A $p$-ADIC SUPERCONGRUENCE CONJECTURE
OF VAN HAMME

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(Communicated by Ken Ono)

Abstract. In this paper we prove a $p$-adic supercongruence conjecture of van Hamme by placing it in the context of the Beukers-like supercongruences of Rodriguez-Villegas. This conjecture is a $p$-adic analog of a formula of Ramanujan.

1. Introduction

Recently, van Hamme [vH] made several conjectures concerning $p$-adic analogs of several formulas of Ramanujan. In this paper we prove one of these conjectures by making a connection between it and one of the Beukers-like supercongruences discovered by Rodriguez-Villegas [FRV].

We begin with numbers Apéry used in his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$:

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$B(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

where

$$(a)_k := a(a+1)\cdots(a+k-1) \quad \text{and} \quad \binom{n}{k} := \frac{(-1)^k(-n)_k}{k!}$$

are the standard notation for the raising factorial and binomial coefficients respectively. For $p$ an odd prime, Beukers made the following two conjectures concerning these numbers and the coefficients of two modular forms:

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2},$$  \hfill (1.1)

$$B\left(\frac{p-1}{2}\right) \equiv b(p) \pmod{p^2},$$  \hfill (1.2)

Received by the editors September 18, 2007, and, in revised form, October 22, 2007.

2000 Mathematics Subject Classification. Primary 33C20; Secondary 11S80.

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where
\[ \sum_{k=0}^{\infty} a(n)q^n := q \cdot \Pi_{n=1}^{\infty} (1 - q^{2n})^4(1 - q^{4n})^4 \in S_4(\Gamma_0(8)) \] and
\[ \sum_{k=0}^{\infty} b(n)q^n := q \cdot \Pi_{n=1}^{\infty} (1 - q^{4n})^6 \in S_3(\Gamma_0(16), (\frac{-d}{d})), \quad q := e^{2\pi iz}. \]

Both of these conjectures have multiple guises. Another form of (1.2) is found in [vH]:
\[ (1.3) \]
\[ \frac{p-1}{2} \sum_{k=0}^{p-1} (-1)^k \left( \frac{-\frac{1}{2}}{k} \right)^3 \equiv b(p) \pmod{p^2}. \]

Beukers proved these modulo \( p \) [Be1], [Be2]. For (1.1), a partial proof was given by Ishikawa [I], and a complete proof was given by Ahlgren and Ono [AO]. For (1.3), proofs have been given by Ishikawa [I], van Hamme [vH], and Ahlgren [A]. (For techniques that yield a computer free version of [A], see Mortenson [M3].) Finite field analogs of classical hypergeometric series [G] play large roles in [AO], [A], [M3], and [McO]. It should be noted that [O1] is the first paper in which the material of [G] is used to obtain such congruences.

In [FRV], Rodriguez-Villegas discovered numerically a number of Beukers-like supercongruences. This was motivated by his joint work with Candelas and de la Ossa [COV], where they studied Calabi-Yau manifolds over finite fields. For proofs of some of these congruences, see [M1], [M2], [M3], and [K], where again the theory of finite field analogs of classical hypergeometric series plays a large role. (The supercongruence of [K] is also found in the list of conjectures in [vH].) For example, we have the following:

**Theorem 1.1** ([M1], [M2]). Let \( p \) be an odd prime \( p \geq 5 \), and denote by \( \phi_p(x) \) the Legendre symbol modulo \( p \). Then
\[ \frac{p-1}{2} \sum_{k=0}^{p-1} \left( \frac{1}{2} \right) k \left( \frac{-\frac{1}{2}}{k} \right) k! \equiv \phi_p(-1) \pmod{p^2}. \]

It should be noted that the proof in [M1] is software package dependent, whereas the proof in [M2] is not.

The motivation for this research comes from a recent paper by McCarthy and Osburn [McO], where they prove the following conjecture of van Hamme [vH, page 226, (A.2)]:

**Conjecture** ([vH] (A.2)). If \( p \) is an odd prime, then
\[ \frac{p-1}{2} \sum_{k=0}^{p-1} (4k + 1) \left( \frac{-\frac{1}{2}}{k} \right)^5 \equiv p \cdot b(p) \pmod{p^3}. \]

