CONVERGENT MARTINGALES OF OPERATORS 
AND THE RADON NIKODYM PROPERTY 
IN BANACH SPACES

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Abstract. We extend Troitsky’s ideas on measure-free martingales on Banach lattices to martingales of operators acting between a Banach lattice and a Banach space. We prove that each norm bounded martingale of cone absolutely summing (c.a.s.) operators (also known as 1-concave operators), from a Banach lattice $E$ to a Banach space $Y$, can be generated by a single c.a.s. operator. As a consequence, we obtain a characterization of Banach spaces with the Radon Nikodým property in terms of convergence of norm bounded martingales defined on the Chaney-Schaefer $l^\infty$-tensor product $E\bar{\otimes} Y$. This extends a classical martingale characterization of the Radon Nikodým property, formulated in the Lebesgue-Bochner spaces $L^p(\mu, Y)$ ($1 < p < \infty$).

1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $Y$ be a Banach space. For $1 \leq p < \infty$, let $L^p(\mu, Y)$ denote the space of (classes of a.e. equal) Bochner $p$-integrable functions $f: \Omega \to Y$ and denote the Bochner norm on $L^p(\mu, Y)$ by $\Delta_p$, i.e.

$$\Delta_p(f) = \left( \int_{\Omega} \|f\|_Y^p \, d\mu \right)^{1/p}.$$  

In the Lebesgue-Bochner spaces, martingale theory provides an important link to the geometric properties of the Banach space $Y$. For example, $Y$ has the Radon Nikodým property if and only if every martingale $(f_i) \subset L^p(\mu, Y)$, which is uniformly $\|\cdot\|_p$-bounded, converges in the $\|\cdot\|_p$-norm for all finite measure spaces $(\Omega, \Sigma, \mu)$ and $1 < p < \infty$ (cf. [9]).

It is well known that $L^p(\mu, Y)$ is isometrically isomorphic to the norm completion $L^p(\mu)\bar{\otimes}_{\Delta_p} Y$ of $L^p(\mu) \otimes_{\Delta_p} Y$, where $\Delta_p$ denotes the induced Bochner norm (cf. [4, 5]). Chaney [1] characterized the Radon Nikodým property on $Y$ in terms of equality of $L^p(\mu)\bar{\otimes}_{\Delta_p} Y$ and the space of Dinculeanu’s operators from $L^q(\mu)$ to $Y$ ($1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). These are the operators $T: L^q(\mu) \to Y$ for which

$$\sup\{\sum_{i=1}^n \|\alpha_i T(\chi_{E_i})\|_Y\},$$

where the supremum is taken over all
simple functions \( f = \sum_{i=1}^{n} \alpha_i \chi_{E_i} \) with \( \chi_{E_1}, \ldots, \chi_{E_n} \in L^q(\mu) \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and \( \|f\|_q < 1 \) (cf. \[6\] \[7\] \[8\]).

The aim of this paper is to extend Chaney’s characterization by using Troitsky’s \[18\] generalized notion of a martingale in a Banach lattice.

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice \( E \) and a Banach space \( Y \); the norm \( \|\cdot\|_t \), given by

\[
\|u\|_t = \inf \left\{ \left\| \sum_{i=1}^{n} y_i \right\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}
\]

for all \( u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes Y \), coincides with the Bochner norm on \( L^p(\mu) \otimes Y \) for all \( \sigma \)-finite measure spaces \( (\Omega, \Sigma, \mu) \) and \( 1 \leq p < \infty \) (see \[1\] \[17\] and \[11\]).

Let \( E \) be a Banach lattice and let \( Y \) be a Banach space. Recall from \[17\] Chapter IV, §3 that a linear map \( T : E \rightarrow Y \) is called cone absolutely summing if for every positive summable sequence \( (x_n) \) in \( E \), the sequence \( (T x_n) \) is absolutely summable in \( Y \). Such operators are known, in the terminology of Krivine, as 1-concave operators (cf. \[13\] p. 45 and \[13\]). The space \( \mathcal{L}^{\text{cas}}(E, Y) = \{ T : E \rightarrow Y : T \text{ is cone absolutely summing} \} \) is a Banach space with respect to the norm defined by

\[
\|T\|_{\text{cas}} = \sup \left\{ \sum_{i=1}^{n} \|Tx_i\| : x_1, \ldots, x_n \in E_+, \|\sum_{i=1}^{n} x_i\| = 1, n \in \mathbb{N} \right\}
\]

for all \( T \in \mathcal{L}^{\text{cas}}(E, Y) \). Cone absolutely summing operators generalize Dinculeanu’s operators, mentioned above.

Cone absolutely summing operators extend the Chaney-Schaefer \( l \)-tensor product in the following sense: The canonical map \( E^* \otimes^l Y \rightarrow \mathcal{L}^{\text{cas}}(E, Y) \) given by \( \sum_{i=1}^{n} x_i^* \otimes y_i = : u \mapsto L_u \), where \( L_u x = \sum_{i=1}^{n} \langle x, x_i^* \rangle y_i \) for all \( x \in E \), is an isometry (cf. \[17\] Chapter IV, §7] and \[11\] \[11\] \[12\] \[13\]). Here, \( E^* \) denotes the continuous dual of \( E \).

2. Preliminaries

We recall the abstract notions in \[18\] for a filtration and a martingale in a Banach space.

