ON A DESINGULARIZATION OF THE MODULI SPACE OF NONCOMMUTATIVE TORI

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Abstract. It is shown that the moduli space of the noncommutative tori \( A_\theta \) admits a natural desingularization by the group \( \text{Ext}(A_\theta, A_\theta) \). Namely, we prove that the moduli space of pairs \( (A_\theta, \text{Ext}(A_\theta, A_\theta)) \) is homeomorphic to a punctured two-dimensional sphere. The proof is based on a correspondence (a covariant functor) between the complex and noncommutative tori.

1. Introduction

A. Let \( 0 < \theta < 1 \) be an irrational number, whose regular continued fraction has the form \( \theta = [a_0, a_1, a_2, \ldots] \). Consider an AF-algebra \( A_\theta \) given by the Bratteli diagram in Figure 1. The \( a_i \) indicate the number of edges in the upper row of the diagram. With a moderate abuse of the terminology, we shall call \( A_\theta \) a noncommutative torus. (Note that a standard definition of the noncommutative torus – a universal \( C^* \)-algebra generated by the unitaries \( u, v \) satisfying the commutation relation \( vu = e^{2\pi i \theta} uv \) – is not an AF-algebra. However, the two objects are isomorphic at the level of their dimension groups [8], [9].)

B. Recall that the noncommutative tori \( A_\theta, A_{\theta'} \) are said to be stably isomorphic, whenever \( A_\theta \otimes K \cong A_{\theta'} \otimes K \), where \( K \) is the AF-algebra of the compact operators. It is well known that the AF-algebras \( A_\theta, A_{\theta'} \) are stably isomorphic if and only if \( \theta' \equiv \theta \mod SL(2, \mathbb{Z}) \), i.e. \( \theta' = (a\theta + b) / (c\theta + d) \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \) [3]. It is easy to see that the stable isomorphism is an equivalence relation, which splits the set \( \{ A_\theta \mid 0 < \theta < 1, \theta \in \mathbb{R} - \mathbb{Q} \} \) into the disjoint equivalence classes. By \( \mathcal{M} \) we shall understand a collection of such classes, or the
“moduli space” of the noncommutative tori. An examination of \(\mathcal{M}\) as a topological space (with the topology induced by \(\mathbb{R}\)) shows that the points of \(\mathcal{M}\) have no disjoint neighborhoods, since each orbit \(\{\theta' \in \mathbb{R} \mid \theta' \equiv \theta \mod SL(2, \mathbb{Z})\}\) is dense in the real line \(\mathbb{R}\). A question arises as to how to “desingularize” the (non-Hausdorff) moduli space \(\mathcal{M}\).

C. Let \(A, B\) be a pair of the \(C^*\)-algebras. Recall that an extension of \(A\) by \(B\) is a \(C^*\)-algebra \(E\) filling the short exact sequence \(0 \to B \to E \to A \to 0\) of the \(C^*\)-algebras. If \(A\) is a separable nuclear \(C^*\)-algebra, the \(\text{Ext} (A, B)\) is an additive abelian group, whose group operation is a sum of the two extensions. The \(\text{Ext} (A, B)\) is a homotopy invariant in both variables. The extensions \(E_1, E_2\) are said to be stably equivalent if there exists an isomorphism \(\psi : E_1 \otimes K \cong E_2 \otimes K\), such that \(\psi \circ \alpha_1(B \otimes K) = \alpha_2(B \otimes K)\), where \(\alpha_i : B \to E_i, i = 1, 2\) \([1]\). We shall further restrict to the case \(A = B = \mathbb{A}_g\) and study the stable equivalence classes of the group \(\text{Ext} (\mathbb{A}_g, \mathbb{A}_g)\). Using the classification results of D. Handelman \([5]\), it will develop that the group \(\text{Ext} (\mathbb{A}_g, \mathbb{A}_g) \cong \text{Hom} (K_0(\mathbb{A}_g), \mathbb{R}) \cong \mathbb{R}\). Moreover, the \(\text{Ext} (\mathbb{A}_g, \mathbb{A}_g)/\text{stable equivalence} \cong \mathbb{R}/\mathbb{Z}\).