Upon examining their proof, one is immediately struck by the shadow of one of Beukers’ supercongruences, e.g. (1.3). In other words, one finds
\[ \frac{p-1}{2} \sum_{k=0}^{p-1} (4k + 1) \left( \frac{-\frac{1}{2}}{k} \right)^5 \equiv p \cdot \sum_{k=0}^{p-1} (-1)^k \left( \frac{-\frac{1}{2}}{k} \right)^3 \pmod{p^3}. \]
This leads one to speculate as to the potential role of other Beukers-like supercongruences, and brings us to the goal of this paper. With this idea, Theorem 1.1, and the arsenal of transformation formulas for generalized hypergeometric series found in Bailey’s tract [B], we prove the next supercongruence conjecture on van Hamme’s list [vH, page 226, (B.2)]:

Conjecture (vH (B.2)). Let \( p \) be an odd prime; then

\[
\sum_{k=0}^{\frac{p-1}{2}} (4k + 1) \left( -\frac{k}{2} \right)^3 \equiv p \cdot \phi_p(-1) \pmod{p^3}.
\]

This is the \( p \)-adic analog of the following formula of Ramanujan [vH, page 226, (B.1)]:

\[
\sum_{k=0}^{\infty} (4k + 1) \left( -\frac{k}{2} \right)^3 = \frac{2}{\pi}.
\]

It should be pointed out that such congruences address questions like Problem 11.40 of [O2].

The paper is organized as follows. In section 3 we recall necessary background information, and in section 4 we prove a technical lemma. In section 5, we prove van Hamme’s conjecture (B.2).

2. Background information

We begin this section by recalling a theorem of Morley.

**Theorem 2.1 ([Mo]).** Let \( p \) be an odd prime. Then

\[
2^{p-2} \equiv \phi_p(-1) \left( \frac{p-1}{2} \right) \pmod{p^3}.
\]

We recall the gamma function as defined by Weierstrass and found in [WW, Ch. XII].

**Definition 2.2.** Let \( z \in \mathbb{C} \); then

\[
\frac{1}{\Gamma(z)} := z^\gamma \prod_{n=1}^{\infty} \left\{ \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right\},
\]

where \( \gamma \) is Euler’s constant.

From this definition it is apparent that \( \Gamma(z) \) is analytic except at the points \( z = 0, -1, -2, \ldots \), where it has simple poles. We also recall some gamma function properties, which are found in [WW, Ch. XII].

**Proposition 2.3.** Let \( z \in \mathbb{C} \). Then the following are true:

1. \( \Gamma(1) = 1 \),
2. \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \),
3. \( \Gamma(z + 1) = z \Gamma(z) \),
4. \( \Gamma(z) \Gamma(1 - z) = \frac{\sin(\pi z)}{\pi z} \),
5. \( 2^{z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \Gamma(\frac{1}{2}) \Gamma(2z) \).
Lastly, we recall some generalized hypergeometric series transformations as found in Bailey’s tract [B]. The first may be viewed as a specialization of Whipple’s famous \( \tau F_6 \) transformation [B, p. 28]. The specialization is

\[
\begin{align*}
\pFq{6}{5}{a, 1 + \frac{1}{2}a, b, c, d, e}{\frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e}{1}
\end{align*}
\]

(2.1)

\[
= \frac{\Gamma(1 + a - d)\Gamma(1 + a - e)}{\Gamma(1 + a)\Gamma(1 + a - d - e)} \pFq{3}{2}{1 + a - b - c, d, e}{1 + a - b, 1 + a - c}{1}
\]

(2.2)

where

\[
F_n^{(a_0, a_1, \ldots, a_n)}(a, b_1, \ldots, b_n | t) = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_n)_k t^k}{k!(b_1)_k \cdots (b_n)_k}
\]

is the classical hypergeometric series.

We recall a transformation formula for a terminating \( \pFq{3}{2} \) [B, page 22, eq. 1]:

\[
\frac{\Gamma(e + m)\Gamma(f + m)}{\Gamma(s)\Gamma(e)\Gamma(f)} \pFq{3}{2}{a, b, -m | 1 - e, f}{1}
\]

(2.3)

\[
= \frac{(-1)^m \Gamma(1 - a)\Gamma(1 - b)}{\Gamma(1 - a)\Gamma(s - m)\Gamma(1 - b - m)} \pFq{3}{2}{e - b, f - b, -m | 1 - b - m, s - m}{1 - b - m}
\]

where \( s := e + f - a - b + m \), and \( m \) is a positive integer. Here we are careful, because we will eventually look at the limit as \( a \) goes to one.

3. A TECHNICAL LEMMA

Here we show a technical lemma, which appears in the proof of Conjecture (B.2).

