and μ be the Lebesgue measure. If the partition $\mathcal{P}_n$ has the equipartition property, where $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 1)\}$, then we expect the size of each atom to be about $1/n$. It is not hard to see that this is not the case. However, it is true that for a given θ, there exists a subsequence $n_k = q_k - 1$ (Section 2) such that for every x

$$\lim_{k\to\infty} \frac{\log \mu(P_{n_k}(x))}{\log n_k} = 1.$$

For $t \in \mathbb{R}$ we denote by $\|t\|$ the distance to the nearest integer, i.e.,

$$\|t\| = \min_{n\in\mathbb{Z}} |t - n|.$$

An irrational number θ, $0 < \theta < 1$, is said to be of type η if

$$\eta = \sup\{t > 0 : \liminf_{j\to\infty} j^t \|j\theta\| = 0\}.$$

Note that $\eta \ge 1$. The set of irrational numbers of type 1 has measure 1 and clearly includes the irrational numbers with bounded partial quotients. There exist numbers of type ∞, called the Liouville numbers.

Note that $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ is the partition of $[0, 1)$ obtained by the orbit $\{-k\theta\}$, $0 \le k \le n$. Let $\mathcal{Q}_n$ be the partition of $[0, 1)$ by the orbit $\{k\theta\}, 0 \le k \le n$, that is, $\mathcal{Q}_n = \bigvee_{i=1}^{n} T^i\mathcal{P}$.

Theorem 1.1. *For a given irrational θ we have the following:*

(i) *For every $x \in [0, 1)$*

$$\frac{1}{\eta} \le \liminf_{n\to\infty} \frac{-\log \mu(Q_n(x))}{\log n} \le 1 \quad \textit{and} \quad 1 \le \limsup_{n\to\infty} \frac{-\log \mu(Q_n(x))}{\log n} \le \eta.$$

(ii) *For almost every* $x \in [0,1)$

$$\liminf_{n\to\infty} \frac{-\log \mu(Q_n(x))}{\log n} = \frac{1}{\eta} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{-\log \mu(Q_n(x))}{\log n} = 1.$$

It is easy to show that the above theorem also holds for the partition $\mathcal{P}_n$. Since $\eta = 1$ for almost every θ, we have

$$\lim_{n\to\infty} \frac{-\log \mu(Q_n(x))}{\log n} = 1 \text{ for a.e. } x \in [0,1).$$

It is known [7] that for every x

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} = \frac{1}{\eta}, \quad \limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} = 1.$$

The following theorem states that the limits of the recurrence time with respect to the partition $\mathcal{P}_n$ and $\mathcal{Q}_n$ show different behavior, and unlike with respect to the balls of radius r, the limits are not necessarily less than 1.

Theorem 1.2. *For almost every* $x \in [0,1)$

(i) $\displaystyle\lim_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} = 1.$

(ii) $\displaystyle\liminf_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1, \quad \limsup_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = \eta.$

With respect to the partition $\mathcal{Q}_n$ we immediately obtain by Theorems 1.1 and 1.2 that for almost every x

$$\liminf_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{\log n} = \frac{1}{\eta} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{\log n} = 1.$$

Careful investigation also leads to the following theorem for the partition $\mathcal{P}_n$.

Theorem 1.3. *For almost every* $x \in [0,1)$

$$\liminf_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{\log n} = \frac{1}{\eta} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{\log n} = 1.$$

In Section 2 we consider the size of the elements of $\mathcal{Q}_n$ and prove Theorem 1.1. In Section 3 the first return time to the partition is discussed and Theorems 1.2 and 1.3 are proven. In Section 4 we compare two kinds of sequences of low complexity, automatic and Sturmian sequences. We show more or less "independence" between the return time property and the complexity via these examples.

2. The partition generated by the irrational rotation

Let θ be an irrational number with partial quotients a_i. Let p_i/q_i be the i-th convergent of θ. It is easy to see [13] that $\|j\theta\| \geq \|q_i\theta\|$ for $0 < j < q_{i+1}$, and

$$\eta = \sup\{t > 0 : \liminf_{i\to\infty} q_i^t \|q_i\theta\| = 0\}. \tag{1}$$

Define

$$D_k = q_k\theta - p_k, \text{ for } k \geq 0.$$

Then $D_0 = \theta$ and $D_k = (-1)^k \|q_k\theta\|$ for $k \geq 1$. For convenience put $D_{-1} = -1$. A real number s not of the form $s = B\theta - A$ for integers A and B and such that $0 < s < 1$ has a unique θ-expansion by Ostrowski representation ([21])

$$s = (c_1 + 1)D_0 + \sum_{k=1}^{\infty} c_{k+1} D_k,$$

where $0 \leq c_1 < a_1$, $0 \leq c_{k+1} \leq a_{k+1}$ for $k \geq 1$, and $c_k = 0$ if $c_{k+1} = a_{k+1}$.