**Definition 2.1.** Let \( E \) be a Banach lattice and \( Y \) a Banach space.

(a) Let \( (T_i) \) be a sequence of contractive projections on \( Y \). If \( T_{ij} = T_i T_j \) for each \( i, j \in \mathbb{N} \), then \( (T_i) \) is called a filtration on \( Y \). In the case where \( (T_i) \) is a filtration on a Banach lattice, we will always assume each \( T_i \) to be positive.

(b) If \( (T_i) \) is a filtration on \( E \) with the property that each \( R(T_i) \) is a (closed) Riesz subspace of \( E \), then \( (T_i) \) will be called a BL-filtration on \( E \) (this terminology is also used in \[2\]).

(c) If \( (T_i) \) is a filtration on \( E \) with each \( T_i \) strictly positive (i.e., \( T_i \) is positive and \( \{ f \in E : T_i(|f|) = 0 \} = \{0\} \) ), then \( (T_i) \) is called a strictly positive filtration.

(d) If \( (T_i) \) is a filtration on \( Y \) and \( (f_i) \subset Y \), then the pair \( (f_i, T_i) \) is called a martingale in \( Y \) if \( T_i f_j = f_i \) for all \( i \leq j \).
(e) If \((f_i, T_i)\) is a martingale in \(Y\), then \((f_i, T_i)\) is called fixed if there exists \(f \in Y\) such that \(f_i = T_i f\) for all \(i \in \mathbb{N}\). In this case, \((f_i, T_i)\) is said to be fixed on \(f\).

(f) If \((T_i)\) is a filtration on \(E\), then \((f_i, T_i)\) is called a submartingale in \(E\) if \(T_i f_j \geq f_i\) for all \(i \leq j\).

(g) Let \((T_i)\) be a filtration on \(Y\). We say that \((T_i)\) is complemented in \(Y\) if there exists a contractive projection \(T_\infty : Y \rightarrow Y\) with \(\mathcal{R}(T_\infty) = \bigcup_{i=1}^\infty \mathcal{R}(T_i)\) and \(T_i T_\infty = T_\infty T_i = T_i\) for all \(i \in \mathbb{N}\).

If \((\Omega, \Sigma, \mu)\) is a finite measure space, \(1 \leq p < \infty\) and \((\Sigma_i)\) an increasing sequence of \(\sigma\)-subalgebras of \(\Sigma\), it follows that the sequence of conditional expectations \((\mathbb{E}(\cdot | \Sigma_i))\) on \(L^p(\mu)\) satisfies the above definition of a BL-filtration. Moreover, \((\mathbb{E}(\cdot | \Sigma_i))\) is complemented by \(\mathbb{E}(\cdot | \bigvee_{i=1}^\infty \Sigma_i) : L^p(\mu) \rightarrow L^p(\mu)\), where \(\bigvee_{i=1}^\infty \Sigma_i\) denotes the \(\sigma\)-algebra generated by \(\bigcup_{i=1}^\infty \Sigma_i\).

In [17] Chapter III, §11, Proposition 11.5 it is shown that if \(T : E \rightarrow E\) is a strictly positive projection on a Banach lattice \(E\), then \(\mathcal{R}(T)\) is a Banach sublattice of \(E\). Consequently, every strictly positive filtration on \(E\) is also a BL-filtration.

It is easily verified that if \((T_i)\) is a filtration on \(E\), then the sequence of adjoint operators \((T_i^*)\) is a filtration on \(E^*\). However, it is not clear that \((T_i^*)\) is a BL-filtration whenever \((T_i)\) is a BL-filtration. We need an additional definition.

**Definition 2.2.** Let \(E\) be a Banach lattice with a non-empty quasi-interior \(Q_+\). A filtration \((T_i)\) on \(E\) is said to be quasi-interior preserving if \(T_i Q_+ \subset Q_+\) for each \(i \in \mathbb{N}\).

For background reading on quasi-interior points in Banach lattices, we refer the reader to [17] Chapter II, §6.

**Lemma 2.3.** Let \(E\) be a Banach lattice with a non-empty quasi-interior \(Q_+\). If \(T : E \rightarrow E\) is a positive projection, then \(T Q_+ \subset Q_+\) if and only if there exists a quasi-interior point \(0 < e \in E_+\) such that \(T e = e\).

**Proof.** Note that \(T \geq 0\) implies that \(T\) is bounded. Suppose there exists \(e \in Q_+\) such that \(T e = e\) and let \(q \in Q_+\). By [17] Chapter II, §6, Theorem 6.3, we have \(\lim_{n \rightarrow \infty} T(nq \wedge e) = T(e) = e\). Also, \(0 \leq T(nq \wedge e) \leq (nT q) \wedge e \leq e\) with \(T(nq \wedge e) \uparrow\). Hence, \(\lim_{n \rightarrow \infty} (nT q) \wedge e = e\). Now let \(p \in E_+\). Using [17] Chapter II, §6, Theorem 6.3] again, we obtain

\[
p = \lim_{m \rightarrow \infty} (me) \wedge p = \lim_{m \rightarrow \infty} (m \lim_{n \rightarrow \infty} (nT q) \wedge e) \wedge p = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (m(nT q) \wedge e) \wedge p.
\]

As \((m(nT q) \wedge e) \wedge p \leq (mnT q) \wedge p \leq p\) for each \(n, m \in \mathbb{N}\), it follows that

\[
\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (mnT q) \wedge p = p.
\]

Thus, \(T q \in Q_+\) by [17] Chapter II, §6, Theorem 6.3, i.e. \(T Q_+ \subset Q_+\). The converse is trivial.

