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INCOMPRESSIBILITY OF TORI TRANSVERSE TO AXIOM A FLOWS

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ABSTRACT. We prove that a torus transverse to an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold is incompressible. This strengthens the works of Brunella (1993), Fenley (1995), and Mosher (1992).

1. INTRODUCTION

It is well known that an Anosov vector field on a closed atoroidal 3-manifold is transitive. This follows from the fact that a torus transverse to an Anosov vector field on a closed 3-manifold is incompressible [2], [3], [15]. In this paper we extend both results to the more general class of vector fields on irreducible 3-manifolds, namely, Axiom A vector fields that do not exhibit sinks, sources or null homotopic periodic orbits. Recall that a *sink* of a C^1 vector field is a hyperbolic attracting closed orbit, while a *source* is a sink for the time-reversed vector field. See [14] where sufficient conditions for the transitivity of Axiom A vector fields on toroidal 3-manifolds are given. Let us present our results in a precise way.

Hereafter M, X and X_t will denote a 3-manifold, a C^1 vector field on M and the flow generated by X in M respectively. We say that M is *closed* if it is compact connected and has empty boundary. We say that M is *irreducible* if every embedded 2-sphere in M is the boundary of a 3-ball in M. A two-side surface S in M is *incompressible* if the homomorphism $\pi_1(S) \to \pi_1(M)$ induced by the inclusion is injective. We say that X is *transitive* if it has a dense orbit. A compact invariant set H of X is *hyperbolic* if there is a tangent bundle splitting $T_HM = E^s_H \oplus E^X_H \oplus E^u_H$ over H such that E^s_H is contracting, E^u_H is expanding and E^X_H is the subbundle generated by X. A *nonwandering point* of X is a point $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. We say that X is *Axiom A* if its nonwandering set is hyperbolic with dense closed orbits, and *Anosov* if M is a hyperbolic set. An Anosov vector field is Axiom A but not conversely. See [8], [6], [10], [17] for details. Our main result is the following.

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Theorem 1.1. *Let X be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold. Then, every torus transverse to X is incompressible.*

Let us state two short corollaries of Theorem 1.1. Recall that a closed 3-manifold is *atoroidal* if it has no incompressible tori [8], [10].

Corollary 1.1. *Let X be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold M. If M is atoroidal, then X is a transitive Anosov vector field.*

The following corollary was proved early in [2], [3], [15].

Corollary 1.2. *Anosov vector fields on closed atoroidal 3-manifolds are transitive.*

Theorem 1.1 becomes false if we remove the hypothesis that X does not exhibit null homotopic periodic orbits [1]. On the other hand, if all closed 3-manifolds exhibiting an Axiom A vector field as in the theorem were irreducible, then we could remove the hypothesis that M is irreducible from the theorem. However, nonirreducible closed 3-manifolds exhibiting such Axiom A vector fields do exist [13]. Finally, Corollary 1.1 becomes false if we remove the hypothesis that M is atoroidal. Indeed, a counterexample can be constructed as in [4].

2. Proofs

A *solid torus* is a compact 3-manifold diffeomorphic to the product of a 2-disk and a circle. The boundary and the interior of a manifold U will be denoted by ∂U and $Int(U)$ respectively. The following elementary lemma deals with solid tori inside a solid torus.

Lemma 2.1. *Let ST^1 and ST^2 be two solid tori contained in a solid torus ST. If $\partial(ST^1) \subset Int(ST^2)$, then $ST^1 \subset Int(ST^2)$.*

