

FIRST EIGENVALUES OF GEOMETRIC OPERATORS UNDER THE RICCI FLOW

XIAODONG CAO

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ABSTRACT. In this paper, we prove that the first eigenvalues of $-\Delta + cR$ ($c \geq \frac{1}{4}$) are nondecreasing under the Ricci flow. We also prove the monotonicity under the normalized Ricci flow for the cases $c = 1/4$ and $r \leq 0$.

1. FIRST EIGENVALUE OF $-\Delta + cR$

Let M be a closed Riemannian manifold, and $(M, \mathbf{g}(t))$ be a smooth solution to the Ricci flow equation

$$\frac{\partial}{\partial t} \mathbf{g}_{ij} = -2R_{ij}$$

on $0 \leq t < T$. In [Cao07], we prove that all eigenvalues $\lambda(t)$ of the operator $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow on manifolds with a nonnegative curvature operator. Assume $f = f(x, t)$ is the corresponding eigenfunction of $\lambda(t)$; that is,

$$\left(-\Delta + \frac{R}{2}\right)f(x, t) = \lambda(t)f(x, t)$$

and $\int_M f^2 d\mu = 1$. More generally, we define

$$(1.1) \quad \lambda(f, t) = \int_M \left(-\Delta f + \frac{R}{2}f\right)f d\mu,$$

where f is a smooth function satisfying

$$\frac{d}{dt} \left(\int_M f^2 d\mu \right) = 0, \quad \int_M f^2 d\mu = 1.$$

We can then derive the monotonicity formula under the Ricci flow.

Theorem 1.1 ([Cao07]). *On a closed Riemannian manifold with a nonnegative curvature operator, the eigenvalues of the operator $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow. In particular,*

$$(1.2) \quad \frac{d}{dt} \lambda(f, t) = 2 \int_M R_{ij} f_i f_j d\mu + \int_M |Rc|^2 f^2 d\mu \geq 0.$$

In (1.2), when $\frac{d}{dt} \lambda(f, t)$ is evaluated at time t , f is the corresponding eigenfunction of $\lambda(t)$.

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Remark 1.2. Clearly, at time t , if f is the eigenfunction of the eigenvalue $\lambda(t)$, then $\lambda(f, t) = \lambda(t)$. By the eigenvalue perturbation theory, we may assume that there is a C^1 -family of smooth eigenvalues and eigenfunctions (for example, see [KL06], [RS78] and [CCG⁺07]). When λ is the lowest eigenvalue, we can further assume that the corresponding eigenfunction f is positive. Since the above formula does not depend on the particular evolution of f , $\frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda(f, t)$.

Remark 1.3. In [Li07a], J. Li used the same technique to prove the monotonicity of the first eigenvalue of $-\Delta + \frac{1}{2}R$ under the Ricci flow without assuming a nonnegative curvature operator. A similar result appeared in the physics literature [OSW05].

Remark 1.4. When $c = \frac{1}{4}$, the monotonicity of the first eigenvalue has been established by G. Perelman in [Per02]. The evolution of the first eigenvalue of the Laplace operator under the Ricci flow has been studied by L. Ma in [Ma06]. The evolution of the Yamabe constant under the Ricci flow has been studied by S.C. Chang and P. Lu in [CL07].

In this paper, we shall study the first eigenvalues of operators $-\Delta + cR$ ($c \geq \frac{1}{4}$) without *curvature assumption* on the manifold. Our first result is the following theorem.

Theorem 1.5. *Let $(M^n, \mathbf{g}(t))$, $t \in [0, T)$, be a solution of the Ricci flow on a closed Riemannian manifold M^n . Assume that $\lambda(t)$ is the lowest eigenvalue of $-\Delta + cR$ ($c \geq \frac{1}{4}$), $f = f(x, t) > 0$ satisfies*

$$(1.3) \quad -\Delta f(x, t) + cRf(x, t) = \lambda(t)f(x, t)$$

and $\int_M f^2(x, t) d\mu = 1$. Then under the Ricci flow, we have

$$\frac{d}{dt}\lambda(t) = \frac{1}{2} \int_M |R_{ij} + \nabla_i \nabla_j \varphi|^2 e^{-\varphi} d\mu + \frac{4c-1}{2} \int_M |Rc|^2 e^{-\varphi} d\mu \geq 0,$$

where $e^{-\varphi} = f^2$.