Assume that k is even. If $c_{k+1}(x) = 0$, then $x \in (-i\theta, -(i+(a_{k+1}-1)q_k+q_{k-1})\theta)$, $0 \leq i < q_k$. If $c_{k+1}(x) = c$, $1 \leq c < a_{k+1}$, then $x \in (-(i + (a_{k+1} - c)q_k + q_{k-1})\theta, -(i + (a_{k+1} - (c+1))q_k + q_{k-1})\theta)$, $0 \leq i < q_k$. If $c_{k+1}(x) = a_{k+1}$, then $x \in (-(i + q_{k-1})\theta, -(i - q_k + q_{k-1})\theta)$, $q_k - q_{k-1} \leq i < q_k$. Note that $q_k - i = (c_1(x) + 1)q_0 + c_2(x)q_1 + \cdots + c_k(x)q_{k-1}$.

The following theorem is well known regarding the lengths of the elements of $\mathcal{Q}_n$ and the number of elements of each length ([2], [23]).

Theorem 2.1. *Let $n = cq_k + q_{k-1} + \ell$, $1 \leq c \leq a_{k+1}$ and $0 \leq \ell < q_k$. Then each length of element of $\mathcal{Q}_n$ has only three values: $\|q_k\theta\|$, $\|q_{k-1}\theta\| - c\|q_k\theta\|$ and $\|q_{k-1}\theta\| - (c-1)\|q_k\theta\|$. Moreover, the number of elements of $\mathcal{Q}_n$ with length $\|q_k\theta\|$, $\|q_{k-1}\theta\| - c\|q_k\theta\|$ and $\|q_{k-1}\theta\| - (c-1)\|q_k\theta\|$ is $n - q_k + 1$, $\ell + 1$ and $q_k - \ell - 1$, respectively.*

Since $q_{k+1} = a_{k+1}q_k + q_{k-1}$, $k \geq 1$, for a given n we can choose k, c and ℓ such that $n = cq_k + q_{k-1} + \ell$ where $1 \leq c \leq a_{k+1}$ and $0 \leq \ell < q_k$.

Let $\mathcal{A}_n$ be the set of elements of $\mathcal{Q}_n$ whose length is $\|q_k\theta\|$ for some k and let $\mathcal{B}_n = \mathcal{Q}_n \setminus \mathcal{A}_n$. Denote $|Q_n(x)|$ by the length of the interval $Q_n(x)$.

Proof of Theorem 1.1. (i) Put $n = q_{k+1} - 1$. We note that $n = cq_k + q_{k-1} + \ell$ with $c = a_{k+1} - 1$, $\ell = q_k - 1$; hence we have $|Q_n(x)| = \|q_k\theta\|$ or $\|q_k\theta\| + \|q_{k+1}\theta\|$ by Theorem 2.1. Therefore

$$\frac{1}{2(n+1)} = \frac{1}{2q_{k+1}} < \|q_k\theta\| \leq |Q_n(x)| < 2\|q_k\theta\| < \frac{2}{q_{k+1}} = \frac{2}{n+1}.$$

Hence for every x

$$\limsup_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \geq 1 \text{ and } \liminf_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \leq 1.$$

Since $|Q_n(x)|$ is a nonincreasing sequence, we have for any n with $q_k \leq n < q_{k+1}$

$$\|q_k\theta\| \leq |Q_{q_{k+1}-1}(x)| \leq |Q_n(x)| \leq |Q_{q_k-1}(x)| < 2\|q_{k-1}\theta\| < \frac{2}{q_k}.$$

