A RIGIDITY THEOREM FOR HOLOMORPHIC GENERATORS ON THE HILBERT BALL

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Abstract. We present a rigidity property of holomorphic generators on the open unit ball $B$ of a Hilbert space $H$. Namely, if $f \in \text{Hol}(B, H)$ is the generator of a one-parameter continuous semigroup $\{F_t\}_{t \geq 0}$ on $B$ such that for some boundary point $\tau \in \partial B$, the admissible limit $K\text{-lim}_{z \to \tau} \frac{f(z)}{\|z - \tau\|} = 0$, then $f$ vanishes identically on $B$.

Let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. If $H$ is finite dimensional, we will identify $H$ with $\mathbb{C}^n$. We denote by $\text{Hol}(D, E)$ the set of all holomorphic mappings on a domain $D \subset H$ which map $D$ into a subset $E$ of $H$, and put $\text{Hol}(D) := \text{Hol}(D, D)$.

We are concerned with the problem of finding conditions for a mapping $F \in \text{Hol}(D, E)$ to coincide identically with a given holomorphic mapping on $D$ when they behave similarly in a neighborhood of a boundary point $\tau \in \partial D$.

A number of basic results in this direction are due to D. M. Burns and S. G. Krantz [6]. They establish conditions at a boundary point for a holomorphic self-mapping $F$ of the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ to coincide with the identity mapping (see Proposition 1 below). Then they generalize this fact to the $n$-dimensional case: for holomorphic self-mappings of the open unit ball (see Proposition 3 below) and of strongly pseudoconvex domains in $\mathbb{C}^n$. Further developments of this theme are presented by X. J. Huang in [13], where he obtains similar results for weakly pseudoconvex domains. More recently, L. Baracco, D. Zaitsev and G. Zampieri [3] have proved local boundary rigidity theorems for mappings defined only on one side as germs at a boundary point, and extended their results from boundaries of domains to submanifolds of higher codimension. More higher-dimensional results can be found, for instance, in [2] and [11].

In this paper we present a rigidity theorem for holomorphic generators on the open unit ball $B$ of a Hilbert space $H$ which generalizes the analogous theorem for the one-dimensional case [8, 17, 7] and properly contains the above-mentioned Burns–Krantz theorem for the open unit ball in $\mathbb{C}^n$.
We begin by recalling the result of D. M. Burns and S. G. Krantz [6] for holomorphic self-mappings of the open unit disk $\Delta$.

**Proposition 1.** Let $F \in \text{Hol}(\Delta)$. If the unrestricted limit

$$\lim_{z \to \tau} \frac{F(z) - z}{(z - \tau)^3} = 0$$

for some $\tau \in \partial \Delta$, then $F \equiv I$ on $\Delta$.

This assertion also holds when the unrestricted limit is replaced with the angular one (see Proposition 2 below) has been proved in [8] and [17]. To formulate it, we first [127x368]F [127x636]Proposition 1. [127x654]morphic self-mappings of the open unit disk $\Delta$.

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Proposition 2. Let \( g \in \text{Hol}(\Delta, \mathbb{C}) \) be the generator of a one-parameter continuous semigroup. Suppose that

\[
\angle \lim_{z \to 1} \frac{g(z)}{|z - 1|^3} = 0.
\]

Then \( g \equiv 0 \) in \( \Delta \).

Here we take this opportunity to present a completely different proof of this assertion.

Proof. Suppose that \( g \) does not vanish identically on \( \Delta \). Condition (2) implies that \( \tau = 1 \) is the Denjoy–Wolff point of the semigroup generated by \( g \) (see Lemma 3 in [10]). So, \( g \) has no null point in \( \Delta \) (see Theorem 1 in [10]). Consequently, \( g \) can be represented by the Berkson–Porta formula

\[
g(z) = -(1 - z)^2 p(z), \quad z \in \Delta,
\]

where \( p \) is a holomorphic function of nonnegative real part which does not vanish in \( \Delta \).

Consider the function

\[
g_1(z) := \frac{-z}{(1 - z)^2} \cdot g(z) = zp(z), \quad z \in \Delta.
\]

This function is the holomorphic generator of a semigroup on \( \Delta \) with its Denjoy–Wolff point at zero.

However, the equality

\[
\angle \lim_{z \to 1} \frac{g_1(z)}{z - 1} = \angle \lim_{z \to 1} \frac{-z}{(1 - z)^3} \cdot g(z) = 0
\]

implies that \( g_1(1) = 0 \) and \( g_1'(1) = 0 \). Therefore \( \tau = 1 \), too, is the Denjoy–Wolff point of the semigroup generated by \( g_1 \) (again by Lemma 3 in [10]). The contradiction we have reached proves that \( g \equiv 0 \) on \( \Delta \). \( \square \)

As we have already mentioned above, D. M. Burns and S. G. Krantz generalize their one-dimensional result for holomorphic self-mappings of \( \Delta \) (Proposition 1) to the open unit ball \( B := \{ x \in \mathbb{C}^n : |x| < 1 \} \), where \( |x| = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \).