Proof. By contradiction assume that $ST^1 \not\subset Int(ST^2)$. We claim that $\partial(ST^2) \subset Int(ST^1)$. Indeed, $ST^1 \cap \partial(ST^2) \neq \emptyset$ since ST^1 is connected and $\partial(ST^2)$ separates ST. Note that $ST^1 \cap \partial(ST^2)$ is closed in $\partial(ST^2)$ since ST^1 and $\partial(ST^2)$ are closed in $\partial(ST^2)$. Moreover, $ST^1 \cap \partial(ST^2)$ is open in $\partial(ST^2)$ since $ST^1 \cap \partial(ST^2) = Int(ST^1) \cap \partial(ST^2)$. As $\partial(ST^2)$ is connected we conclude that $ST^1 \cap \partial(ST^2) = \partial(ST^2)$ and then $\partial(ST^2) \subset ST^1$. Actually we have $\partial(ST^2) \subset Int(ST^1)$ since $\partial(ST^1) \subset Int(ST^2)$, and so $\partial(ST^2) \cap \partial(ST^1) = \emptyset$. This proves the claim.

As $\partial(ST^2) \subset Int(ST^1)$ (by the claim) and $\partial(ST^1) \subset Int(ST^2)$ (by hypothesis), we conclude that $ST^1 \cup ST^2$ is a closed 3-manifold. Obviously $ST^1 \cup ST^2 \subset ST$, and so $ST^1 \cup ST^2$ is open and closed in ST. Since ST is connected we conclude that $ST^1 \cup ST^2 = ST$. But this is a contradiction since ST has boundary and $ST^1 \cup ST^2$ does not. The result follows. $\square$

The following lemma deals with compact 3-manifolds with toral boundary inside a solid torus.

Lemma 2.2. *Let ST be a solid torus and let $U \subset ST$ be a compact connected 3-manifold whose boundary $\partial U = T_1 \cup \cdots \cup T_n$ is a union of tori. If each T_i bounds a solid torus ST_i in ST, then there is $i_0 \in \{1, \cdots, n\}$ such that $ST_i \cap U = T_i$, for* *all* $i \neq i_0$ *in* $\{1, \cdots, n\}$, *and*

$$U \cup \left(\bigcup_{i \in \{1, \cdots, n\}, i \neq i_0} ST_i \right) \subseteq ST_{i_0}.$$

In particular, if $n = 1$, *then* U *is a solid torus.*

Proof. Note that each T_i divides ST in two connected components, one of which is the solid torus ST_i. Note also that $Int(U)$ is contained in one of these components since it is connected and $T_i \subset \partial U$. If $Int(U)$ is not contained in ST_i, for all i, then we would obtain a closed 3-manifold inside ST by just capping each T_i with ST_i. But no such manifolds inside ST exist by the argument in the last part of the proof of Lemma 2.1. So, $U \subset ST_{i_0}$ for some i_0; thus $T_i \subset Int(ST_{i_0})$ for each $i \neq i_0$ in $\{1, \cdots, n\}$. Then, $ST_i \subset Int(ST_{i_0})$ for all $i \neq i_0$ in $\{1, \cdots, n\}$ by Lemma 2.1. This proves the first part of the lemma.

Now assume that ∂U is formed by a single torus T_1 which bounds a solid torus ST_1 in ST. By the first part we get $i_0 = 1$, hence $U \subset ST_1$. Since ST_1 and U have common boundary T_1 we conclude that U is an open subset of ST_1 *relative to* ST_1. But U is also a closed subset of ST_1 relative to ST_1; therefore $U = ST_1$ since ST_1 is connected. Therefore U is a solid torus since ST_1 is also. This proves the lemma. □

Next we introduce some basic concepts in hyperbolic dynamics [6], [17]. Let X be a C^1 vector field on a closed 3-manifold M. The *omega-limit set* of $x \in M$ is defined by

$$\omega(x) = \left\{ y = \lim_{n \to \infty} X_{t_n}(x) : t_n \text{ is a sequence converging to } \infty \text{ as } n \to \infty \right\}.$$

A compact invariant set Λ is *transitive* if $\Lambda = \omega(x)$ for some $x \in \Lambda$. An *attractor* is a transitive set Λ such that

$$\Lambda = \bigcap_{t \geq 0} X_t(U)$$

for some compact neighborhood U. This neighborhood is a *basin of attraction* of Λ if it is a compact 3-manifold with nonempty boundary ∂U transverse to X. A proper attractor always has a basin of attraction. A *repeller* of X is an attractor for its time reversed field $-X$. A *basin of repulsion* of a repeller R of X is a basin of attraction of R viewed as an attractor of $-X$.