For any $\epsilon > 0$, by (1), we have for sufficiently large k:

$$\frac{1}{n^{\eta+\epsilon}} \leq \frac{1}{q_k^{\eta+\epsilon}} < \|q_k\theta\| \leq |Q_n(x)| < \frac{2}{q_k} < \|q_k\theta\|^{1/(\eta+\epsilon)} < \frac{1}{q_{k+1}^{1/(\eta+\epsilon)}} < \frac{1}{n^{1/(\eta+\epsilon)}}.$$

Hence for every x

$$\limsup_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \leq \eta \text{ and } \liminf_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \geq \frac{1}{\eta}.$$

(ii) Take an $\epsilon > 0$ and let

$$E_k = \{x : |Q_n(x)| < \frac{1}{n^{1+\epsilon}} \text{ for some } q_k \leq n < q_{k+1}\}.$$

By Theorem 2.1 if $q_k \le n < q_{k+1}$, then $\min|Q_n(x)| = \|q_k\theta\|$. If $x \in E_k$, then $Q_n(x) = \|q_k\theta\|$ for some n with $n < \|q_k\theta\|^{-\frac{1}{1+\epsilon}}$ for sufficiently large k. Thus we have

$$E_k \subset \{x : |Q_n(x)| = \|q_k\theta\|, n = \lfloor \|q_k\theta\|^{-\frac{1}{1+\epsilon}} \rfloor\}.$$

Since we have at most $n - q_k + 1$ elements of length $\|q_k\theta\|$ in $\mathcal{Q}_n$, we have

$$\sum_k \mu(E_k) < \sum_k \|q_k\theta\|^{\frac{\epsilon}{1+\epsilon}} < \infty \tag{2}$$

for all ϵ. Hence by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \le 1 \quad \text{a.e.}$$

Let

$$m(k) = (\lfloor \frac{a_{k+1}}{2} \rfloor + 1)q_k + q_{k-1} - 1$$

and

$$F_k = \{x : Q_{m(k)}(x) \in \mathcal{B}_{m(k)}\} = \{-x : 0 \le c_{k+1}(x) < a_{k+1} - \lfloor \frac{a_{k+1}}{2} \rfloor\}$$

for $k \ge 1$. Then, by Theorem 2.1, for $x \in F_k$

$$|Q_{m(k)}(x)| = \|q_{k-1}\theta\| - \lfloor \frac{a_{k+1}}{2} \rfloor \|q_k\theta\|.$$

Fix k and let h_s be the number of elements of $\mathcal{B}_{m(k+s)}$ in each element of $\mathcal{B}_{m(k)}$ for $s \ge 0$. Note that $h_0 = 1$, $h_1 = a_{k+1} - \lfloor \frac{a_{k+1}}{2} \rfloor$, and $h_{s+1} = a_{k+s+1}h_s + h_{s-1}$ for $s \ge 1$. Then we have

$$\mu(F_{k+s} \cap F_k) = \frac{q_k h_s}{q_{k+s}} \mu(F_{k+s}), \quad \mu(F_k) = q_k(\|q_{k-1}\theta\| - \lfloor \frac{a_{k+1}}{2} \rfloor \|q_k\theta\|).$$

It is not difficult to show by induction that for any $s \ge 0$

$$h_s = q_{k+s}(\|q_{k-1}\theta\| - \lfloor \frac{a_{k+1}}{2} \rfloor \|q_k\theta\|) + (-1)^s(\lfloor \frac{a_{k+1}}{2} \rfloor q_k - q_{k-1})\|q_{k+s}\theta\|.$$

Therefore, we have

$$|\mu(F_{k+s} \cap F_k) - \mu(F_k)\mu(F_{k+s})| = \frac{q_k}{q_{k+s}}(\lfloor \frac{a_{k+1}}{2} \rfloor q_k - q_{k-1})\|q_{k+s}\theta\|\mu(F_{k+s}). \tag{3}$$

By the definition of the type for any $\epsilon > 0$ we can choose a subsequence $\{k_i\}$ as

$$q_{k_i+1} > \frac{1}{2\|q_{k_i}\theta\|} > q_{k_i}^{\eta-\epsilon}.$$

Then, by Theorem 2.1, for $x \in F_{k_i}$

$$\begin{aligned} |Q_{m(k_i)}(x)| &= \|q_{k_i-1}\theta\| - \lfloor \frac{a_{k_i+1}}{2} \rfloor \|q_{k_i}\theta\| \ge \|q_{k_i-1}\theta\| - \frac{a_{k_i+1}}{2}\|q_{k_i}\theta\| \\ &> \frac{\|q_{k_i-1}\theta\|}{2} > \frac{1}{4q_{k_i}} > \frac{1}{4q_{k_i+1}^{1/(\eta-\epsilon)}} \ge \frac{1}{4 \cdot 2^{1/(\eta-\epsilon)} \cdot m(k_i)^{1/(\eta-\epsilon)}}. \end{aligned}$$