D. An objective of the note is to show that the moduli of the pairs \((\mathbb{A}_g, \text{Ext} (\mathbb{A}_g, \mathbb{A}_g))\) under the stable equivalence is no longer a non-Hausdorff topological space, but a two-dimensional orbifold (a punctured sphere). To prove this result we shall use the Teichmüller space of a torus (a space of the complex structures on the torus) \([6]\). Namely, Hubbard and Masur established a homeomorphism between the Teichmüller space \(T_g\) of a surface of genus \(g \geq 1\) and the space of quadratic differentials on it. We shall use the homeomorphism to extend the action of the modular group \(SL(2, \mathbb{Z})\) from the upper half-plane \(\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}\) \(\cong T_1\) to the space \((\mathbb{A}_g, \text{Ext} (\mathbb{A}_g, \mathbb{A}_g))\). Denote by \(\widetilde{\mathcal{M}}\) the set of pairs \((\mathbb{A}_g, \text{Ext} (\mathbb{A}_g, \mathbb{A}_g))\) modulo the stable equivalence. One obtains the following (natural) desingularization of the moduli space of the noncommutative tori.

**Theorem 1.** \(\widetilde{\mathcal{M}}\) is a punctured two-dimensional sphere.

2. **Proof**

We shall split the proof into two lemmas. The background material is mostly standard, and we shall recall in passing some important notation and ideas.

**Lemma 1.** \(\mathcal{M}\) is a two-dimensional orbifold.

**Proof of Lemma** \([1]\). We shall use a standard dictionary existing between the \(AF\)-algebras and their dimension groups \([2]\). Instead of dealing with the \(AF\)-algebra \(\mathbb{A}_g\), we shall work with its dimension group \(G_\theta = (G, G^+)\), where \(G \cong \mathbb{Z}^2\) is the lattice and \(G^+ = \{(x, y) \in \mathbb{Z}^2 \mid x + \theta y \geq 0\}\) is a positive cone of the lattice. The \(G_\theta\) is the additive abelian group with an order, which defines the \(AF\)-algebra \(\mathbb{A}_g\) up to a stable isomorphism.

Under the dictionary, the extension problem for the \(AF\)-algebra \(\mathbb{A}_g\) translates as an extension problem for the dimension groups \(G_\theta \to E \to G_\theta\) (we omit the zeros in the exact sequence). An important result of Handelman establishes the intrinsic classification of the extensions of the simple dimension group by a simple dimension group; see Theorem III.5 of \([3]\). Let us recall the classification as it is exposed in \([3]\) Theorem 17.5 and Corollary 17.7. We shall adopt the same notation as in the cited work.
Let $H$ be a dense subgroup of the real line $\mathbb{R}$ and $K$ a nonzero dimension group. Let $E$ be the abelian group $H \oplus K$, and let $\tau : H \to E$ and $\pi : E \to K$ be a natural injection and projection maps. Assume that $f : K \to \mathbb{R}$ is a homomorphism of the dimension group. Then: (i) $E$ is a dimension group with the positive cone
\[ E^+_f = \{(0,0)\} \cup \{(x,y) \in E \mid y \geq 0 \text{ and } x + f(y) > 0\}, \]
which gives an extension $H \xrightarrow{\tau} (E,E^+_f) \xrightarrow{\pi} K$ of $H$ by $K$; (ii) if $f,f' : K \to \mathbb{R}$ are the group homomorphisms, then the extensions $E_f,E_{f'}$ are equivalent if and only if $(f-f')(K) \subseteq H$.

