A NOTE ON SCALING ASYMPTOTICS FOR
BOHR-SOMMERFELD LAGRANGIAN SUBMANIFOLDS

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Abstract. This paper deals with the asymptotic expansions describing the quantum states associated to Bohr-Sommerfeld Lagrangian submanifolds of a compact Kähler manifold, in the context of geometric quantization. More precisely, it provides an improvement on a result of the work of Debernardi and the author (2006), describing a natural factorization of the expansion and providing certain remainder estimates.

Introduction

The purpose of this paper is to improve an expansion in [5] for the asymptotics associated to Bohr-Sommerfeld Lagrangian submanifolds of a compact Hodge manifold, in the context of geometric quantization (see e.g. [1], [2], [6], [9]). We adopt the general framework for quantizing Bohr-Sommerfeld Lagrangian submanifolds presented in [2], based on applying the Szegő kernel of the quantizing line bundle to certain delta functions concentrated along the submanifold.

Let $M$ be a $d$-dimensional complex projective manifold, with complex structure $J$; consider an ample line bundle $A$ on it, and let $h$ be a Hermitian metric on $A$ such that the unique compatible connection has curvature $\Omega = -2i\omega$, where $\omega$ is a Kähler form. Then the unit circle bundle $X \subseteq A^*$, endowed with the connection one-form $\alpha$, is a contact manifold. A Bohr-Sommerfeld Lagrangian submanifold of $M$ (or, more precisely, of $(M, A, h)$) is then simply a Legendrian submanifold $\Lambda \subseteq X$, conceived as an immersed submanifold of $M$.

In a standard manner, $X$ inherits a Riemannian structure for which the projection $\pi : X \to M$ is a Riemannian fibration. In view of this, we shall implicitly identify (generalized) functions, densities and half-densities on $X$ at certain places.

Referring to §2 of [5] for a more complete description of the preliminaries involved, we recall that if $\Lambda \subseteq X$ is a compact Legendrian submanifold and $\lambda$ is a half-density on it, there is a naturally induced generalized half-density $\delta_{\Lambda, \lambda}$ on $X$ supported on $\Lambda$. Following [2], we can then define a sequence of CR functions

$$u_k := P_k(\delta_{\Lambda, \lambda}) \in \mathcal{H}(X)_k,$$

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where $\mathcal{H}(X)_k$ is the $k$-th isotype of the Hardy space with respect to the $S^1$-action, and $P_k : L^2(X) \to \mathcal{H}(X)_k$ is the orthogonal projector (extended to $\mathcal{D}'(X) \to \mathcal{H}(X)_k$). In the present setting there are natural unitary structures on $\mathcal{H}(X)_k$ and the space of global holomorphic sections $H^0 \left(M, A^{\otimes k}\right)$, and a natural unitary isomorphism $\mathcal{H}(X)_k \cong H^0 \left(M, A^{\otimes k}\right)$. One thinks of the $u_k$’s as representing the quantizations of $(\Lambda, \lambda)$ at Planck’s constant $1/k$. It is easily seen that $u_k$ is rapidly decaying as $k \to +\infty$ on the complement of $S^1 \cdot \Lambda = \pi^{-1} \left(\pi(\Lambda)\right)$; here $\pi : X \to M$ is the projection. On the other hand, the asymptotic concentration of the $u_k$’s along $S^1 \cdot \Lambda$ poses an interesting problem, already considered in Theorem 3.12 of [2].

This theme was revisited in [5] in a somewhat different technical setting; in particular, Corollary 1.1 of [5] shows that the scaling asymptotics of $u_k$ (to be defined shortly) near any $x \in S^1 \cdot \Lambda$ admit an asymptotic expansion, and explicitly computes the leading order term. We shall presently give a more precise description of this expansion as a function on the tangent space of $M$ at $m = \pi(x)$. Namely, we shall show that this asymptotic expansion may be factored as an exponentially decaying term in the component $w^\perp$ of $w \in T_mM$ orthogonal to $\Lambda$ times an asymptotic expansion with polynomial coefficients in $w$ (more precisely, the expansion is generally given by a finite sum of terms of this form, one from each branch of $\Lambda$ projecting to $m$); the exponential term also contains a symplectic pairing between $w^\perp$ and the component of $w$ along $\Lambda$, $w^\parallel$. Furthermore, we shall give some relevant remainder estimates not mentioned in [5].