Since q_k increases and $\|q_k\theta\|$ decreases exponentially, by (3) we have for any n

$$\sum_{j=1}^{n} \left|\mu(F_{k_{i+j}} \cap F_{k_i}) - \mu(F_{k_i})\mu(F_{k_{i+j}})\right| \le \sum_{j=1}^{n} C\mu(F_{k_{i+j}}), \tag{4}$$

for some constant C. By a version of the Borel-Cantelli lemma ([24], p. 45) for almost every x we have $x \in F_{k_i}$ for infinitely many i's. Therefore,

$$\liminf_{n\to\infty} \frac{-\log|Q_n(x)|}{\log n} \le \frac{1}{\eta} \quad \text{a.e.} \qquad \square$$

3. The first return time for irrational rotations

Having Theorem 1.1, it is natural to ask if we can compare the asymptotic sizes of atoms and the return times as in the case of positive entropy. If $h(T,\mathcal{P}) > 0$, then the Shannon-McMillan-Breiman theorem and the Ornstein-Weiss theorem state that

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1 \text{ for a.e. } x \in [0,1).$$

We will show that for every θ, the above statement holds for the partition $\mathcal{Q}_n$ but not for $\mathcal{P}_n$. For the notational simplicity, write $R_n(x) = R_{P_n(x)}(x)$ and $\bar{R}_n(x) = R_{Q_n(x)}(x)$.

Generally there are three values for the recurrence time of an interval [23], but for some specific length of interval the recurrence time has only two values.

Theorem 3.1 ([16]). *Let $b = \|q_{k-1}\theta\| - c\|q_k\theta\|$, $0 \le c < a_{k+1}$. If k is even, then*

$$R_{[0,b)}(x) = \begin{cases} q_k, & 0 \le x < b - \|q_k\theta\|, \\ (c+1)q_k + q_{k-1}, & b - \|q_k\theta\| \le x < b. \end{cases}$$

If k is odd, then

$$R_{[0,b)}(x) = \begin{cases} (c+1)q_k + q_{k-1}, & 0 \le x < \|q_k\theta\|, \\ q_k, & \|q_k\theta\| \le x < b. \end{cases}$$

Proof of Theorem 1.2. (i) If $Q_n(x) \in \mathcal{A}_n$, then $|Q_n(x)| = \|q_k\theta\|$ for some k. By Theorem 3.1 we have $\bar{R}_n(x) = q_{k+1}$ or $q_{k+1} + q_k$. Therefore, if $Q_n(x) \in \mathcal{A}_n$, then

$$\frac{1}{2|Q_n(x)|} = \frac{1}{2\|q_k\theta\|} < q_{k+1} \le \bar{R}_n(x) < 2q_{k+1} < 2\|q_k\theta\| = \frac{2}{|Q_n(x)|}. \tag{5}$$

For a fixed $\epsilon > 0$ let

$$G_k = \{x : \frac{\log \bar{R}_n(x)}{-\log|Q_n(x)|} > 1 + \epsilon \text{ for some } q_k + q_{k-1} \le n < q_{k+1} + q_k\}.$$

By (5) for large k, if $x \in G_k$, then $Q_n(x) \in \mathcal{B}_n$ for some $q_k + q_{k-1} \le n < q_{k+1} + q_k$. Moreover, each element of $\mathcal{B}_n$ is of the form $[(cq_k + q_{k-1} + i)\theta, i\theta)$ or $[i\theta, (cq_k + q_{k-1} + i)\theta)$ with $0 \le i < q_k$ and $1 \le c < a_{k+1}$, for $q_k + q_{k-1} \le n < q_{k+1} + q_k$. By Theorem 3.1, if $(c+1)q_k + q_{k-1} > (\|q_{k-1}\theta\| - c\|q_k\theta\|)^{-(1+\epsilon)}$ for some $1 \le c < a_{k+1}$, then we have

$$G_k = \bigcup_{i=0}^{q_k-1} [(i-q_k)\theta, i\theta) \text{ or } \bigcup_{i=0}^{q_k-1} [i\theta, (i-q_k)\theta).$$