We have to specialize the above theorem to the case $H = K = G_\theta$. It is immediate from (i) that $E \cong \mathbb{Z}^2$. Note that the group homomorphisms $f : G_\theta \to \mathbb{R}$ are bijective with the reals $\mathbb{R}$. Indeed, we have to find all the linear maps $f : \mathbb{R}^2 \to \mathbb{R}$, such that $\ker f = x + \theta y$. (The last equation follows from the condition $f(G_\theta^+ > 0)$.) Such maps have the form $f(p) = (p,t)$, $p,t \in \mathbb{R}^2$, where $(p,t)$ is the dot product of the two vectors. Let $t = (t_1,t_2)$. Then $f(-p,y) = t_1(-\theta y) + t_2 y = y(t_2 - \theta) = 0$ for all $y \in \mathbb{R}$. Therefore, $t_2 = \theta t_1$ and $f_1(x,y) = t_1 x + \theta t_1 y = t_1 (x + \theta y)$, $t_1 \in \mathbb{R}$. Thus, all linear maps $f : \mathbb{R}^2 \to \mathbb{R}$ with $\ker f = x + \theta y$ are bijective with the reals $t_1 \in \mathbb{R}$. In other words, $\text{Ext} (k_\theta,k_\theta)$ and $\mathbb{R}$ are isomorphic as additive abelian groups.

Let us find when the two extensions $E,E'$ are equivalent. Since $H = G_\theta$ is a subgroup of $\mathbb{R}$, one can write $H = \mathbb{Z} + \theta \mathbb{Z}$. Let $t,t'$ be the real numbers corresponding to the homomorphisms $f,f'$. Then $f(G_\theta) = t(\mathbb{Z} + \theta \mathbb{Z})$ and $f'(G_\theta) = t'(\mathbb{Z} + \theta \mathbb{Z})$. The condition $(f-f')(K) \subseteq H$ of the item (ii) will take the form $(t-t')(\mathbb{Z} + \theta \mathbb{Z}) \subseteq \mathbb{Z} + \theta \mathbb{Z}$. One gets immediately that $t = t' + n,n \in \mathbb{Z}$ as a necessary and sufficient condition for the last inclusion. In other words, the extensions $E,E'$ are equivalent if and only if $t' = t \mod \mathbb{Z}$. Thus, the equivalence classes of $\text{Ext} (k_\theta,k_\theta)$ are bijective with the factor space $\mathbb{R}/\mathbb{Z}$ (a unit interval).

To finish the proof of Lemma 11 let us extend the domain of definition of $\theta$ from the interval $(0,1)$ to the real line $\mathbb{R}$ by allowing $a_0$ to take on any integer value. In this way, one can identify the pairs $(k_\theta,\text{Ext} (k_\theta,k_\theta))$ with the points of $\mathbb{R}^2$ equipped with the usual Euclidean topology. We have seen that the points $(\theta,t) \sim (\theta',t') \in \mathbb{R}^2$ are equivalent if and only if $\theta' \equiv \theta \mod SL(2,\mathbb{Z})$ and $t' \equiv t \mod \mathbb{Z}$. Note that the action of the modular group on the second coordinate is always free. Therefore, the points $x,y$ of the space $\mathcal{M} \cong \mathbb{R}^2/\sim$ admit the disjoint neighborhoods defined, e.g., by the open balls of radius $1/3$ centered in $x$ and $y$, respectively. The balls are locally homeomorphic to the Euclidean plane, and therefore $\mathcal{M}$ is a two-dimensional orbifold.\[\square\]

Lemma 11 gives a (partial) desingularization of the space $\mathcal{M}$. Indeed, we have seen that the group $SL(2,\mathbb{Z}) \times \mathbb{Z}$ acts in the plane $(k_\theta,\text{Ext} (k_\theta,k_\theta))$ by the formula $(\theta,t) \to (\theta + \frac{a_0}{\theta + d},t + n)$, where $ad - bc = 1$ and $a,b,c,d,n \in \mathbb{Z}$. However, the last formula does not specify the action on the parameter plane $(\theta,t)$ of the modular group $SL(2,\mathbb{Z})$ alone, since the function $n = n(a,b,c,d)$ is unknown. To find how the integer $n$ depends on the integers $a,b,c,d$, we would need a special construction which involves a correspondence (a covariant functor) between the complex and noncommutative tori. Such a construction will be given in the next paragraph and is encapsulated in the following lemma.

\[\text{1 That is, } f \text{ preserves the positive cone of } K \text{ and } \mathbb{R} : f(K^+) > 0.\]
Lemma 2. There exists a homeomorphism \( h : \tilde{\mathcal{M}} \to \mathbb{H}/SL(2, \mathbb{Z}) \), where \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) is the Lobachevsky plane endowed with a hyperbolic metric.

