THE LIMITING DISTRIBUTION OF THE COEFFICIENTS OF THE $q$-CATALAN NUMBERS

WILLIAM Y. C. CHEN, CAROL J. WANG, AND LARRY X. W. WANG

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Abstract. We show that the limiting distributions of the coefficients of the $q$-
Catalan numbers and the generalized $q$-Catalan numbers are normal. Despite
the fact that these coefficients are not unimodal for small $n$, we conjecture
that for sufficiently large $n$, the coefficients are unimodal and even log-concave
except for a few terms of the head and tail.

Introduction

The main objective of this paper is to show that the limiting distribution of the
coefficients of the $q$-Catalan numbers is normal. The Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

have many combinatorial interpretations; see Stanley [10]. The usual $q$-analog of
the Catalan numbers is given by

$$C_n(q) = \frac{1}{[n+1]} \left[\frac{2n}{n}\right],$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$, and

$$\left[\frac{n}{k}\right] = \frac{[n]!}{k![n-k]!}.$$

There are also other types of $q$-analogs of the Catalan numbers; see, for example,
Andrews [2], Gessel and Stanton [4], Krattenthaler [5].

We further consider the limiting distribution of the coefficients of the quotient of
two products, which includes the result for the $q$-Catalan numbers as a special case.
We conclude this paper with two conjectures on the unimodality and log-concavity
for almost all the coefficients of the $q$-Catalan numbers and the generalized $q$-
Catalan numbers provided that $n$ is sufficiently large.

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THE LIMITING DISTRIBUTION

In this section, we use the moment generating function technique to obtain the limiting distribution of the coefficients of the \( q \)-Catalan numbers. We introduce the random variable \( \xi_n \) corresponding to the probability generating function

\[
\phi_n(q) = \frac{C_n(q)}{C_n}.
\]

As far as the computations are concerned, we will not need the following combinatorial interpretation of \( C_n(q) \). However, for the sake of completeness, we recall that \( \xi_n \) reflects the distribution of the major indices of Catalan words of length \( 2n \); see, for example, \([3]\). Moreover, we write

\[
C_n(q) = \sum_{m} m_n(k) q^k,
\]

where \( m_n(k) \) stands for the number of Catalan words of length \( 2n \) with major index \( k \). The following lemma gives the expectation and variance of \( \xi_n \).

**Lemma 2.1.** We have

(2.1) \[
E(\xi_n) = \frac{n(n-1)}{2} \quad \text{and} \quad \text{Var}(\xi_n) = \frac{n(n-1)(n+1)}{6}.
\]

**Proof.** By the definition of \( C_n(q) \), it is easy to check the following symmetry property of \( m_n(k) \):

\[
m_n(k) = m_n(n(n-1) - k).
\]

Hence

\[
E(\xi_n) = \frac{n(n-1)}{2}.
\]

Let

\[
F = F(q) = \prod_{i=1}^{n-1} (1 + q + \cdots + q^{n+i}) \quad \text{and} \quad G = G(q) = \prod_{i=1}^{n-1} (1 + q + \cdots + q^i).
\]

It is easily verified that \( C_n(q) = F/G \). Since

\[
C_n(q)''|_{q=1} = \left( \frac{F''G - F'G'}{G^2} - \frac{2G'F'}{G^2} + \frac{2G''F}{G^3} \right)|_{q=1} = \frac{1}{12} n(n-1)(3n^2 - n - 4)C_n,
\]

we obtain

\[
\text{Var}(\xi_n) = \frac{C_n(q)''|_{q=1}}{C_n} + E(\xi_n) - E(\xi_n)^2 = \frac{1}{6} n(n-1)(n+1).
\]

This completes the proof. \( \square \)

**Lemma 2.2.** When \( n \to \infty \), we have

\[
\sum_{k=2}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!} \sigma^2 \sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) \to 0
\]

uniformly for \( t \) from any bounded set, where the \( B_j \)'s are the Bernoulli numbers and \( \sigma^2 \) is the variance of \( \xi_n \) as given in (2.1).
Proof: The second summation can be expanded as follows:

\[
\sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) = \sum_{i=2}^{n} \sum_{j=1}^{2k} \binom{2k}{j} n^{j} i^{2k-j} = \sum_{j=1}^{2k} \binom{2k}{j} \left( \sum_{i=2}^{n} n^{j} i^{2k-j} \right).
\]