**Lemma 3.1.** Let \( n = 2t + 1 \) be a positive odd integer, and \( \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \). Then

\[
\frac{\Gamma(1 + \omega^n)\Gamma(1 + \omega^{2n})}{\Gamma(\frac{1}{2} - \omega^n)\Gamma(\frac{1}{2} - \omega^{2n})} = \frac{2^{2n-2}}{\prod_{k=1}^{n-1}((2k - 1)^2 + 3n^2)}
\]

**Proof.** We use Proposition 2.3 (4) to write

\[
\frac{\Gamma(1 + \omega^n)\Gamma(1 + \omega^{2n})}{\Gamma(\frac{1}{2} - \omega^n)\Gamma(\frac{1}{2} - \omega^{2n})} = \frac{\pi \sin(-\pi \omega^n) \sin(-\pi \omega^{2n})}{\Gamma(-\omega^n)\Gamma(-\omega^{2n})\Gamma(\frac{1}{2} - \omega^n)\Gamma(\frac{1}{2} - \omega^{2n})}.
\]

Writing sine in terms of the exponential function and simplifying, we obtain

\[
\frac{\pi \sin(-\pi \omega^n) \sin(-\pi \omega^{2n})}{\Gamma(-\omega^n)\Gamma(-\omega^{2n})\Gamma(\frac{1}{2} - \omega^n)\Gamma(\frac{1}{2} - \omega^{2n})} = \frac{1}{4\pi^2} \cdot \frac{1}{\Gamma(-\omega^n)\Gamma(\frac{1}{2} - \omega^n)\Gamma(-\omega^{2n})\Gamma(\frac{1}{2} - \omega^{2n})}.
\]

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Using the duplication formula of Proposition 2.3 (5) yields

\[
\frac{4\pi^2}{e^{\pi \sqrt{3/2}} + e^{-\pi \sqrt{3/2}}} \cdot \frac{1}{\Gamma(-\omega^2/3)} \cdot \frac{1}{\Gamma(-\omega/3)} \cdot \frac{1}{\Gamma(-\omega^2/3)} = \frac{1}{\pi} \frac{2^n}{\Gamma(t + \frac{1}{2} - \frac{n}{2} i \sqrt{3}) \Gamma(t + \frac{1}{2} + \frac{n}{2} i \sqrt{3})} = \frac{1}{\pi} \prod_{k=1}^{n-1} \left( (k - \frac{1}{2})^2 + \frac{3}{4} n^2 \right) \frac{2^n \sin(\frac{n}{2} i \sqrt{3})}{\prod_{k=1}^{n-1} (k - \frac{1}{2})^2 + \frac{3}{4} n^2},
\]

where in the last step we have used \( \Gamma(\frac{1}{2})^2 = \pi \). Recalling \( n = 2t + 1 \), where \( t \) is a nonnegative integer, we find

\[
\frac{\pi}{e^{\pi \sqrt{3/2}} + e^{-\pi \sqrt{3/2}}} \cdot \frac{2^n}{\Gamma(-\omega n) \Gamma(-\omega^2 n)}.
\]

where the penultimate line follows from the repeated application of Proposition 2.3 (3), and the ultimate line follows from Proposition 2.3 (4). The conclusion of the lemma easily follows.

\[\square\]

4. PROOF OF CONJECTURE (B.2)

Using an idea of McCarthy and Osburn [McO], we make the following choice of variables in equation (2.1). We let \( a = \frac{1}{2}, b = \frac{1}{2} - \omega \frac{e}{3}, c = \frac{1}{2} - \omega^2 \frac{e}{3}, d = \frac{1}{2} - \frac{e}{3}, \) and \( e = 1 \), where \( \omega = -\frac{1}{2} + i \sqrt{2} \), to produce

\[
e_{F_b} \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} - \omega \frac{e}{3}, \frac{1}{2} - \omega^2 \frac{e}{3}, \frac{1}{2} - \frac{e}{3}, \frac{1}{2} \right) = \frac{\Gamma(1 + \frac{e}{3}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{e}{3})} 3F_2 \left( \frac{1}{2} - \frac{e}{3}, \frac{1}{2} - \frac{e}{3}, 1 \mid \frac{1}{2}, 1 + \omega \frac{e}{3}, 1 + \omega^2 \frac{e}{3} \right).
\]

This one can consider \( \pmod{p^3} \), and by using appropriate gamma function properties, one obtains

\[
\sum_{k=0}^{p-1} \left( 4k + 1 \right)^3 \equiv -\frac{1}{k} \pmod{p^3}.
\]