The spaces \(L^p(\mu), 1 \leq p < \infty\), have non-empty quasi-interior (e.g., \(1\) is a quasi-interior point of \(L^p(\mu)\)) and if \((\Sigma_i)\) is an increasing sequence of \(\sigma\)-subalgebras of \(\Sigma\), then the sequence of conditional expectations \((\mathbb{E}(\cdot | \Sigma_i))\) is a filtration with \(\mathbb{E}(1 | \Sigma_i) = 1\) for each \(i \in \mathbb{N}\). By the above lemma, \((\mathbb{E}(\cdot | \Sigma_i))\) is a filtration that is quasi-interior preserving.
Proposition 2.4. Suppose that $E$ is a Banach lattice possessing a non-empty quasi-interior $Q_+$ and $T : E \to E$ is a bounded linear operator. Then $TQ_+ \subset Q_+$ if and only if $T^* : E^* \to E^*$ is strictly positive.

Proof. By [17, Chapter II, §6, Theorem 6.3], $q \in Q_+$ if and only if $\langle q, f^* \rangle > 0$ for all $0 < f^* \in E^*$. With this in mind, assume that $TQ_+ \subset Q_+$. Since $Q_+$ is dense in $E_+$, it follows that $T \geq 0$, which implies $T^* \geq 0$. To show strict positivity, suppose that $T^*f^* = 0$ for some $f^* \in E^*_+$. Then $0 = \langle q, T^*f^* \rangle = \langle Tq, f^* \rangle$ for $q \in Q_+$. Since $Tq \in Q_+$, it follows that $f^* = 0$. Conversely, if $T^*$ is strictly positive, then for all $0 < f^* \in E^*$ we have $T^*f^* > 0$. Thus, for $q \in Q_+$, it follows that $\langle Tq, f^* \rangle = \langle q, T^*f^* \rangle > 0$ for all $f^* \in E^*$. Consequently, $Tq \in Q_+$. □

Corollary 2.5. Suppose that $E$ is a Banach lattice possessing non-empty quasi-interior. Then for any quasi-interior preserving filtration $(T_i)$ on $E$, we have that $(T_i^*)$ is a BL-filtration on $E^*$.

The following convergence result was shown in [2].

Proposition 2.6. Let $Y$ be a Banach space and $(T_i)$ a filtration on $Y$. Then the following statements hold:

(a) $f \in \bigcup_{i=1}^{\infty} R(T_i)$ if and only if $\|T_i f - f\| \to 0$ as $i \to \infty$.

(b) If $(f_i)$ is a martingale relative to $(T_i)$, then $(f_i)$ converges to $f$ if and only if $f \in \bigcup_{i=1}^{\infty} R(T_i)$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.

Consider the case where a filtration $(T_i)$ on a Banach lattice $E$ is complemented in $E$ by a contractive projection $T_\infty : E \to E$. Since each $T_i$ is assumed to be positive, it follows that $T_\infty$ is also positive. Indeed, if $f \in E_+$, then $T_i f \in E_+$ for each $i \in \mathbb{N}$. Thus, Proposition 2.6 implies $\lim_{i \to \infty} T_i f = \lim_{i \to \infty} T_i T_\infty f = T_\infty f \geq 0$.

Definition 2.7. Let $Y$ be a Banach space and $(T_i)$ a filtration on $Y$.

(a) Define the space of norm bounded martingales as

$$\mathcal{M}(Y, T_i) = \{(f_i, T_i) \text{ a martingale in } Y : \sup_i \|f_i\| < \infty\},$$

一起对所有 $(f_i, T_i) \in \mathcal{M}(Y, T_i)$ 定义

(b) Define the space of norm convergent martingales by

$$\mathcal{M}_{nc}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is norm convergent in } Y\}.$$

(c) Define the space of fixed martingales by

$$\mathcal{M}_f(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : \exists f \in Y \text{ so that } T_i f = f, \forall i \in \mathbb{N}\}.$$

It is easily shown that $\mathcal{M}(Y, T_i)$ and $\mathcal{M}_{nc}(Y, T_i)$ are Banach spaces. By Proposition 2.6(b), we have the inclusions $\mathcal{M}_{nc}(Y, T_i) \subset \mathcal{M}_f(Y, T_i) \subset \mathcal{M}(Y, T_i)$. Note that if $(T_i)$ is a complemented filtration on $Y$, then $\mathcal{M}_{nc}(Y, T_i) = \mathcal{M}_f(Y, T_i)$. If $(f_i, T_i)$ is fixed, then it is also fixed on an element in $\bigcup_{i=1}^{\infty} R(T_i)$ and thus convergent by Proposition 2.6.

In the case where $Y$ is reflexive, Troitsky showed in [18, Corollary 18] that $\mathcal{M}_f(Y, T_i) = \mathcal{M}(Y, T_i)$. If $(T_i)$ is also complemented in $Y$, then $\mathcal{M}_{nc}(Y, T_i) = \mathcal{M}_f(Y, T_i) = \mathcal{M}(Y, T_i)$.