Assume that $(c+1)q_k + q_{k-1} > (\|q_{k-1}\theta\| - c\|q_k\theta\|)^{-(1+\epsilon)}$ for some $1 \le c < a_{k+1}$. Then we have $q_{k+1} = a_{k+1}q_k + q_{k-1} > \|q_{k-1}\theta\|^{-(1+\epsilon)}$, which implies that

$$\mu(G_k) = q_k\|q_k\theta\| < \frac{q_k}{q_{k+1}} < q_k\|q_{k-1}\theta\|^{1+\epsilon} < \frac{1}{q_k^{\epsilon}} \text{ and } \sum_k \mu(G_k) < \infty.$$

For a fixed $\epsilon > 0$ let

$$H_k = \{x : \frac{\log \bar{R}_n(x)}{-\log |Q_n(x)|} < 1 - \epsilon \text{ for some } q_k + q_{k-1} \le n < q_{k+1} + q_k\}.$$

By (5) and Theorem 3.1 we have

$$H_k \subset \{x : Q_n(x) \in \mathcal{B}_n \text{ and } q_k < \frac{1}{|Q_n(x)|^{1-\epsilon}} \text{ for some } q_k + q_{k-1} \le n < q_{k+1} + q_k\}.$$

Since $|\mathcal{B}_n| \le q_k$ for $q_k + q_{k-1} \le n < q_{k+1} + q_k$, we have

$$\mu(H_k) < q_k \cdot q_k^{-\frac{1}{1-\epsilon}} = q_k^{\frac{\epsilon}{1-\epsilon}} \text{ and } \sum_k \mu(H_k) < \infty.$$

By the Borel-Cantelli lemma for $\{G_k\}$ and $\{H_k\}$, we have

$$\limsup_{n\to\infty} \frac{\log \bar{R}_n(x)}{-\log \mu(Q_n(x))} = 1 \quad \text{a.e.}$$

(ii) In the same way as in the proof of (i), we have

$$\liminf_{n\to\infty} \frac{\log R_n(x)}{-\log \mu(P_n(x))} \ge 1 \quad \text{a.e.}$$

Let $n = q_{k+1} + q_k - 1$. Then since $|P_n(x)| = \|q_k\theta\|$ or $\|q_{k+1}\theta\|$, we have

$$\frac{\log R_n(x)}{-\log \mu(P_n(x))} \le \frac{\log(q_{k+1} + q_k)}{-\log \|q_k\theta\|} \text{ or } \frac{\log(q_{k+2} + q_{k+1})}{-\log \|q_{k+1}\theta\|}.$$

By the fact $q_{k+1} < 1/\|q_k\theta\|$, for every x we have

$$\liminf_{n\to\infty} \frac{\log R_n(x)}{-\log \mu(P_n(x))} \le 1.$$

Since $|P_n(x)|$ is of the form $\|q_{k-1}\theta\| - c\|q_k\theta\|$ for some k, $0 \le c < a_{k+1}$, by Theorem 3.1 and (1) we have for any $\epsilon > 0$

$$R_n(x) \le q_{k+1} < \frac{1}{2\|q_k\theta\|} < q_k^{\eta+\epsilon} < \frac{1}{\|q_{k-1}\theta\|^{\eta+\epsilon}} < \frac{1}{|P_n(x)|^{\eta+\epsilon}}$$

for sufficiently large k. Hence for every x

$$\limsup_{n\to\infty} \frac{\log R_n(x)}{-\log \mu(P_n(x))} \le \eta.$$

If $\eta = 1$, then the proof is completed. Assume that $\eta > 1$. By Theorem 3.1, if $n = cq_k + q_{k-1} - 1$, $1 \le c \le a_{k+1}$, then for x with $c_{k+1}(x) = a_{k+1} - c$ we have

$$\frac{\log R_n(x)}{-\log \mu(P_n(x))} = \frac{\log(cq_k + q_{k-1})}{-\log(\|q_{k-1}\theta\| - (c-1)\|q_k\theta\|)} > \frac{\log q_{k+1} + \log(c/a_{k+1})}{\log q_k + \log(2a_{k+1}/(a_{k+1} - c))}.$$