For \( k > 1 \), the second factor in the preceding summation is bounded by the following integral:

\[
\sum_{i=2}^{n} n^{j} i^{2k-j} < n^{j} \int_{1}^{n+1} t^{2k-j} dt = n^{j} \cdot \frac{(n+1)^{2k-j+1} - 1}{2k-j+1}.
\]

Consequently,

\[
\sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) < 2^{2k} (n+1)^{2k+1} - 8^{2k} n^{2k+1}.
\]

Since \( \sigma^2 = \frac{n^3 - n}{6} > \frac{n^3}{8} \) when \( n \) is sufficiently large, we have

\[
\sigma^{-2k} \sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) < 64^{2k} n^{1-k} \leq n^{-1/3} 64^{2k} n^{-k/3},
\]

for large \( n \) and \( k > 1 \). Thus

\[
\left| \sum_{2k, k \geq 3} B_{2k} \frac{t^{2k}}{2k(2k)!} \sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) \right| < n^{-1/3} \sum_{2k, k \geq 3} |B_{2k}| \frac{t^{2k}}{2k(2k)!} 64^{2k} n^{-k/3}.
\]

In view of the following asymptotic expansion of the Bernoulli numbers [1],

\[
|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}},
\]

the convergent radius \( R \) of the series \( \sum_{2k, k \geq 3} |B_{2k}| \frac{t^{2k}}{2k(2k)!} \) equals \( 2\pi \). Since \( t \) is from a bounded set, when \( n \) is large enough, the series

\[
\sum_{2k, k \geq 3} |B_{2k}| \frac{64tn^{-\frac{k}{3}}t^{2k}}{2k(2k)!}
\]

converges. Moreover, it is evident that \( 64tn^{-\frac{k}{3}} < 1 \); we can bound the above summation by the constant

\[
M_1 = \sum_{2k, k \geq 3} |B_{2k}| \frac{1}{2k(2k)!}.
\]

Similarly, it can be deduced that

\[
\sum_{2k, k \geq 3} B_{2k} \frac{t^{2k}}{2k(2k)!} \sigma^{-2k} \sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) < \frac{M_2}{n^{\frac{k}{3}}},
\]
where \( M_2 = \sum_{2k, k \geq 2} B_{2k} \frac{1}{2k(2k)!} \) is a constant. Hence
\[
\sum_{k=2}^{\infty} B_{2k} \frac{e^{2k}}{2k(2k)!} \sigma^{2k} \left( \sum_{i=2}^{n} ((n + i)^{2k} - i^{2k}) \right) < \frac{M_1 + M_2}{n^{1/3}},
\]
which tends to zero as \( n \to \infty \). This completes the proof. \( \square \)

In [7], Margolius used Bernoulli numbers to show that the distribution of the number of inversions in a random permutation is asymptotically normal. In [6], Louchard and Prodinger used the saddle point method to derive some stronger results. Based on Lemma 2.2, we obtain the following theorem.

**Theorem 2.3.** When \( n \to \infty \), the random variable
\[
\eta_n = \frac{\xi_n - E(\xi_n)}{\text{Var}(\xi_n)^{1/2}}
\]
has the standard normal distribution.