For a BL-filtration $(T_i)$ on a Banach lattice $E$, [2, Proposition 3.7] implies that $\mathcal{M}_{nc}(E, T_i)$ is a Banach lattice with respect to the ordering defined by $(f_i, T_i) \geq (g_i, T_i)$ as $f_i \geq g_i$. Conversely, if $(f_i, T_i)$ is a martingale in $\mathcal{M}(E, T_i)$, then $f_i$ is a martingale relative to $(T_i), 0 \leq f_i \leq e_i \geq 0$ for some $e_i \in E_+$.

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Let $\summing$ operators from a Banach lattice $E$ be a Banach lattice and $(T_i)$ is a filtration on $E$, Troitsky showed in [18, Theorem 7] that $\mathcal{M}(E, T_i)$ is also a Banach lattice under this ordering. Moreover, Troitsky showed the following, which is key to our main result (cf. [18, p. 446]).

**Proposition 2.8.** Let $E$ be a KB-space and $(T_i)$ a filtration on $E$. If $(s_i, T_i)$ is a norm bounded submartingale in $E$, then there exists a unique least martingale $(f_i, T_i) \in \mathcal{M}(E, T_i)$ such that $s_i \leq f_i$ for each $i \in \mathbb{N}$. Moreover, $\|(f_i, T_i)\| \leq \|(s_i, T_i)\|$.

The notion of a filtration $(T_i)$ on a Banach lattice $E$ can be extended to the $l_1$ tensor product $E \otimes_1 Y$ for any Banach space $Y$. Indeed, it follows from the proof of [2, Lemma 4.2] that $(T_i \otimes \text{id}_Y)$ is a sequence of commuting contractive projections on $E \otimes_1 Y$ with increasing range. This extension is consistent with a classical filtration on the Lebesgue-Bochner spaces.

### 3. Cone absolutely summing martingales

To characterize the Radon Nikodým property, we require a fair amount of preparation. We first consider a canonical filtration on the space of cone absolutely summing operators from a Banach lattice $E$ to a Banach space $Y$.

**Proposition 3.1.** Let $E$ be a Banach lattice and $Y$ a Banach space. Suppose that $(T_i)$ is a BL-filtration on $E$. Then the sequence $(\hat{T}_i)$ of maps $\hat{T}_i : \mathcal{L}^{\text{cas}}(E, Y) \to \mathcal{L}^{\text{cas}}(E, Y)$, defined by $\hat{T}_iF = F \circ T_i$ for each $F \in \mathcal{L}^{\text{cas}}(E, Y)$ and $i \in \mathbb{N}$, is a sequence of contractive projections on $\mathcal{L}^{\text{cas}}(E, Y)$ with $\hat{T}_{i \wedge j} = \hat{T}_i \hat{T}_j$.

**Proof.** Since $(T_i)$ is a filtration, $F \circ T_i \in \mathcal{L}^{\text{cas}}(E, Y)$ and $\hat{T}_i$ is a well-defined, linear projection for each $i \in \mathbb{N}$. It also follows from

$$\|\hat{T}_iF\|_{\text{cas}} = \sup \left\{ \sum_{j=1}^{n} \|FT_i x_j\| : (x_j)_{j=1}^{n} \subset E_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}$$

$$= \sup \left\{ \sum_{j=1}^{n} \|Fx_j\| : (x_j)_{j=1}^{n} \subset \mathcal{R}(T_i)_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}$$

$$\leq \sup \left\{ \sum_{j=1}^{n} \|Fx_j\| : (x_j)_{j=1}^{n} \subset E_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}$$

$$= \|F\|_{\text{cas}}$$

that each $\hat{T}_i$ is bounded and $\sup_{i \in \mathbb{N}} \|\hat{T}_i\| = 1$. Moreover, $\hat{T}_i \hat{T}_j F = F \circ T_j \circ T_i = F \circ T_{i \wedge j} = \hat{T}_i \hat{T}_j F$ for each $F \in \mathcal{L}^{\text{cas}}(E, Y)$ and $i, j \in \mathbb{N}$. This completes the proof. \qed

In view of the above proposition, we are justified in making the following definition.
Definition 3.2. Let $E$ be a Banach lattice and $Y$ a Banach space. Suppose that $(T_i)$ is a BL-filtration on $E$. Then $(\tilde{T}_i)$, as defined in Proposition 3.1, is called the filtration on $L^\text{cas}(E,Y)$ induced by $(T_i)$.

We exhibit a known characterization of cone absolutely summable operators.

Lemma 3.3. Let $E$ be a Banach lattice, $Y$ a Banach space and $l > 0$. For any bounded operator $T: E \to Y$ the following statements are equivalent:

(a) $T$ is cone absolutely summing with $\|T\|_{\text{cas}} \leq l$.
(b) There exists $x_T^* \in E_1^*$ so that $\|x_T^*\| \leq l$ and $\|Tx\| \leq \langle |x|, x_T^* \rangle$ for all $x \in E$.
(c) There exist an $\ell^1$-space $L$, $0 \leq T_1 \in \mathcal{L}(E, L)$ and $T_2 \in \mathcal{L}(L, Y)$ such that

$$T = T_2 \circ T_1, \quad \text{where } \|T_1\| \leq l \text{ and } \|T_2\| \leq 1.$$  

In the case where $E$ is separable, we may take $L = L^1(\mu)$ in (c), where $(\Omega, \Sigma, \mu)$ is a finite measure space.