Let

$$I_k = \{x \in [0,1) : a_{k+1} - 2\lfloor \frac{a_{k+1}}{3} \rfloor \le c_{k+1}(x) \le a_{k+1} - \lfloor \frac{a_{k+1}}{3} \rfloor\}.$$

For a fixed k and $0 < c \le a_{k+1}$ let r_s be the number of sets of the form $\{x : c_{k+s}(x) = d_{k+s}, \dots, c_{k+2} = d_{k+2}, c_{k+1} = c\}$ for some $d_{k+2}, \dots, d_{k+s}$. Then $r_1 = 1$, $r_2 = a_{k+2}$ and $r_{s+1} = a_{k+s+1} r_s + r_{s-1}$ for $s > 1$, which implies that

$$r_s = q_{k+s}\|q_k\theta\| - (-1)^s q_k \|q_{k+s}\theta\|, \quad \text{for } s \ge 1.$$

Then we have for $s \ge 1$

$$\frac{\mu(I_{k+s} \cap I_k)}{\mu(I_k)} = \frac{r_s \mu(I_{k+s})}{q_{k+s}\|q_k\theta\|}.$$

Note that $\mu(I_k) = q_k(\lfloor \frac{a_{k+1}}{3} \rfloor + 1)\|q_k\|$. We have

$$|\mu(I_{k+s} \cap I_k) - \mu(I_{k+s})\mu(I_k)| = \frac{q_k\|q_{k+s}\theta\|}{q_{k+s}\|q_k\theta\|}\mu(I_{k+s})\mu(I_k). \tag{6}$$

We choose a sequence $\{k_i\}$ such that

$$q_{k_i+1} > q_{k_i}^{\eta-\epsilon}.$$

Then for large i, $a_{k_i+1} \ge 3$, and if $x \in I_{k_i}$, then there is an n such that $q_{k_i} + q_{k_i-1} - 1 \le n < q_{k_i+1} - 1$ and

$$\frac{\log R_n(x)}{-\log \mu(P_n(x))} > \frac{\log q_{k_i+1} - \log 9}{\log q_{k_i} + \log 6} > \frac{(\eta - \epsilon)\log q_{k_i} - \log 9}{\log q_{k_i} + \log 6}.$$

Since q_k increases and $\|q_k\theta\|$ decreases exponentially, by (6) we have for any $n > k$

$$\sum_{j=1}^{n} \left|\mu(I_{k_{i+j}} \cap I_{k_i}) - \mu(I_{k_i})\mu(I_{k_{i+j}})\right| \le \sum_{j=1}^{n} C\mu(I_{k_{i+j}}) \tag{7}$$

for some constant C. By a version of the Borel-Cantelli lemma ([24], p. 45), for almost every x we have $x \in I_{k_i}$ for infinitely many i's. Hence we have for almost every x

$$\limsup_n \frac{\log R_n(x)}{-\log \mu(P_n(x))} \ge \eta - \epsilon. \qquad \square$$

Proof of Theorem 1.3. Since the proof is similar to those in previous sections, we will be brief. For $k \ge 1$ let

$$\bar{F}_k = \{x : 0 \le c_{k+1}(x) < a_{k+1} - \lfloor \frac{a_{k+1}}{2} \rfloor\}$$

as in the proof of Theorem 1.1 (ii). Choose a subsequence $\{k_i\}$ as $q_{k_i+1} > 2q_{k_i}^{\eta-\epsilon}$ for an $\epsilon > 0$. If we put $m'(k) = \lfloor \frac{a_{k+1}}{2} \rfloor q_k + q_{k-1} - 1$, then for $x \in \bar{F}_{k_i}$

$$\frac{\log R_{m'(k_i)}(x)}{\log m'(k_i)} < \frac{\log q_{k_i}}{\log(q_{k_i+1} - q_{k_i}) - \log 2} < \frac{\log q_{k_i}}{(\eta - \epsilon)\log q_{k_i} - \log 2}.$$

In the same way as in (4), for almost every x we have $x \in \bar{F}_{k_i}$ for infinitely many k's. By Theorem 1.1 for the partition $\mathcal{P}_n$ and by Theorem 1.2,

$$\liminf_{n\to\infty} \frac{\log R_n(x)}{\log n} = \frac{1}{\eta} \quad \text{a.e. } x.$$