A *hyperbolic attractor* (resp. *hyperbolic repeller*) of X is a hyperbolic set which is also an attractor (resp. a repeller) of X. In particular, a sink (resp. a source) is a hyperbolic attractor (resp. repeller), but not conversely. It follows from Smale's Spectral Decomposition Theorem that every Axiom A vector field exhibits a hyperbolic attractor and a hyperbolic repeller which are proper if the vector field is not transitive.

On the other hand, if $x \in M$ belongs to a hyperbolic set of X, then the so-called Invariant Manifold Theory [9] asserts that the strong stable manifold of x defined by

$$W^{ss}(x) = \left\{ y \in M : \lim_{t \to \infty} d(X_t(y), X_t(x)) = 0 \right\}$$

is a C^1 immersed submanifold of M. Consequently, the stable manifold

$$W^s(x) = \bigcup_{t \in \mathbb{R}} W^{ss}(X_t(x))$$

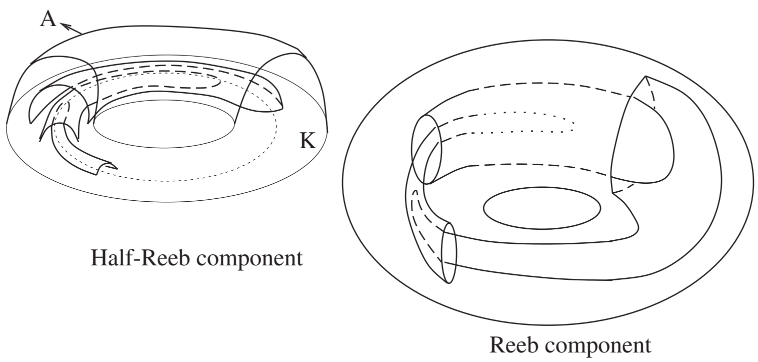

FIGURE 1

is an immersed submanifold as well.

Now we state the concept of a half-Reeb component, which will be useful in the next lemma (for basic concepts in foliation theory including the definition of Reeb component; see [5] and the references therein). Let $\mathcal{F}$ be a foliation in a solid torus ST transverse to $\partial(ST)$. A *half-Reeb component* of $\mathcal{F}$ is a saturated subset $H \subset ST$, bounded by an annulus leaf A and an annulus $K \subset \partial(ST)$ with $\partial K = \partial A$, such that the double manifold $2H$ is a Reeb component of the double foliation $2\mathcal{F}$ (see [2] and Figure 1).

The following lemma was proved in [11]. We outline its proof here for the sake of completeness.

Lemma 2.3. *A hyperbolic attractor having a solid torus as a basin of attraction is an attracting periodic orbit (hence it is a sink).*

Proof. It suffices to prove that the expanding subbundle of the attractor is zero dimensional. By contradiction we assume that such a subbundle is not zero dimensional. Then, both the stable and unstable subbundles of the attractor are one dimensional. It follows that the stable manifolds $W^s(x)$ form an invariant foliation $\mathcal{F}$ in the solid torus which is transverse to the boundary.

Clearly $\mathcal{F}$ has no closed leaves; hence $\mathcal{F}$ has no Reeb components. On the other hand, $\mathcal{F}$ has no half-Reeb components too, due to an argument based on [2]. It follows that $\mathcal{F}$ has neither Reeb nor half-Reeb components, and so the double foliation $2\mathcal{F}$ defined in $S^2 \times S^1$ (the double of the solid torus) has no Reeb components.