The proof of the equivalence of (a), (b) and (c) in the above lemma may be found in [17, Chapter IV, §6, Proposition 6.4]. However, the last part of the lemma requires a proof: Assume $E$ is separable. Then, by [17, Chapter II, §6, Proposition 6.2], there exists a quasi-interior point $0 \leq e \in E$. By construction, the map $T_1: E \to L$ is a Riesz homomorphism with dense range (see [17, p. 243, §3]). Thus, it follows by [17, Chapter II, §6, Proposition 6.4] that $T_1e$ is a quasi-interior point of $L$. Hence, $T_1e$ is also a weak order unit of $L$. It follows by Kakutani’s representation theorem for $\ell^1$-spaces (cf. [12] or [13, Theorem 1.b.2]) that $L$ is Riesz and isometrically isomorphic to $L^1(\mu)$, where $(\Omega, \Sigma, \mu)$ may be chosen to be finite.

For our next result, we recall that the functional $x_T^* \in E^*_1$ in Lemma 3.3(b) is the extension of the additive map $\rho_T: E_+ \to E_+$, defined by

$$\rho_T(x) = \sup \left\{ \sum_{i=1}^{\infty} \|Tx_i\| : (x_i) \in (\ell^1 \otimes_\pi E)_+, \sum_{i=1}^{\infty} x_i = x \right\}$$

for each $x \in E_+$ (cf. [17, Chapter IV, §2, Theorem 2.7]). Here, $\ell^1 \otimes_\pi E$ denotes the space of unconditionally summable sequences in $E$.

Proposition 3.4. Let $E$ be a Banach lattice with order continuous dual and $Y$ a Banach space. Suppose that $(T_i)$ is a BL-filtration on $E$ and $(\tilde{T}_i)$ is the filtration on $L^\text{cas}(E,Y)$ induced by $(T_i)$. If $(F_i, \tilde{T}_i) \in \mathcal{M}(L^\text{cas}(E,Y), \tilde{T}_i)$, then there exists $0 \leq (F^*_i, T^*_i) \in \mathcal{M}(E^*, T^*_i)$ such that $\|F_i x\| \leq \langle |x|, F^*_i \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Moreover, $\sup_{i \in \mathbb{N}} \|F_i^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$.

Proof. By Lemma 3.3 there exists, for each $F_i$, a positive functional $x_{F_i}^* \in E^*$ with $\|x_{F_i}^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}} := l$ and $\|F_i x\| \leq \langle |x|, x_{F_i}^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Define $s_i^* \in E_+^*$ by $\langle x, s_i^* \rangle = \langle T_i x, x_{F_i}^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Then, $\sup_{i \in \mathbb{N}} \|s_i^*\| \leq l$ and, since $x_{F_i}^* \geq 0$, we get

$$\|F_i x\| = \|\tilde{T}_i F_i x\| = \|F_i T_i x\| \leq \langle |T_i x|, x_{F_i}^* \rangle \leq \langle |T_i x|, x_{F_i}^* \rangle = \langle |x|, s_i^* \rangle.$$
for all \( x \in E \) and \( i \in \mathbb{N} \). We now show that \((s_i^*, T_i^*)\) is a submartingale. Let \( i \leq j \) and \( x \in E_+ \). Then,
\[
\langle x, T_i^* s_j^* \rangle = \langle T_i x, s_j^* \rangle = \langle T_j T_i x, x_{F_j}^* \rangle = \langle T_i x, x_{F_j}^* \rangle
\]
\[
= \sup \left\{ \sum_{n=1}^{\infty} \| F_j x_n \| : (x_n) \in (L^1_\infty E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
\[
\geq \sup \left\{ \sum_{n=1}^{\infty} \| F_j x_n \| : (x_n) \in (L^1_\infty \mathcal{R}(T_i))_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
as \((L^1_\infty \mathcal{R}(T_i))_+ \subset (L^1_\infty E)_+\). Since \( T_i \) is a projection, it follows for \( x \in E \) that \( x \in \mathcal{R}(T_i) \) if and only if \( x = T_i x \). In addition, the positivity of \( T_i \) implies \((T_i x_n) \in (L^1_\infty \mathcal{R}(T_i))_+ \) for all \((x_n) \in (L^1_\infty E)_+\). Consequently, for \((x_n) \in (L^1_\infty E)_+\), we have \((x_n) \in (L^1_\infty \mathcal{R}(T_i))_+ \) if and only if \((x_n) = (T_i x_n)\). Thus,
\[
\sup \left\{ \sum_{n=1}^{\infty} \| F_j x_n \| : (x_n) \in (L^1_\infty \mathcal{R}(T_i))_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
\[
= \sup \left\{ \sum_{n=1}^{\infty} \| F_j T_i x_n \| : (x_n) \in (L^1_\infty E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
\[
= \sup \left\{ \sum_{n=1}^{\infty} \| \hat{T}_i F_j x_n \| : (x_n) \in (L^1_\infty E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
\[
= \sup \left\{ \sum_{n=1}^{\infty} \| F_i x_n \| : (x_n) \in (L^1_\infty E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
\]
\[
= \langle T_i x, x_{F_i}^* \rangle = \langle x, s_i^* \rangle.
\]
Since \( s_i^*(x) \leq T_i^* s_j^*(x) \) for all \( x \in E_+ \), it follows that \( s_i^* \leq T_i^* s_j^* \). Consequently, \((s_i^*, T_i^*)\) is a submartingale. Since \( E^* \) is order continuous, it follows that \( E^* \) is a KB-space (cf. [18, Theorem 2.4.14]). Thus, by Proposition 2.8, there exists a unique least martingale \( 0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*) \) that dominates the submartingale \((s_i^*, T_i^*)\), with \( \sup_{i \in \mathbb{N}} \| f_i^* \| \leq \sup_{i \in \mathbb{N}} \| s_i^* \| \leq l = \sup_{i \in \mathbb{N}} \| F_i \|_{\text{cas}} \). Hence,
\[
\| F_i \|_{\text{cas}} \leq \langle |x|, s_i^* \rangle \leq \langle |x|, f_i^* \rangle
\]
for all \( x \in E \), and the proof is complete. \( \square \)