Before stating the results of this paper, let us recall that for any $x \in X$ we can find a Heisenberg local chart for $X$ centered at $x$,

$$
\rho : B_{2d}(\epsilon) \times (-\pi, \pi) \to X, \quad (p, q, \theta) \mapsto r_{e^{i\theta}} \left(\epsilon(p, q)\right);
$$

here $B_{2d}(\epsilon) \subseteq \mathbb{R}^{2d}$ is a ball of radius $\epsilon$ centered at the origin, $\varrho : B_{2d}(\epsilon) \to M$ is a preferred local chart for $M$ centered at $m =: \pi(x)$, meaning that it trivializes the holomorphic and symplectic structures at $m$, and $\epsilon$ is a unitary local frame of $A^*$, given by the unitarization of a preferred local frame (complete definitions are in [8]). Finally, $r_{e^{i\theta}} : X \to X$ is the diffeomorphism induced by the circle action. It is in this kind of local coordinate that the scaling limits of Szegő kernels exhibit their universal nature (Theorem 3.1 of [8]). If $\rho$ is a system of Heisenberg local coordinates centered at $x$, and $p, q \in \mathbb{R}^d, w = p + iq$, one poses

$$
x + w =: \rho((p, q), 0).
$$

For any $\theta$, we have

$$
u_k(\rho(p, q, \theta)) = e^{ik\theta} u_k(\rho(p, q, 0)) = e^{ik\theta} u_k(x + w).
$$

Given that a system of Heisenberg local coordinates induces a unitary isomorphism of $T_mM$ and $\mathbb{C}^d$, with this understanding we can also consider the expression $x + w$ with $w \in T_mM$.

If $x \in S^1 \cdot \Lambda$, there are only finitely many elements $h_1, \ldots, h_{N_{\Lambda}} \in S^1$ such that $x_j =: r_{h_j}(x) \in \Lambda$. Since $\Lambda$ is Legendrian, hence horizontal for the connection, for any $j$ we may naturally identify the tangent space $T_{x_j}\Lambda \subseteq T_{x_j}X$ with a subspace of $T_{\pi(x)}M$. With this in mind, if $w \in T_{\pi(x)}M$ we can write $w = w^\parallel + w^\perp$ for unique $w^\parallel \in T_{x_j}\Lambda$ and $w^\perp \in T_{x_j}\Lambda^\perp$; the latter denotes the orthocomplement of $T_{x_j}\Lambda$ in $T_{\pi(x)}M$ in the Riemannian metric of $M$. 


Finally, let $\text{dens}_{\Lambda}^{(1/2)}$ be the Riemannian half-density on $\Lambda$ (for the induced metric); thus if $\lambda$ is a $C^\infty$ half-density on $\Lambda$ we can write $\lambda = F_\lambda \cdot \text{dens}_{\Lambda}^{(1/2)}$ for a unique $F_\lambda \in C^\infty(\Lambda)$.