We now focus on the coefficient of \( p \) on the right-hand side of equation (4.1). We recall equation (2.2) and let \( a = 1 + e, b = \frac{1}{2} - \frac{e}{3}, m = \frac{4k-1}{2}, e = 1 + \omega \frac{e}{3}, \) and
\[ f = 1 + \omega^2 \frac{p}{2} \] to obtain\[ \frac{\Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right) \Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right)}{\Gamma\left(\frac{1}{2} - \epsilon\right) \Gamma\left(\frac{1}{2} + \omega^2 \frac{p}{2}\right) \Gamma\left(1 + \omega^2 \frac{p}{2}\right) \cdot 3F_2 \left(1 + \epsilon, \frac{1}{2} - \omega^2 \frac{p}{2}, \frac{1}{2} - \omega^2 \frac{p}{2}; 1 + \omega^2 \frac{p}{2}, 1 + \omega^2 \frac{p}{2} \right) \]

\[ = \frac{(-1)^{p-1}}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{1}{2} + \omega^2 \frac{p}{2}\right) \Gamma\left(1 + \omega^2 \frac{p}{2}\right)}{\Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right) \Gamma\left(1 - \omega^2 \frac{p}{2}\right)} \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{(1 + \omega^2 \frac{p}{2})k} \] (mod \(p^2\)).

Isolating the hypergeometric series on the left, taking the limit as \(\epsilon\) goes to 0, and then examining everything (mod \(p^3\)) yields,

\[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \frac{1}{2}k)k}{(1 + \omega^2 \frac{p}{2})k(1 + \omega^2 \frac{p}{2})k} \equiv \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{(1 + \omega^2 \frac{p}{2})k(1 + \omega^2 \frac{p}{2})k} \]

\[ \equiv (-1)^{p-1} \cdot 2^{1-p}(p-1)! \cdot \frac{\Gamma\left(1 + \omega^2 \frac{p}{2}\right) \Gamma\left(1 + \omega^2 \frac{p}{2}\right)}{\Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right) \Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right)} \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2} \] (mod \(p^2\)).

However, we need only be concerned with the above equation (mod \(p^2\)). Employing Proposition 3.3 and (5) produces

\[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \frac{1}{2}k)k}{(1 + \omega^2 \frac{p}{2})k(1 + \omega^2 \frac{p}{2})k} \equiv (-1)^{p-1} \cdot 2^{1-p}(p-1)! \cdot \frac{\Gamma\left(1 + \omega^2 \frac{p}{2}\right) \Gamma\left(1 + \omega^2 \frac{p}{2}\right)}{\Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right) \Gamma\left(\frac{1}{2} - \omega^2 \frac{p}{2}\right)} \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2} \] (mod \(p^2\)).

From Lemma 3.1 we have

\[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \frac{1}{2}k)k}{(1 + \omega^2 \frac{p}{2})k(1 + \omega^2 \frac{p}{2})k} \equiv (-1)^{p-1} \cdot 2^{1-p}(p-1)! \cdot \frac{\sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2}}{\prod_{k=1}^{p-1} ((2k-1)^2 + 3p^2)} \cdot \frac{\sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2}}{\prod_{k=1}^{p-1} ((2k-1)^2)} \] (mod \(p^2\))

\[ \equiv (-1)^{p-1} \cdot 2^{p-1}(p-1)! \cdot \frac{\sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2}}{\prod_{k=1}^{p-1} ((2k-1)^2)} \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2} \] (mod \(p^2\))

\[ \equiv (-1)^{p-1} \cdot 2^{2p-2} \cdot \left(\frac{p}{2}\right)^{-1} \cdot \frac{\sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2}}{\prod_{k=1}^{p-1} ((2k-1)^2)} \] (mod \(p^2\))

\[ \equiv \sum_{k=0}^{p-1} \frac{(\frac{1}{2}k)k}{k!^2} \] (mod \(p^2\)),

where the last line follows from Morley’s result, Theorem 2.1.
We now return to equation (4.1) and see
\[
\sum_{k=0}^{p-1} (4k+1)\left(\frac{-1}{k}\right)^3 \equiv p \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \frac{p}{k})(\frac{1}{2} - \frac{p}{k})}{(1 + \omega \frac{p}{2})_k(1 + \omega^2 \frac{p}{2})_k} \pmod{p^3}
\]
\[
\equiv p \cdot \sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^3} \equiv p \cdot \phi_p(-1) \pmod{p^3},
\]
where the last congruence follows from Theorem (4.1).

Acknowledgements
The author would like to thank George Andrews, Karl Mahlburg, Ken Ono, and the referee for their helpful comments and suggestions.

References


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