**Theorem 3.5.** Let \( E \) be a Banach lattice with order continuous dual and \( Y \) a Banach space. Suppose that \((T_i)\) is a BL-filtration on \( E \) and \((\hat{T}_i)\) is the filtration on \( \mathcal{L}^{\text{cas}}(E, Y) \) induced by \((T_i)\). Then \( \mathcal{M}_i(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) = \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) \).

**Proof.** The inclusion \( \mathcal{M}_i(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) \subset \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) \) is obvious. For the reverse inclusion, let \((F_i, \hat{T}_i) \in \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) \). By Proposition 2.8, there exists \( 0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*) \) such that \( \sup_{i \in \mathbb{N}} \| f_i^* \| \leq \sup_{i \in \mathbb{N}} \| F_i \|_{\text{cas}} \) and \( \| F_i x \| \leq \langle |x|, f_i^* \rangle \) for each \( x \in E \) and \( i \in \mathbb{N} \). Let \( \sup_{i \in \mathbb{N}} \| f_i^* \| := K \) and define \( f^* : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \to \mathbb{R} \) by \( \langle x, f^* \rangle = \lim_{i \to \infty} \langle x, f_i^* \rangle \) for each \( x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \). Observe that \( f^* \) is well defined, as for \( x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \) there exists \( i \in \mathbb{N} \) such that \( x \in \mathcal{R}(T_i) \). Consequently, \( i \leq j \) implies \( \langle x, f_i^* \rangle = \langle x, T_i^* f_j^* \rangle = \langle T_i x, f_j^* \rangle = \langle x, f_j^* \rangle \). Thus, \( \langle x, f^* \rangle = \lim_{i \to \infty} \langle x, f_i^* \rangle \) exists for each \( x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \). Evidently, \( f^* \) is positive, linear and the inequality \( \langle x, f^* \rangle = \lim_{i \to \infty} \| x, f_i^* \| \leq \lim_{i \to \infty} \| f_i^* \| |x| = K \|x\| \) shows that \( f^* \) is also bounded with norm \( \| f^* \| \leq K \).
Now define a map $F : \bigcup_{i=1}^{\infty} R(T_i) \to Y$ by $Fx = \lim_{i \to \infty} F_ix$ for each $x \in \bigcup_{i=1}^{\infty} R(T_i)$. The map $F$ is well defined because, for each $x \in \bigcup_{i=1}^{\infty} R(T_i)$, there is some $i \in \mathbb{N}$ for which $x \in R(T_i)$. Thus, $i \leq j$ implies $F_ix = \hat{T}_iF_jx = F_jT_ix = F_jx$ so that $Fx = \lim_{i \to \infty} F_ix$ exists for each $x \in \bigcup_{i=1}^{\infty} R(T_i)$. It is now evident that $F$ is linear. Moreover, since $\bigcup_{i=1}^{\infty} R(T_i)$ is a Riesz subspace of $E$, we have

$$\|Fx\| = \lim_{i \to \infty} \|F_ix\| \leq \lim_{i \to \infty} \langle |x|, f^*_i \rangle = \langle |x|, f^* \rangle \leq K \|x\|$$

for all $x \in \bigcup_{i=1}^{\infty} R(T_i)$. Thus, $F$ is bounded. Let $\overline{f}^*$ and $\overline{F}$ denote the unique continuous extensions of $f^*$ and $F$ respectively to the Banach sublattice $\bigcup_{i=1}^{\infty} R(T_i)$ of $E$. Then we have $\|\overline{F}x\| \leq \langle |x|, \overline{f}^* \rangle$ for all $x \in \bigcup_{i=1}^{\infty} R(T_i)$. Consequently, Lemma 3.3 implies $\overline{F} \in \mathcal{L}^{\text{cas}}(\bigcup_{i=1}^{\infty} R(T_i), Y)$. By Chapter IV, §3, Proposition 3.9), $\overline{F}$ possesses an extension $\overline{F}_\infty \in \mathcal{L}^{\text{cas}}(E, Y)$ with $\|\overline{F}_\infty\|_{\text{cas}} = \|\overline{F}\|_{\text{cas}}$. Finally, $\overline{T}_i\overline{F}_\infty x = F\hat{T}_ix = \lim_{j \to \infty} F_j\hat{T}_ix = \lim_{j \to \infty} \hat{T}_iF_jx = F\hat{T}_ix$ for all $x \in E$ and $i \in \mathbb{N}$. Thus, $(F_i, \hat{T}_i) \in \mathcal{M}_1(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i)$.

We continue our preparations with the next lemma, which is a simple restatement of well-known facts about order continuity of the norm in dual Banach lattices.

**Lemma 3.6.** Let $E$ be a Banach lattice such that $E^*$ has order continuous norm. If $T : E \to \ell^1$ is a positive linear operator, then $T$ is compact.