**Theorem 1.** Let $\Lambda \subseteq X$ be a compact Legendrian submanifold, and suppose $\lambda$ is a smooth half-density on it. For every $k = 1, 2, \ldots$, let $u_k := P_k(\delta_\Lambda, \lambda)$. Suppose $x \in S^1 \cdot \Lambda$, and choose a system of Heisenberg local coordinates for $X$ centered at $x$. Let $h_1, \ldots, h_N \in S^1$ be the finitely many elements such that $r_{h_j}(x) \in \Lambda$. Then:

1. Suppose $a > 0$. Uniformly for $\min_j \{\|w_j^+\|\} \gtrsim k^a$, we have
   \[ u_k \left( x + \frac{w}{\sqrt{k}} \right) = O(k^{-\infty}). \]

2. There exist polynomials $a_{ij}$ on $\mathbb{C}^d$ such that the following holds: for $w \in T_{\pi(x)}M$ and $k, \ell = 1, 2, \ldots$, let us define
   \[ R_{k, \ell} (x, w) := u_k \left( x + \frac{w}{\sqrt{k}} \right) \]
   \[ - \left( \frac{2k}{\pi} \right)^{d/2} \sum_{j=1}^n h_j^{-k} e^{-\|w_j^+\|^2 - i\omega_{\pi(x)}(w_j^+, w_j^-)} F_\lambda(x_j) \]
   \[ \cdot \left( 1 + \sum_{i=1}^{\ell} k^{-1/2} a_{ij}(w) \right). \]

Then uniformly for $\|w\| \lesssim k^{1/6}$ we have
\[ |R_{k, \ell} (x, w)| \leq C_{\ell} k^{(d-\ell-1)/2} \sum_{j=1}^N e^{\frac{-1-k}{k} \|w_j^+\|^2}. \]

**Corollary 1.** $\forall w \in T_{\pi(x)}M$, the following asymptotic expansion holds as $k \to +\infty$:
\[ u_k \left( x + \frac{w}{\sqrt{k}} \right) \sim \left( \frac{2k}{\pi} \right)^{d/2} \sum_{j=1}^n h_j^{-k} e^{-\|w_j^+\|^2 - i\omega_{\pi(x)}(w_j^+, w_j^-)} F_\lambda(x_j) \]
\[ \cdot \left( 1 + \sum_{i=1}^{\ell} k^{-1/2} a_{ij}(w) \right). \]

This agrees with Corollary 1.1 of [5] to leading order but gives a clearer picture of the asymptotic expansion, as well as an explicit estimate on the remainder.

The proof of Theorem 1 is based on the scaling asymptotics of Szegö kernels in Theorem 3.1 of [3], whereas the proofs in [3] are based on microlocal arguments that encompass the equivariant setting, involving a direct use of the parametrix developed in [3]. In view of the scaling asymptotics of equivalent Szegö kernels proved in [7], factorizations akin to Theorem 1 also hold in the equivariant setting; we shall not discuss this here.

**Proof of Theorem 1**

Let us first prove item 2. Thus, we want to investigate the asymptotics of $u_k \left( x + \frac{w}{\sqrt{k}} \right)$ as $k \to +\infty$, assuming that $w \in T_{\pi(x)}M$, $\|w\| \leq C k^{1/6}$ for some fixed $C > 0$. 

Let \( \Pi_k \in \mathcal{C}^\infty(X \times X) \) be the Schwartz kernel of \( \mathcal{P}_k \); explicitly, if \( \{ s_r^{(k)} \} \) is an orthonormal basis of \( \mathcal{H}(X) \), then
\[
\Pi_k(y, y') = \sum_r s_r^{(k)}(y) \cdot s_r^{(k)}(y') \quad (y, y' \in X).
\]

Let \( \text{dens}_X \) and \( \text{dens}_\Lambda \) denote, respectively, the Riemannian density on \( X \) and \( \Lambda \). Then, in standard distributional shorthand, by definition of \( \delta_{\Lambda, \lambda} \) for any \( x' \in X \) we have
\[
u_k(x') = \int_X \Pi_k(x', y) \delta_{\Lambda, \lambda}(y) \text{dens}_X(y)
= \langle \delta_{\Lambda, \lambda}, \Pi_k(x', \cdot) \rangle = \int_\Lambda \Pi_k(x', y) F_\lambda(y) \text{dens}_\lambda(y).
\]