**Proof.** Let \( M_n(q) \) denote the moment generating function of \( \xi_n \). Then we have \( M_n(q) = \phi_n(e^q) \); see Sachkov [8]. Hence
\[
M_n(q) = \frac{n + 1}{\binom{2n}{n}} \frac{1 - e^q}{1 - e^{(n+1)q}} \prod_{i=1}^{n} \frac{1 - e^{(n+i)q}}{1 - e^{iq}}
= \prod_{i=2}^{n} \frac{i}{n+i} \prod_{i=2}^{n} \frac{1 - e^{(n+i)q}}{1 - e^{iq}}
= \prod_{i=2}^{n} \frac{(1 - e^{(n+i)q})/(n + i)}{(1 - e^{iq})/i}
= \exp \left\{ \frac{1}{2} \sum_{i=2}^{n} \left( (n + i)q - iq \right) \right\} \prod_{i=2}^{n} \left( \frac{e^{(n+i)q/2} - e^{-(n+i)q/2}}{\sinh (iq/2)} \right)^{n+i/2}
= \exp \left\{ \frac{n(n - 1)q}{2} \right\} \prod_{i=2}^{n} \frac{\sinh ((n + i)q/2)}{\sinh (iq/2)}^{n+i/2}.
\]
Recalling the following relation on the Bernoulli numbers [7]
\[
\ln \left( \frac{\sinh(x/2)}{x/2} \right) = \sum_{k=1}^{\infty} \frac{x^{2k}}{2k(2k)!},
\]
we find that
\[
\ln M_n(q) = \frac{n(n - 1)}{2} q + \sum_{i=2}^{n} \left( \ln \left( \frac{\sinh ((n + i)q/2)}{(n + i)/2} \right) - \ln \left( \frac{\sinh (iq/2)}{i/2} \right) \right)
= \frac{n(n - 1)}{2} q + \sum_{k=1}^{\infty} B_{2k} \frac{q^{2k}}{2k(2k)!} \sum_{i=2}^{n} ((n + i)^{2k} - i^{2k}) \, .
\]
Setting \( q = t/\sigma \), where \( \sigma \) is the standard deviation of \( \xi_n \) as given in Lemma 2.1, we are led to the expansion
\[
\ln M_n(t/\sigma) = \frac{n(n - 1)t}{2\sigma} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \sum_{i=2}^{n} ((n + i)^{2k} - i^{2k}) \, .
\]
Applying Lemma 2.2, we have, when \( n \rightarrow \infty \),
\[
\sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k)!} \sum_{i=2}^{n} (n+i)^{2k} - (i^{2k}) \rightarrow 0
\]
uniformly for \( t \) from any bounded set. Finally,
\[
\lim_{n \rightarrow \infty} M_n(t/\sigma) \exp \left\{ -\frac{n(n-1)t}{2\sigma} \right\} = \lim_{n \rightarrow \infty} \exp \left\{ \sum_{k=1}^{\infty} \frac{B_{2k} t^{2k}}{2k(2k)!} \sum_{i=2}^{n} ((n+i)^{2k} - i^{2k}) \right\} = e^{t^2/2},
\]
which coincides with the moment generating function of the standard normal distribution. Employing Curtiss's theorem \([8]\), we reach the conclusion that \( \eta_n \) has the standard normal distribution when \( n \) approaches infinity. \( \square \)

A general setting

In this section, we will determine the limiting distribution of the coefficients of a quotient of products and will give two special cases.

**Theorem 3.1.** Let \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) be two sequences of positive numbers, and let
\[
\phi_n(x) = \sum_k p_n(k)x^k = \frac{(1-q^{a_1}))(1-q^{a_2})\cdots(1-q^{a_n})}{(1-q^{b_1})(1-q^{b_2})\cdots(1-q^{b_n})}.
\]
Suppose that \( \xi_n \) is the random variable corresponding to the generating function \( \phi_n(x) \), that is,
\[
P(\xi_n = k) = p_n(k) \sum_k p_n(k).
\]
Then \( \xi_n \) is normally distributed as \( n \rightarrow \infty \) if and only if
\[
\sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k)!} \left( \sum_{i=1}^{n} (a_i^2k - b_i^2k) \right) \frac{1}{\left( \sum_{i=1}^{n} (a_i^2 - b_i^2) \right)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

**Proof.** The expectation of \( \xi_n \) is easy to compute, as given below:
\[
E(\xi_n) = \phi_n(x)_{q=1} = \frac{1}{2} \sum_{i=1}^{n} (a_i - b_i).
\]

Proceeding analogously as in the proof of Lemma 2.1, we find
\[
\sigma^2 = \text{Var}(\xi_n) = \frac{1}{12} \sum_{i=1}^{n} (a_i^2 - b_i^2),
\]