Proof of Lemma 2. Let \( X \) be a topological surface of genus \( g \geq 0 \). The Teichmüller space \( \text{Mod} \mathcal{X} \) of \( X \) consists of the equivalence classes of the complex structures on \( X \). The space \( \text{Mod} \mathcal{X} \) is an open ball of the (real) dimension \( 6g - 6 \) if \( g \geq 2 \) and \( 2g \) if \( g = 0, 1 \). By \( \text{Mod} X \) we designate a group of the orientation-preserving diffeomorphisms of \( X \) modulo the trivial ones. The points \( S, S' \in \text{Mod} \mathcal{X} \) are equivalent if there exists a conformal map \( f \in \text{Mod} X \) such that \( S' = f(S) \). The moduli of conformal equivalence is denoted by \( \mathcal{M}_g = \text{Mod} \mathcal{X} / \text{Mod} X \). The space \( \mathcal{M}_g \) is a (classical) moduli space, whose definition dates back to Riemann.

Let \( S \in \text{Mod} \mathcal{X} \) be a Riemann surface thought of as a point in the Teichmüller space, and let \( H^0(S, \Omega^{\otimes 2}) \) be the space of the holomorphic quadratic forms on \( S \). The fundamental theorem of Hubbard and Masur says that there exists a homeomorphism \( h_S : H^0(S, \Omega^{\otimes 2}) \to T_g \) [6, p. 224]. The space \( H^0(S, \Omega^{\otimes 2}) \) is a real vector space of dimension \( 6g - 6 \), where \( g \geq 2 \). It has been shown in the above cited work that \( H^0(S, \Omega^{\otimes 2}) \cong \text{Hom} (H_1(X, \tilde{\Gamma})^-, \mathbb{R}) \) defined by the formula

\[
\omega \mapsto \left( \gamma \mapsto \text{Im} \left( \int_{\gamma} \omega \right) \right),
\]

where \( H_1(X, \tilde{\Gamma})^- \) is the odd part in the homology of a double cover \( \tilde{X} \) of \( X \) ramified at the zeroes \( \tilde{\Gamma} \) of the odd multiplicity of the quadratic form [6, p. 232]. (The symbols and formulas will simplify as we come to the complex torus – our principal case.) It has been proved that \( H_1(X, \tilde{\Gamma})^- \cong \mathbb{Z}^{6g - 6} \).

Let \( X = T^2 \), i.e. \( g = 1 \). In this case each quadratic differential form is the square of a holomorphic abelian form (a one-form), i.e. \( H^0(S, \Omega^{\otimes 2}) = H^0(S, \Omega) \). Therefore \( \tilde{\mathcal{M}} \cong T^2 \), \( \tilde{\Gamma} = \emptyset \) and \( H_1(X, \tilde{\Gamma})^- = H_1(T^2) \cong \mathbb{Z}^2 \). In other words, one gets a homeomorphism \( h_S : \text{Hom} (\mathbb{Z}^2, \mathbb{R}) \to T_1 \). As we have seen earlier, \( \text{Hom} (\mathbb{Z}^2, \mathbb{R}) = \{ t_1 \mathbb{Z} + t_2 \mathbb{Z} \mid t_1, t_2 \in \mathbb{R} \} = \{ t(z + \theta z) \mid \theta, t \in \mathbb{R} \} \), where \( t = t_1, \theta = t_2/t_1 \). On the other hand, the Teichmüller space \( T_1 \cong \mathbb{H} \), where \( \mathbb{H} = \{ \tau = x + iy \in \mathbb{C} \mid y > 0 \} \) is a (Lobachevsky) upper half-plane and \( \tau \) is a modulus of the complex torus \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) [10] pp. 6-14. Thus, we have a homeomorphism \( h_S : \langle \mathbb{H}, \text{Ext} \rangle (\mathbb{H}, \mathbb{H}) \to \mathbb{H} \).