But $2\mathcal{F}$ is supported in $S^2 \times S^1$, which is a manifold with nonzero second homotopy group. So, $2\mathcal{F}$ is the trivial product foliation of $S^2 \times S^1$ by spheres (e.g. Theorem 1.10-(iii), p. 92 in [5]), and so $\mathcal{F}$ is the trivial product foliation by meridian disks on ST. This implies that the leaves of $\mathcal{F}$ are invariant disks. Applying the classical Poincare-Bendixon Theorem [16] to one of these disks, we could find a singularity in the solid torus. However, such singularities cannot exist due to the continuity of the hyperbolic splitting. This contradiction proves the result. $\square$

The previous lemmas will be used to prove the following one.

Lemma 2.4. *Let X be an Axiom A vector field on a closed 3-manifold M. If there is a solid torus in M whose boundary is transverse to X, then X exhibits either a sink or a source or a null homotopic periodic orbit.*

Proof. Let ST be the solid torus in the statement of the lemma. We can assume that X points inward to ST in $\partial(ST)$, for, otherwise, we replace X by $-X$ in the argument below. Then, the Spectral Decomposition Theorem [6] implies that there is a hyperbolic attractor Λ of X in ST. Of course we can assume that Λ is not a sink, for, otherwise, we are done. Note that X is not transitive since it points inward to ST in $\partial(ST)$. Therefore Λ is proper, and so it exhibits a basin of attraction U contained in ST. As Λ is not a sink we have that ∂U is a union of tori transverse to X.

We claim that if X does not exhibit null homotopic periodic orbits, then every torus in ∂U bounds a solid torus in ST. By contradiction, assume that there is a torus in ∂U which does not bound a solid torus in ST. As ST is irreducible atoroidal we have that such a torus is contained in a 3-ball in ST (e.g. [8], [10] or (4), p. 11 in [7]). Then such a torus divides ST in two connected components, one of which contained in the ball (see for instance [13]). Hence we could find a periodic orbit inside the ball, a contradiction because X does not exhibit null homotopic periodic orbits. This contradiction proves the claim.

Hereafter we assume that X does not exhibit null homotopic periodic orbits. Then, the previous claim implies that every torus in ∂U bounds a solid torus in ST. Therefore we can apply Lemma 2.2 to U. Replacing ST by the solid torus ST_{i_0} in that lemma, we can assume that $\partial(ST)$ is one of the boundary components of U. If U has no more boundary components apart from $\partial(ST)$, then U is a solid torus by Lemma 2.2 since $n = 1$ in such a case. So, Lemma 2.3 would imply that Λ is a sink, a contradiction since we have assumed that Λ is not a sink. Therefore, there is another torus T^1 in ∂U. Set $ST^0 = ST$ and let ST^1 be the solid torus bounded by T^1 in ST^0 (it exists by the claim). Note that X points outward from ST^1 at T^1; therefore there is a repeller Λ^1 in the spectral decomposition of X inside ST^1. Of course we can assume that Λ^1 is not a source, for, otherwise, we are done.

By repeating the above argument we get a nested sequence of solid tori

$$ST^0 \subset ST^1 \subset ST^2 \subset \cdots \subset ST^k \subset \cdots$$

with $ST^i \setminus Int(ST^{i+1})$ containing a hyperbolic attractor of X (for i even) or a hyperbolic repeller of X (for i odd). As the number of attractors and repellers in the spectral sequence of X is finite, we have that the sequence must stop. But this occurs precisely when some of the solid tori ST^k is the basin of attraction (or repulsion) of a hyperbolic attractor (or repeller) of X. Therefore, X exhibits a sink or a source by Lemma 2.3. This finishes the proof. □

Proof of Theorem 1.1. Let T be a torus transverse to an Axiom A vector field X in a closed irreducible 3-manifold. In addition, suppose that X does not exhibit sinks, sources or null homotopic periodic orbits. As the manifold is irreducible we have that T either bounds a solid torus or is contained in a 3-ball or is incompressible. If T bounds a solid torus, then X would exhibit either a sink or a source or a null homotopic periodic orbit by Lemma 2.4. If T is contained in a 3-ball, then there is a periodic orbit in the ball, and so X would exhibit a null homotopic periodic orbit. In both cases we get a contradiction; therefore T is incompressible. This finishes the proof. □