\( (3.1) \)
Hence,

\[
B_2 \frac{t^2}{2(2)!\sigma^2} \left( \sum_{i=1}^{n} (a_i^2 - b_i^2) \right) = \frac{1}{6} \cdot \frac{t^2}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} (a_i^2 - b_i^2) \right) = \frac{t^2}{2}.
\]

By the same procedure as in the proof of Theorem 2.3, we obtain

\[
\lim_{n \to \infty} M_n(t/\sigma) \exp \left\{ \frac{1}{2} \sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) \right\} = e^{t^2/2} \lim_{n \to \infty} \exp \left\{ \sum_{k=2}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \left( \sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) \right) \right\}.
\]

It follows that the limiting distribution of \(p_n(k)\) is normal if and only if

\[
\sum_{k=2}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \left( \sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

for \(t\) from any bounded set. By virtue of the variance formula (3.1), the condition (3.2) is equivalent to

\[
\sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \left( \sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

for \(t\) from any bounded set. Thus (3.2) is verified. This completes the proof. \(\Box\)

**Corollary 3.2.** Let \(p_n(k)\) be given as in the above theorem. Suppose that for \(k \geq 2\),

there exist constants \(\alpha > 0\), \(\beta < 0\) and \(\gamma < 0\) such that

\[
\frac{\sum_{i=1}^{n} (a_i^{2k} - b_i^{2k})}{\left( \sum_{i=1}^{n} (a_i^{2} - b_i^{2}) \right)^{\gamma}} < n^{\gamma}(\alpha n^\beta)^{2k},
\]

for \(t\) from any bounded set. Then the limiting distribution of \(p_n(k)\) is normal.

**Proof.** Note that the convergent radius \(R\) of the series

\[
\sum_{2k,k \geq 3} |B_{2k}| \frac{x^{2k}}{2k(2k)!}
\]

is \(2\pi\). If (3.4) holds for \(k > 1\), then for \(t\) from any bounded set, and for sufficiently large \(n\), we have

\[
\left| \frac{t^{2k}}{2k(2k)!} \sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) / \sigma^{2k} \right| \leq n^{\gamma}(\tan^\beta)^{2k},
\]

where \(\tan^\beta < 2\pi\). It is clear that \(n^{\gamma} \rightarrow 0\) since \(\gamma < 0\). \(\Box\)

If we choose \(\alpha = 32\sqrt{3}/3, \beta = \gamma = -\frac{1}{3}\), Corollary 3.2 contains Theorem 2.3 as a special case. We now give two more examples. One is the following \(q\)-analog of the Catalan numbers:

\[
c_n(q) = \frac{[2]}{[2n]} \left[ \frac{2n}{n-1} \right].
\]
which are symmetric and unimodal; see Stanley [9].

Using Theorem 3.1, we reach the following assertion.

**Corollary 3.3.** The distribution of the coefficients in \( c_n(q) \) is asymptotically normal.

**Proof.** First, we write \( c_n(q) \) in the following form:

\[
\prod_{i=1}^{n} \frac{(1 - q^{n+i-1})}{(1 - q) \prod_{i=3}^{n-1} (1 - q^i)}.
\]

Set \( a_1 = a_2 = 1, \ a_i = n + i - 1, \ 3 \leq i \leq n, \) and \( b_1 = b_2 = 1, \ b_i = i - 1, \ 4 \leq i \leq n. \) Then we have

\[
\sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) = (a_3^{2k} - b_3^{2k}) + \sum_{i=4}^{n} (a_i^{2k} - b_i^{2k})
\]

\[
= (n + 2)^{2k} - 1 + \sum_{i=3}^{n-1} ((n + i)^{2k} - i^{2k})
\]

and

\[
\left( \sum_{i=1}^{n} (a_i^2 - b_i^2) \right)^k = \left( (n + 2)^2 - 1 + \sum_{i=3}^{n-1} ((n + i)^2 - i^2) \right)^k
\]

\[
= (n - 1)^k (n + 1)^k (2n - 3)^k.
\]

By the same arguments as in the proof of Lemma 2.2, we may set \( \alpha = 32 \sqrt{3}/3 \) and \( 2 \beta = -\frac{\gamma}{4} \) such that the condition (3.4) is satisfied. Therefore, Theorem 3.1 implies the limiting distribution of the coefficients of \( c_n(q) \).