Let us show that \( h_S \) is equivariant in the first coordinate with respect to the action of \( \text{Mod} (T^2) \cong SL(2, \mathbb{Z}) \); i.e., \( \tau' = \tau \mod SL(2, \mathbb{Z}) \) if and only if \( \theta' = \theta \mod SL(2, \mathbb{Z}) \). Indeed, since \( \text{Hom} (H_1(T^2); \mathbb{R}) \cong \mathbb{H} \) the modular group \( SL(2, \mathbb{Z}) \) acts on the right-hand side by the formula \( \tau \mapsto (a\tau + b)/(c\tau + d) \) and on the left-hand side by a linear transformation \( p_1 \mapsto ap_1 + bp_2, \quad p_2 \mapsto cp_1 + dp_2 \), where \( p = (p_1, p_2) \in H_1(T^2) \) and \( ad - bc = 1 \). The \( f_p(t) = p_1t_1 + p_2t_2 \) will become \( f_p(t') = t_1(ap_1 + bp_2) + t_2(cp_1 + dp_2) = p_1t_1' + p_2t_2', \) where \( t_1' = at_1 + ct_2 \) and \( t_2' = bt_1 + dt_2 \). Therefore \( \theta = t_2/t_1 \) goes to \( \theta' = t_2'/t_1' = (b + d\theta)/(a + c\theta) \) and \( \theta' \equiv \theta \mod SL(2, \mathbb{Z}) \). (The ‘only if’ part of the statement is obtained likewise by an inversion of the formulas.)

Recall that the Lobachevsky plane \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) carries a hyperbolic metric \( ds = |dz|/y \) such that \( SL(2, \mathbb{Z}) \) acts on it by the isometries (linear-fractional transformations). The tessellation of \( \mathbb{H} \) by the fundamental regions is shown in Figure 2. Let \( \tau' = \frac{\tau + \theta}{\tau + d\theta} = T(\tau), \ \tau \in \mathbb{H} \). The number \( n = n(a, b, c, d) \in \mathbb{Z} \) we shall call a height of the transformation \( T \in SL(2, \mathbb{Z}) \) if \( n \) is equal to the number of intersections of the vertical segment \( \text{Im} (\tau' - \tau) \) issued from \( \tau \) with the lines of tiling \( \mathbb{H}/SL(2, \mathbb{Z}) \). (In other words, \( n \) shows how many fundamental regions apart
Let us now define an action of the modular group on \((\mathbb{A} \theta, \text{Ext}_t (\mathbb{A} \theta, \mathbb{A} \theta))\). The action is given by the formula \((\theta, t) \mapsto (\frac{ag+tb}{cg+td}, t+n)\), where \(n = n(a, b, c, d)\) is the height of the transformation \(T = T(a, b, c, d)\). Under the homeomorphism \(h_S\), the tesselation of \(\mathbb{H}\) maps into a tesselation of the plane \((\theta, t)\). As we have shown earlier, the action of the modular group \(SL(2, \mathbb{Z})\) on \(\mathbb{H}\) is equivariant with the action on \((\theta, t)\). On the other hand, it is known that \(\mathbb{H}/SL(2, \mathbb{Z})\) is a punctured two-dimensional sphere \([10, \text{p. 15}]\). Lemma 2 and Theorem 1 follow.

3. Remarks

Let \(E_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})\) be a complex torus and \(h_S(E_\tau) = (\mathbb{A} \theta, \text{Ext}_t (\mathbb{A} \theta, \mathbb{A} \theta))\) its image under the homeomorphism \(h_S\). Let us call the respective reals \(\theta = \theta(\tau)\) and \(t = t(\tau)\) a projective curvature and an area of the complex torus \(E_\tau\). The projective curvature of the complex tori with a nontrivial group of endomorphisms (complex multiplication) is a quadratic irrationality. In the latter case, the noncommutative torus is said to have a real multiplication. The noncommutative tori with real multiplication can be used to construct the abelian extensions of the real quadratic number fields, as was suggested by Yu. Manin [7]. It seems challenging at this point to write a formula for the projective curvature and the area as a function of the complex modulus \(\tau\). It is likely that the functions will be of the class \(C^0\).

Problem 1. Find a formula (if any) for the functions \(\theta(\tau)\) and \(t(\tau)\).

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References


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