Proof of Corollary 1.1. Let X be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible atoroidal 3-manifold. Let us prove that X is transitive. By contradiction assume that it is not so. Then, X exhibits a proper hyperbolic attractor in its spectral decomposition. Because such an attractor is proper we have that it has a basin of attraction U. But the attractor cannot be a sink by hypothesis, so ∂U is formed by finitely many disjoint tori transverse to X. Then, there is an incompressible torus by Theorem 1.1, a contradiction since the manifold is atoroidal. This contradiction proves that X is transitive. As every transitive Axiom A vector field is Anosov we get the result. □

Proof of Corollary 1.2. The result follows directly from Corollary 1.1, since every Anosov vector field on a closed atoroidal 3-manifold satisfies the hypotheses of Corollary 1.1. □

References

[1] Apaza., E., Soares, R., *Axiom A flows without sinks nor sources on 3-manifolds*, Discrete Contin. Dyn. Syst. 21 (2008), no. 2, 393–401.

[2] Brunella, M., *Separating the basic sets of a nontransitive Anosov flow*, Bull. London Math. Soc. 25 (1993), no. 5, 487–490. MR1233413 (94g:58163)

[3] Fenley, S. R., *Quasigeodesic Anosov flows and homotopic properties of flow lines*, J. Differential Geom. 41 (1995), no. 2, 479–514. MR1331975 (96f:58118)

[4] Franks, J., Williams, B., *Anomalous Anosov flows*, Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979), pp. 158–174, Lecture Notes in Math., 819, Springer, Berlin, 1980. MR591182 (82e:58078)

[5] Gabai, D., *3 lectures on foliations and laminations on 3-manifolds*, Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 87–109, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001. MR1810537 (2002g:57032)

[6] Hasselblatt, B., Katok, A., *Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza*, Encyclopedia of Mathematics and Its Applications, 54, Cambridge University Press, Cambridge, 1995. MR1326374 (96c:58055)

[7] Hatcher, A., *Notes on Basic 3-Manifold Topology*, available at Hatcher's homepage.

[8] Hempel, J., 3-*Manifolds*, Ann. of Math. Studies, No. 86, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1976. MR0415619 (54:3702)

[9] Hirsch, M., Pugh, C., Shub, M., *Invariant Manifolds*, Lec. Not. in Math. 583, Springer-Verlag, 1977. MR0501173 (58:18595)

[10] Jaco, W., *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, 43, American Mathematical Society, Providence, R.I., 1980. MR565450 (81k:57009)

[11] Morales, C., *Poincaré-Hopf index and partial hyperbolicity*, Ann. Fac. Sci. Toulouse Math. XVII (2008), no. 1, 193–206.

[12] Morales, C., *Examples of singular-hyperbolic attracting sets*, Dyn. Syst. 22 (2007), no. 3, 339–349.

[13] Morales, C., *Incompressibility of torus transverse to vector fields*, Spring Topology and Dynamical Systems Conference. Topology Proc. 28 (2004), no. 1, 219–228. MR2105459 (2005j:57028)

[14] Morales, C., *Axiom A flows with a transverse torus*, Trans. Amer. Math. Soc. 355 (2003), no. 2, 735–745. MR1932723 (2003g:37049)

[15] Mosher, L., *Dynamical systems and the homology norm of a* 3*-manifold. I. Efficient intersection of surfaces and flows*, Duke Math. J. 65 (1992), no. 3, 449–500. MR1154179 (93g:57018a)

[16] Palis, J., de Melo, W., *Geometric theory of dynamical systems. An introduction.* Translated from the Portuguese by A. K. Manning. Springer-Verlag, New York- Berlin, 1982. MR669541 (84a:58004)

[17] Robinson, C., *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, Second edition. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1999. MR1792240 (2001k:37003)

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