The **m-Catalan numbers** are defined by

\[
C_{n,m} = \frac{1}{(m-1)n+1} \binom{mn}{n},
\]

for \( n \geq 1 \). Accordingly, the generalized \( q \)-Catalan numbers are given by

\[
C_{n,m}(q) = \frac{1}{[(m-1)n+1]} \left[ \binom{mn}{n} \right].
\]

Theorem 3.1 has the following consequence.

**Corollary.** The coefficients of the generalized \( q \)-Catalan numbers \( C_{n,m}(q) \) are normally distributed when \( n \to \infty \).

**Proof.** First, express \( C_{n,m}(q) \) as follows:

\[
\prod_{i=2}^{n} \frac{1 - q^{(m-1)n+i}}{1 - q^i}.
\]

Set \( a_1 = 1, \ a_i = (m-1)n + i, \ 2 \leq i \leq n, \) and \( b_1 = 1, \ b_i = i, \ 2 \leq i \leq n. \) Then we have

\[
\sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) = \sum_{i=2}^{n} (a_i^{2k} - b_i^{2k}) = \sum_{i=2}^{n} \sum_{j=1}^{2k} \binom{2k}{j} ((m-1)n)^{2k-j} i^j.
\]
The same argument as in the proof of Lemma 2.2 yields the following bound:
\[
\sum_{i=1}^{n} (a_i^{2k} - b_i^{2k}) < 8^{2k} (m - 1) n^{2k+1}.
\]

Now,\[
\left(\sum_{i=1}^{n} (a_i^2 - b_i^2)\right)^k = \left(\sum_{i=1}^{n} ((m-1)n+i)^2 - i^2\right)^k > (m-1)^{2k} n^{2k} (n-1)^k > (m-1)^{2k+1} n^{3k}/(2m)^k.
\]

It follows that\[
\frac{\sum_{i=1}^{n} (a_i^{2k} - b_i^{2k})}{\left(\sum_{i=1}^{n} (a_i^2 - b_i^2)\right)^k} < (8\sqrt{2m})^{2k} n^{1-k}.
\]

Again, by the same arguments as in the proof of Lemma 2.2, we may set\[
\alpha = 8\sqrt{2m} \quad \text{and} \quad 2\beta = \gamma = -\frac{1}{3},
\]
such that the condition (3.4) holds. Finally, we may use Theorem 3.1 to get the desired distribution. \[\square\]

**Open problems**

While the \(q\)-Catalan numbers are not unimodal for small \(n\), see Stanley [9], the limiting distribution suggests that the coefficients are almost unimodal in a certain sense for sufficiently large \(n\). Obviously, the first and the last term should not be taken into account; otherwise one can never expect to have unimodality. In fact, an easy computation indicates that \(C_n(q)\) are unimodal for \(n \geq 16\).

**Conjecture 4.1.** The sequence \(\{m_n(1), \ldots, m_n(n(n-1)-1)\}\) is unimodal when \(n\) is sufficiently large.

When \(n > 70\), numerical evidence is suggestive of a stronger conjecture:

**Conjecture 4.2.** There exists an integer \(t\) such that when \(n\) is sufficiently large, the sequence \(\{m_n(t), \ldots, m_n(n(n-1)-t)\}\) is log-concave, namely,
\[
(m_n(k))^2 \geq m_n(k+1)m_n(k-1)
\]
for \(t+1 \leq k \leq n(n-1) - t - 1\). Moreover, the minimum value of \(t\) seems to be 75.

We also conjecture that similar properties hold for the generalized \(q\)-Catalan numbers.

**References**


Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, People’s Republic of China

E-mail address: chen@nankai.edu.cn

E-mail address: wangjian@cfc.nankai.edu.cn

E-mail address: wxw@cfc.nankai.edu.cn