Let us write \( \text{dist}_M \) for the Riemannian distance function on \( M \), pulled back to a smooth function on \( X \times X \) by the projection \( \pi \times \pi \). Let us set:
\[
\begin{align*}
V_k &= \{ x' \in X : \text{dist}_M(x, x') < 4C \sqrt{k}^{-1/3} \}, \\
V'_k &= \{ x' \in X : \text{dist}_M(x, x') > 3C \sqrt{k}^{-1/3} \}.
\end{align*}
\]
If \( y \in V'_k \) and \( \|w\| \leq C \sqrt{k}^{1/6} \), then \( \text{dist}_M \left( x + \frac{w}{\sqrt{k}}, y \right) \geq C \sqrt{k}^{-1/3} \) for \( k \gg 0 \); by the off-diagonal estimates on Szegö kernels in \([4]\), therefore, \( \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) = O \left( k^{-\infty} \right) \) uniformly for \( y \in V'_k \).

For \( k \gg 0 \), \( \Lambda \cap V_k \) has \( N_x \) connected components:
\[
\Lambda \cap V_k = \bigcup_{j=1}^{N_x} \Lambda_{kj},
\]
where \( \Lambda_{kj} \) is the connected component containing \( x_j \). Let \( \{ s_k, s'_k \} \) be an \( S^1 \)-invariant partition of unity on \( X \) subordinate to the open cover \( \{ V_k, V'_k \} \). In view of (2) and the previous discussion, we obtain
\[
u_k \left( x + \frac{w}{\sqrt{k}} \right) = \int_{\Lambda} \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) F_\lambda(y) \text{dens}_\lambda(y)
\sim \sum_{j=1}^{N_x} \int_{\Lambda_{kj}} \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) F_\lambda(y) s_k(y) \text{dens}_\lambda(y),
\]
where \( \sim \) means that the two terms have the same asymptotics. Let us now evaluate the asymptotics of the \( j \)-th summand in (3).

To this end, recall that the Heisenberg local chart \( \rho \) centered at \( x \) depends on the choice of the preferred local chart \( \varrho \) at \( \pi(x) \) and of the local frame \( \mathfrak{e} \) of \( A^* \). We obtain a Heisenberg local chart \( \rho' \) centered at \( x \) by setting \( \rho'(p, q, \theta) := f_{h,j} (\rho(p, q, \theta)) \).

By the discussion in §2 of \([3]\) and (48) of the same paper, we can compose \( \rho' \) with a suitable transformation in \( (p, q) \) (that is, a change of preferred local chart for \( M \)) so as to obtain a Heisenberg local chart \( \rho_j(p, q, \theta) \) centered at \( x_j \) with the following property: \( \Lambda \) is locally defined near \( x_j \) by the conditions \( \theta = f_j(q) \) and \( p = 0 \), where \( f_j \) vanishes to third order at the origin. By construction, we have \( \rho_j(p, q, \theta) = f_{h,j} (\rho(p', q', \theta)) \) for a certain local diffeomorphism \( (p, q) \mapsto (p', q') \).

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Thus $\Lambda$ is locally parametrized, near $x_j$ and in the chart $\rho_j$, by the imaginary vectors $iq$; viewing the $q$’s as local coordinates on $\Lambda$ near $x_j$, we have $\text{dens}_\Lambda = D_\Lambda \cdot |dq|$ for a unique locally defined smooth function $D_\Lambda$. By construction of Heisenberg local coordinates, $D_\Lambda(0) = 1$.