Take an $\epsilon > 0$ and let

$$J_k = \{x : R_n(x) > n^{1+\epsilon} \text{ for some } q_k + q_{k-1} \le n < q_{k+1} + q_k\}.$$

Then for sufficiently large k

$$\begin{aligned} J_k &\subset \{x : |P_n(x)| = \|q_k\theta\|,\ q_{k+1} + q_k > n^{1+\epsilon} \text{ for some } q_k \le n < q_{k+1}\} \\ &= \{x : |P_n(x)| = \|q_k\theta\|,\ n = \lfloor (q_{k+1} + q_k)^{\frac{1}{1+\epsilon}} \rfloor\}. \end{aligned}$$

As in (2) in the proof of Theorem 1.1, we have $\sum_k \mu(J_k) < \infty$. Combined with Theorem 1.1 (ii) for the partition $\mathcal{P}_n$ and Theorem 1.2 (ii), we have

$$\limsup_{n\to\infty} \frac{\log R_n(x)}{\log n} = 1 \quad \text{a.e. } x. \qquad \square$$

4. Automatic sequences

An infinite sequence $u = u_0u_1u_2\dots$ is called k-automatic if it is generated by a k-automaton. An infinite sequence is k-automatic if and only if it is the image under a coding of a fixed point of a k-uniform morphism σ [8]. The Morse sequence (or Prouhet-Thue-Morse sequence), defined as $u_0 = 0$ and $u_{2n} = u_n$, $u_{2n+1} = 1-u_n$ for all $n \ge 0$, is a well-known example of the automatic sequences, since the Morse sequence is a fixed point of the 2-uniform morphism of $\sigma(0) = 01$ and $\sigma(1) = 10$.

A sequence u is called Sturmian if $p_u(n) = n + 1$, where $p_u(n)$ denotes the number of different words of length n occurring in u. It is well known (see e.g. [17]) that u is Sturmian if and only if u is an infinite $\mathcal{P}$-naming of an irrational rotation. Theorem 1.3 says that there exist many Sturmian sequences where the property $\log R_n(u)/\log n$ does not converge, while it has $\lim p(n)/n = 1$.

The following theorem states that for the Morse sequence we have that $\log R_n(u)/\log n$ converges, while it is known [6] that

$$\limsup \frac{p_u(n)}{n} = \frac{10}{3},\ \liminf \frac{p_u(n)}{n} = 3.$$

Theorem 4.1. *Let u be a noneventually periodic automatic sequence on alphabet A. Then we have*

$$\lim_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1.$$

The k-kernel of a sequence $u = (u_n)_{n\ge0}$ is defined as the set $N_k(u)$ of all sequences $(u_{k^i\cdot n+j})_{n\ge0}$, where $i \ge 0$ and $0 \le j < k^i$. A sequence is k-automatic if and only if its k-kernel is finite [9]. Let $A^* = \bigcup_{n\ge0} A^n$ and $A^+ = \bigcup_{n\ge1} A^n$, where $A^0 = \{e\}$ is the set of the empty word.

Lemma 4.2 ([1])**.** *Let u be a noneventually periodic k-automatic sequence on A. Let $x \in A^*$, $y \in A^+$, and $z \in A^*$ be such that xyz is a prefix of u and z is a prefix of yz. Let $m = \#N_k(x)$. Then*

$$\frac{|xyz|}{|xy|} < k^m.$$

Assume that $R_n(u) < \infty$. If we put $z = u_1^n$, then yz is a prefix of u and $|y| = R_n(u)$. By Lemma 4.2 we have

$$\frac{|yz|}{|y|} < k^m,$$

which implies that

$$(k^m - 1)R_n(u) > n. \tag{8}$$

Proof of Theorem 4.1. Let k be the width of the morphism σ. Let $a \in A$ be the first letter of u. Since u is recurrent, we let axa be a prefix of u. Let $c = |x|$. Since u is the fixed point of σ, $\sigma^k(a)\sigma^k(x)\sigma^k(a)$ is also a prefix of u. For any n, choose ℓ as $k^{\ell-1} = |\sigma^{\ell-1}(a)| < n \leq |\sigma^\ell(a)| = k^\ell$. Then we have

$$R_n(u) \leq |\sigma^\ell(a)| + |\sigma^\ell(x)| = k^\ell + |x|k^\ell < (|x|+1)kn. \tag{9}$$

By (8) and (9), we complete the proof. □

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