Proof of Theorem 1.5. Let φ be a function satisfying $e^{-\varphi(x)} = f^2(x)$. We proceed as in [Cao07]; we have

$$(1.4) \quad \begin{aligned} \frac{d}{dt}\lambda(t) &= (2c - \frac{1}{2}) \int R_{ij} \nabla_i \varphi \nabla_j \varphi e^{-\varphi} d\mu \\ &\quad - (2c - 1) \int R_{ij} \nabla_i \nabla_j \varphi e^{-\varphi} d\mu + 2c \int |Rc|^2 e^{-\varphi} d\mu. \end{aligned}$$

Integrating by parts and applying the Ricci formula, it follows that

$$(1.5) \quad \int R_{ij} \nabla_i \nabla_j \varphi e^{-\varphi} d\mu = \int R_{ij} \nabla_i \varphi \nabla_j \varphi e^{-\varphi} d\mu - \frac{1}{2} \int R \Delta e^{-\varphi} d\mu$$

and

$$(1.6) \quad \begin{aligned} &\int R_{ij} \nabla_i \nabla_j \varphi e^{-\varphi} d\mu + \int |\nabla \nabla \varphi|^2 e^{-\varphi} d\mu \\ &= - \int \Delta e^{-\varphi} (\Delta \varphi + \frac{1}{2}R - \frac{1}{2}|\nabla \varphi|^2) d\mu \\ &= (2c - \frac{1}{2}) \int R \Delta e^{-\varphi} d\mu. \end{aligned}$$

In the last step, we use

$$2\lambda(t) = \Delta\varphi + 2cR - \frac{1}{2}|\nabla\varphi|^2.$$

Combining (1.5) and (1.6), we arrive at

$$(1.7) \quad \int |\nabla\varphi|^2 e^{-\varphi} d\mu = 2c \int R\Delta e^{-\varphi} d\mu - \int R_{ij}\nabla_i\varphi\nabla_j\varphi e^{-\varphi} d\mu.$$

Plugging (1.7) into (1.4), we have

$$\begin{aligned} \frac{d}{dt}\lambda(t) &= \int R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi} d\mu + 2c \int |Rc|^2 e^{-\varphi} d\mu \\ &\quad + c \int R\Delta(e^{-\varphi})d\mu - \frac{1}{2} \int R_{ij}\nabla_i\varphi\nabla_j\varphi e^{-\varphi} d\mu \\ &= \int R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi} d\mu + 2c \int |Rc|^2 e^{-\varphi} d\mu + \frac{1}{2} \int |\nabla\varphi|^2 e^{-\varphi} d\mu \\ &= \frac{1}{2} \int |R_{ij} + \nabla_i\nabla_j\varphi|^2 e^{-\varphi} d\mu + (2c - \frac{1}{2}) \int |Rc|^2 e^{-\varphi} d\mu \geq 0. \end{aligned}$$

This proves the theorem as desired. □

2. FIRST EIGENVALUE UNDER THE NORMALIZED RICCI FLOW

In this section, we derive the evolution of $\lambda(t)$ under the normalized Ricci flow equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \frac{2}{n}rg_{ij}.$$

Here $r = \frac{\int_M R d\mu}{\int_M d\mu}$ is the average scalar curvature. It follows from Equation (1.3) that $\lambda \leq cr$. We now compute the derivative of the lowest eigenvalue of $-\Delta + cR$.

Theorem 2.1. *Let $(M^n, g(t))$, $t \in [0, T)$, be a solution of the normalized Ricci flow on a closed Riemannian manifold M^n . Assume that $\lambda(t)$ is the lowest eigenvalue of $-\Delta + cR$ ($c \geq \frac{1}{4}$) and that $f > 0$ is the corresponding eigenfunction. Then under the normalized Ricci flow, we have*

$$(2.1) \quad \frac{d}{dt}\lambda(t) = -\frac{2r\lambda}{n} + \frac{1}{2} \int_M |R_{ij} + \nabla_i\nabla_j\varphi|^2 e^{-\varphi} d\mu + \frac{4c-1}{2} \int_M |Rc|^2 e^{-\varphi} d\mu;$$

where $e^{-\varphi} = f^2$. Furthermore, if $c = \frac{1}{4}$ and $r \leq 0$, then

$$(2.2) \quad \frac{d}{dt}\lambda(t) = \frac{2}{n}r(\lambda - \frac{r}{4}) + \frac{1}{2} \int_M |R_{ij} + \nabla_i\nabla_j\varphi - \frac{r}{n}g_{ij}|^2 e^{-\varphi} d\mu \geq 0.$$

Remark 2.2. After we submitted our paper, J. Li suggested to us that (2.2) is true for all $c \geq 1/4$, with an additional nonnegative term

$$\frac{4c-1}{2} \int_M |Rc - \frac{r}{n}g_{ij}|^2 e^{-\varphi} d\mu;$$

see [Li07b] for a similar result.

Remark 2.3. As a consequence of the above monotonicity formula of $\lambda(t)$, we can prove that both compact steady and expanding Ricci breathers (cf. [Ive93], [Per02]) must be trivial. Such results have been discussed by many authors (for example, see [Ive93], [Ham95], [Ham88], [Per02], [Cao07] and [Li07a], etc.).

When M is a two-dimensional surface, r is a constant. We have the following corollary.

Corollary 2.4. *Let $(M^2, \mathbf{g}(t))$, $t \in [0, T)$, be a solution of the normalized Ricci flow on a closed Riemannian surface M^2 . Assume that $\lambda(t)$ is the lowest eigenvalue of $-\Delta + cR$ ($c \geq \frac{1}{4}$); we have $e^{rt}\lambda$ is nondecreasing under the normalized Ricci flow. Moreover, if $r \leq 0$, then λ is nondecreasing.*

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853-4201
E-mail address: cao@math.cornell.edu