Applying a rescaling by $k^{-1/2}$, we obtain

$$\int_{\Lambda, x_j} \Pi_k \left( x + \frac{w}{\sqrt{k}} y \right) F_\lambda(y) \text{dens}_\lambda(y)$$

$$= k^{-d/2} \int_{\mathbb{R}^d} \Pi_k \left( x + \frac{w}{\sqrt{k}}, r e^{i f_j(q/\sqrt{k})} \right) \left( x_j + \frac{iq}{\sqrt{k}} \right)$$

$$\cdot F_\lambda \left( \frac{q}{\sqrt{k}} \right) s_k \left( \frac{iq}{\sqrt{k}} \right) D_\lambda \left( \frac{q}{\sqrt{k}} \right) dq$$

(4)

$$= k^{-d/2} \int_{\mathbb{R}^d} e^{-ik f_j \left( \frac{q}{\sqrt{k}} \right)} \Pi_k \left( x + \frac{w}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right) \cdot F_\lambda \left( \frac{q}{\sqrt{k}} \right) s_k \left( \frac{iq}{\sqrt{k}} \right) D_\lambda \left( \frac{q}{\sqrt{k}} \right) dq.$$ 

Here, $x + \frac{w}{\sqrt{k}} = \rho \left( \frac{\Re(w)}{\sqrt{k}}, \frac{\Im(w)}{\sqrt{k}}, 0 \right)$ (we use the Heisenberg chart to unitarily identify $T_m M$ with $\mathbb{C}^d$) and $x_j + \frac{iq}{\sqrt{k}} = \rho_j \left( 0, \frac{q}{\sqrt{k}}, 0 \right)$. Notice that $s_k \left( \frac{iq}{\sqrt{k}} \right) = 1$ for $||q|| \lesssim k^{1/6}$ and $s_k \left( \frac{q}{\sqrt{k}} \right) = 0$ for $||q|| \gtrsim k^{1/6}$. In particular, integration takes place over a ball of radius $\sim k^{1/6}$. Also, Taylor expanding $F_\lambda$ and $f_j$ at the origin, we have asymptotic expansions

$$F_\lambda \left( \frac{q}{\sqrt{k}} \right) \sim F_\lambda(x_j) + \sum_{r \geq 1} k^{-r/2} b_r(q), \quad D_\lambda \left( \frac{q}{\sqrt{k}} \right) \sim 1 + \sum_{r \geq 1} k^{-r/2} c_r(q),$$

and, since $f_j$ vanishes to third order at the origin,

$$f_j \left( \frac{q}{\sqrt{k}} \right) \sim \sum_{r \geq 0} k^{-(3+r)/2} d_r(q), \quad e^{-ik f_j \left( \frac{q}{\sqrt{k}} \right)} \sim 1 + \sum_{r \geq 1} k^{-r/2} e_r(q),$$

for suitable polynomials $b_r, c_r, d_r, e_r$.

Let $w_j \in \mathbb{C}^d$ correspond to $w$ in the Heisenberg local coordinates $\rho_j$. By the above, Taylor expanding the transformation $(p, q) \mapsto (p', q')$, we obtain $x + \frac{w}{\sqrt{k}} = r_{h_j^{-1}} \left( x_j + \frac{w}{\sqrt{k}}, H(w, k) \right)$, where $H(w, k) \sim \sum_{i \geq 2} k^{-i/2} h_i(w)$. Without affecting the leading order term of the resulting asymptotic expansion, we may pretend for simplicity that $x + \frac{w}{\sqrt{k}} = r_{h_j^{-1}} \left( x_j + \frac{w}{\sqrt{k}} \right)$.

Write $w_j = p_j + iq_j$, with $p_j, q_j \in \mathbb{R}^d$. Thus $w_j^\perp = p_j$, $w_j^\parallel = iq_j$. In view of Theorem 3.1 of [5], we have

$$\Pi_k \left( x + \frac{w}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right)$$

$$= \Pi_k \left( r_{h_j^{-1}} \left( x_j + \frac{w}{\sqrt{k}} \right), x_j + \frac{iq}{\sqrt{k}} \right) = h_{j^{-1}}^{-k} \Pi_k \left( x_j + \frac{w}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right)$$

$$\sim h_j^{-k} \left( \frac{k}{\pi} \right)^d e^{-ip_j \cdot q - \frac{1}{2} ||p_j||^2 - \frac{1}{2} ||q_j||^2} \left( 1 + \sum_{r \geq 1} k^{-r/2} R_j(w, q) \right),$$