**Proof.** Let $T : E \to \ell^1$ be a positive operator. Denote the restriction of $T^*$ to $c_0$ by $T^*|_{c_0}$. Then $T^*|_{c_0} : c_0 \to E^*$ is positive. But $E^*$ is a KB-space by [16] Theorem 2.4.14]; thus, $T^*|_{c_0}$ is weakly compact (cf. [17] Chapter II, §5, Proposition 5.15]). Consequently, $(T^*|_{c_0})^* : E^{**} \to \ell^1$ is compact because $\ell^1$ has the Schur property. Hence, $T = (T^*|_{c_0})^*|_E$ is compact. □

Lastly, we need the following characterization of the $t$-tensor product, which is shown in [13] Theorem 5.2].

**Theorem 3.7.** Let $E$ be a Banach lattice, $Y$ a Banach space and $T \in \mathcal{L}(E, Y)$. Then $T \in E^* \hat{\otimes} Y$ if and only if there exist $0 \leq S \in \mathcal{L}(E, \ell^1)$ and $R \in \mathcal{L}(\ell^1, Y)$ such that $S$ is compact and $T = R \circ S$. Further, $\|T\|_{\text{cas}} = \inf \|R\| \|S\|_r$, where the infimum is taken over all such factorizations of $T$.

We are now prepared to characterize the Radon Nikodym property:

**Theorem 3.8.** Let $Y$ be a Banach space. Then the following statements are equivalent:

(a) $Y$ has the Radon Nikodym property.

(b) $E^* \hat{\otimes} Y = \mathcal{L}^{\text{cas}}(E, Y)$ for all separable Banach lattices $E$ with order continuous dual.

(c) $\mathcal{M}(E^* \hat{\otimes} Y, T^*_c \hat{\otimes} id_Y) = \mathcal{M}_1(E^* \hat{\otimes} Y, T^*_c \hat{\otimes} id_Y)$ for all separable Banach lattices $E$ with order continuous dual and all BL-filtrations $(T_i)$ on $E$.

(d) $\mathcal{M}(E \hat{\otimes} Y, T_i \hat{\otimes} id_Y) = \mathcal{M}_{\text{pc}}(E \hat{\otimes} Y, T_i \hat{\otimes} id_Y)$ for all separable reflexive Banach lattices $E$ and all complemented, quasi-interior preserving BL-filtrations $(T_i)$ on $E$.

(e) $\mathcal{M}(E \hat{\otimes} Y, T_i \hat{\otimes} id_Y) = \mathcal{M}(E, T_i \hat{\otimes} Y)$ for all separable reflexive Banach lattices $E$ and all complemented, quasi-interior preserving BL-filtrations $(T_i)$ on $E$. 

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Proof. (a)⇒(b) Let $E$ be a separable Banach lattice with order continuous dual. Let $T \in L^{\text{cas}}(E,Y)$. By Lemma 3.3 there exist a finite measure space $(\Omega, \Sigma, \mu)$ and operators $0 \leq T_1 \in \mathcal{L}(E,L^1(\mu))$ and $T_2 \in \mathcal{L}(L^1(\mu), Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \leq \|T\|_{\text{cas}}$ and $\|T_2\| \leq 1$. Since $Y$ has the Radon Nikodym property, the Lewis-Stegall Theorem (cf. [2] Chapter III, §1, Theorem 8) guarantees the existence of operators $0 \leq S_1 \in \mathcal{L}(L^1(\mu), \ell^1)$ and $S_2 \in \mathcal{L}(\ell^1, Y)$ such that $T_2 = S_2 \circ S_1$. Since $E^{*}$ is order continuous, the positive operator $S_1 \circ T_1 : E \to \ell^1$ is compact by Lemma 3.6 Hence, $T \in E^{*} \hat{\otimes}_i Y$ by Theorem 5.7.

(b)⇒(c) Suppose $E$ is a separable Banach lattice with order continuous dual and $(T_i)$ is a filtration on $E$. Let $(f_i, T_i^{*} \otimes \text{id}_Y) \in \mathcal{M}(E^{*} \hat{\otimes}_i Y, T_i^{*} \otimes \text{id}_Y)$. By (b), $E^{*} \hat{\otimes}_i Y$ is isometric to $L^{\text{cas}}(E,Y)$ under the continuous extension of the canonical isometry $E^{*} \hat{\otimes}_i Y \to L^{\text{cas}}(E,Y)$, given by $u \mapsto L_u$, where $L_u = \sum_{i=1}^n \langle \cdot, x_i^{*}\rangle y_i$ for $u = \sum_{i=1}^n x_i^{*} \otimes y_i$. Let $(F_i) \subset L^{\text{cas}}(E,Y)$ be the sequence corresponding to the martingale $(f_i, T_i^{*} \otimes \text{id}_Y)$. Observe that, for $u = \sum_{k=1}^n x_k^{*} \otimes y_k \in E^{*} \hat{\otimes}_i Y$, we have

$$(T_i^{*} \otimes \text{id}_Y)u \mapsto \sum_{i=1}^k \langle \cdot, T_i^{*}x_k^{*}\rangle y_k = L_u \circ T_i = \hat{T}_i L_u$$

for all $k \in \mathbb{N}$. It follows that $(F_i, \hat{T}_i) \in \mathcal{M}(L^{\text{cas}}(E,Y), \hat{T}_i)$. By Theorem 3.5 there exists $F_\infty \in L^{\text{cas}}(E,Y)$ such that $\hat{T}_i F_\infty = F_i$ for each $i \in \mathbb{N}$. Consequently, $(f_i, T_i^{*} \otimes \text{id}_Y) \in \mathcal{M}(E^{*} \hat{\otimes}_i Y, T_i^{*} \otimes \text{id}_Y)$.

(c)⇒(d) Since $E$ is a separable reflexive Banach lattice, $E$ has non-empty quasi-interior, $E^{**}$ is order continuous and $E^*$ is separable (cf. [17] Chapter II, §5, Theorem 5.16). Moreover, by Corollary 2.5 $(T_i^{*})$ is a BL-filtration on $E^*$. Consequently, by (c) and [13] Theorem 6.1], we have

$$\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E^{**} \hat{\otimes}_i Y, T_i^{**} \otimes \text{id}_Y)$$

$$= \mathcal{M}_t(E^{**} \hat{\otimes}_i Y, T_i^{**} \otimes \text{id}_Y) = \mathcal{M}_t(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y).$$

Since the BL-filtration $(T_i)$ is complemented by a (positive) contractive projection $T_\infty : E \to E$, we have by [2] Lemma 4.2 and Lemma 5.1 that $\mathcal{R}(T_\infty \otimes \text{id}_Y) = \mathcal{R}(T_\infty) \hat{\otimes}_i Y = \bigcup_{n=1}^\infty \mathcal{R}(T_n \otimes \text{id}_Y)$. It now follows by a continuity argument that $(T_i \otimes \text{id}_Y)$ is a filtration on $E \hat{\otimes}_i Y$ complemented by $T_\infty \otimes \text{id}_Y$. Thus, $\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}_{nc}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y)$, as required.

(d)⇒(a) For all finite measure spaces $(\Omega, \Sigma, \mu)$ and $1 < p < \infty$, the Banach lattice $L^p(\mu)$ is separable and reflexive. By (d), it follows that $\mathcal{M}(L^p(\mu, Y), \Sigma_i) = \mathcal{M}_{nc}(L^p(\mu, Y), \Sigma_i)$ for every filtration $(\Sigma_i)$. Thus, $Y$ has the Radon Nikodym property by [2] Theorem II.2.2.2).

(d)⇒(e) Suppose $\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}_{nc}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y)$. Since $(T_i)$ is a BL-filtration, it follows that $\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}_{nc}(E, T_i) \hat{\otimes}_i Y$ by [2] Corollary 5.2. Since $E$ is reflexive, $\mathcal{M}_{nc}(E, T_i)$ is Riesz and isometrically isomorphic to $\mathcal{M}(E, T_i)$. Thus, $\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes}_i Y$ by [14] Theorem 6.1. Conversely, using [2] Corollary 5.2 and [14] Theorem 6.1] again, we obtain $\mathcal{M}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes}_i Y = \mathcal{M}_{nc}(E, T_i) \hat{\otimes}_i Y = \mathcal{M}_{nc}(E \hat{\otimes}_i Y, T_i \otimes \text{id}_Y)$, as required. □
The above theorem allows us to generalize [3 Chapter IV, §1, Theorem 1], which characterizes the “Asplund spaces”. A Banach space $Y$ is called an Asplund space if $Y^*$ has the Radon Nikodým property.

**Corollary 3.9.** Let $Y$ be a Banach space. Then $Y$ is an Asplund space if and only if $E^* \hat{\otimes} Y^* = (E \hat{\otimes} Y)^*$ for all separable Banach lattices $E$ with order continuous dual.

**Proof.** By Theorem 3.8, $Y^*$ has the Radon Nikodým property if and only if $E^* \hat{\otimes} Y^* = \mathcal{L}^{cas}(E,Y^*)$ for all separable Banach lattices $E$ with order continuous dual. But, by a theorem of Jacobs, we have $\mathcal{L}^{cas}(E,Y^*) = (E \hat{\otimes} Y)^*$ (cf. [10, 17 Chapter IV, §7, Theorem 7.4] and [3]). Thus, $E^* \hat{\otimes} Y^* = \mathcal{L}^{cas}(E,Y^*) = (E \hat{\otimes} Y)^*$, which completes the proof.

It is important to note that the above theorem does not include the case $E = L^1(\mu)$. However, by [3, Chapter IV, §1, Theorem 1], $Y$ is an Asplund space if and only if $L^1(\mu,Y)^* = L^{\infty}(\mu,Y^*)$ for all finite measure spaces $(\Omega, \Sigma, \mu)$.

Combining Theorem 3.8 with [2, Theorem 5.3] yields another corollary:

**Corollary 3.10.** Let $Y$ be a Banach space. Then the following conditions are equivalent:

(a) $Y$ has the Radon Nikodým property.

(b) For every separable reflexive Banach lattice $E$ and every complemented, quasi-interior preserving BL-filtration $(T_i)$ on $E$, we have $(f_n) \in \mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y)$ if and only if for each $i \in \mathbb{N}$, there exist $(x_i^{(n)}, T_n)_{n=1}^\infty \in \mathcal{M}_{nc}(E, T_i)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have $f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i$, where $\| \sum_{i=1}^{\infty} \lim_{n \to \infty} x_i^{(n)} \| < \infty$ and $\lim_{n \to \infty} \| y_i \| = 0$.

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