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for certain polynomials $R_j$ in $w$ and $q$. Furthermore, by the large ball estimate on the remainder discussed in §5 of [3], the remainder after summing over $1 \leq r \leq R$ is bounded by

\begin{equation}
C_R \left( \frac{k}{R+1} \right) e^{-\frac{1}{2} \|q\|} e^{-\left(\|q\| + \|q - q\|\right)^2}.
\end{equation}

It follows that the product of these asymptotic expansions can be integrated term by term; given this, we only lose a contribution which is $O(k^{-\infty})$ by setting $s_k = 1$ and integrating over all of $\mathbb{R}^d$.

We have

\begin{equation}
\int_{\mathbb{R}^d} e^{-iP_j \cdot q - \frac{1}{2} \|q\|^2} dq = e^{-iP_j \cdot q_j} \int_{\mathbb{R}^d} e^{-iP_j \cdot s - \frac{1}{2} \|s\|^2} ds = (2\pi)^{d/2} e^{-iP_j \cdot q_j - \frac{1}{2} \|P_j\|^2}.
\end{equation}

Given (5), this implies that (4) is given by an asymptotic expansion with leading order term

\begin{equation}
h_j^{-k} \left( \frac{2k}{\pi} \right)^{d/2} e^{-\|w_j\|^2 - i\omega(x)(w_j^+,w_j^-)} F_\lambda(x_j).
\end{equation}

To determine the general term of the expansion, on the other hand, we are led to computing integrals of the form

\begin{equation}
\int_{\mathbb{R}^d} q^{\lambda} e^{-iP_j \cdot q - \frac{1}{2} \|q\|^2} dq = e^{-iP_j \cdot q_j} \int_{\mathbb{R}^d} (s + q_j)^{\lambda} e^{-iP_j \cdot s - \frac{1}{2} \|s\|^2} ds,
\end{equation}

where $q^{\lambda}$ is some monomial. Thus we are led to a sum of terms of the form

\begin{equation}
e^{-iP_j \cdot q_j} C_\gamma(q_j) \int_{\mathbb{R}^d} s^{\gamma} e^{-iP_j \cdot s - \frac{1}{2} \|s\|^2} ds,
\end{equation}

and the integral is the evaluation at $p_j$ of the Fourier transform of $s^{\gamma} e^{-\frac{1}{2} \|s\|^2}$. Up to a scalar factor, the latter is an iterated derivative to $e^{-\frac{1}{2} \|s\|^2}$; therefore we are left with a summand of the form $e^{-iP_j \cdot q_j} T(q_j,q_j) e^{-\frac{1}{2} \|P_j\|^2} ds$, where $T$ is a polynomial in $p_j,q_j$. Given (5), this implies that the general term of the asymptotic expansion for (4) has the form

\begin{equation}
h_j^{-k} \left( \frac{2k}{\pi} \right)^{d/2} k^{-l/2} e^{-\|w_j\|^2 - i\omega(x)(w_j^+,w_j^-)} F_\lambda(x_j) \cdot a_{lj}(w)
\end{equation}

for an appropriate polynomial $a_{lj}(w)$. Finally, (1) (at $x_j$) follows by integrating (6).

To complete the proof of item 2, we need only sum over $j$.

Turning to the proof of item 1, by definition of preferred local coordinates, if $w_j^+ \geq C k^a$, say, then

\begin{equation}
\text{dist}_M \left( x + \frac{v}{\sqrt{k}}, \Lambda_{kJ} \right) \geq \frac{C}{2} k^{a - \frac{1}{2}},
\end{equation}

for all $k \gg 0$. By the off-diagonal estimates of [4], $\Pi_k \left( x + \frac{w_j}{\sqrt{k}}, y \right) = O(k^{-\infty})$ uniformly for $y \in \Lambda_{kJ}$. \hfill $\square$
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