Proposition 3 (see [6]). Let \( B \subset \mathbb{C}^n \) be the open unit ball. Let \( \Phi : B \to B \) be a holomorphic mapping of the ball to itself such that

\[
\Phi(x) = 1 + (x - 1) + O \left( ||x - 1||^4 \right)
\]
as \( x \to 1 \). (Here \( 1 \) denotes the distinguished boundary point \( 1 = (1, 0, \ldots, 0) \) of the ball.) Then \( \Phi(x) = x \) on the ball.

At this juncture, a natural question arises: does the rigidity result for generators (Proposition 2) admit an analogous generalization to the open unit balls of either \( \mathbb{C}^n \) or a Hilbert space \( H \)? The following theorem gives an affirmative answer to this question. Moreover, we show that it is sufficient to consider the \( K \)-limit instead of the unrestricted one in the assumption of the theorem.

Let \( B \) be the open unit ball of the Hilbert space \( H \). For \( \alpha > 1 \), we denote by

\[
D_0(\tau) := \left\{ x \in B : |\langle x, \tau \rangle| < \frac{\alpha}{2} (1 - ||x||^2) \right\}
\]
To this end, we fix a point \( y \) in the Korányi approach regions at \( \tau \in \partial \mathbb{B} \) and say that a mapping \( f : \mathbb{B} \to H \) has a \( K \)-limit \( M \) at \( \tau \) if it tends to \( M \) along every curve ending at \( \tau \) and lying in a Korányi region \( D_\alpha(\tau) \).

**Theorem.** Let \( f \in \text{Hol}(\mathbb{B}, H) \) be the generator of a one-parameter continuous semigroup on \( \mathbb{B} \). If for some \( \tau \in \partial \mathbb{B} \), the \( K \)-limit

\[
\lim_{x \to \tau} \frac{f(x)}{\|x - \tau\|^3} = 0,
\]

then \( f \equiv 0 \) on \( \mathbb{B} \).

**Proof.** We prove this assertion by reduction to the one-dimensional case. Namely, we consider the restriction of the orthogonal projection of an appropriate modification of the generator \( f \) to a one-dimensional disk touching \( \mathbb{B} \) at the point \( \tau \in \partial \mathbb{B} \).

To this end, we fix a point \( y \in \mathbb{B} \) and define the mapping

\[
M_y(x) := \frac{y - P_y x - sQ_y x}{1 - \langle x, y \rangle}, \quad x \in \mathbb{B},
\]

where \( P_y \) is the orthogonal projection of \( H \) onto the subspace generated by \( y \) (i.e. \( P_0 \equiv 0 \)) and \( P_y x = \frac{\langle x, y \rangle}{\|y\|^2} y \) for \( y \neq 0 \), \( Q_y = I - P_y \) and \( s = \sqrt{1 - \|y\|^2} \). This mapping is an automorphism of \( \mathbb{B} \) satisfying \( M_y^{-1} = M_y \) (cf. p. 98 in [12] and p. 25 in [20]).

Denote by \( U_y \) a unitary operator on \( \mathbb{B} \) such that \( U_y \tau = M_y \tau \). Then the mapping

\[
m := M_y \circ U_y
\]

is a biholomorphism of \( \mathbb{B} \) onto \( \mathbb{B} \). Therefore, by Lemma 3.7.1 on p. 30 of [9], the mapping

\[
f_m(w) = [m'(w)]^{-1} f(m(w)), \quad w \in \mathbb{B},
\]

is a holomorphic generator on \( \mathbb{B} \).

Substituting

\[
[m'(w)]^{-1} = [m^{-1}(x)]_{x=m(w)}' = U_y^* M_y'(m(w))
\]

in (4), we have

\[
f_m(w) = U_y^* M_y'(m(w)) f(m(w)), \quad w \in \mathbb{B}.
\]

Now we define a holomorphic function \( g \) on the unit disk \( \Delta \) of the complex plane \( \mathbb{C} \) by

\[
g(z) := \langle f_m(z \tau), \tau \rangle, \quad z \in \Delta.
\]

This function \( g \) is a holomorphic generator on \( \Delta \). To see this, note that by the Theorem in [4], the generator \( f_m \) satisfies the inequality

\[
\text{Re}(f_m(x) - (1 - \|x\|^2)f_m(0), x) \geq 0 \quad \text{for all} \quad x \in \mathbb{B}.
\]

In particular, for \( x = z \tau \), where \( z \in \Delta \),

\[
\text{Re}((f_m(z \tau), \tau)\overline{\tau}) \geq (1 - |z|^2) \text{Re}((f_m(0), \tau)\overline{\tau});
\]

i.e.,

\[
\text{Re}(g(z)\overline{\tau}) \geq (1 - |z|^2) \text{Re}(g(0)\overline{\tau}) \quad \text{for all} \quad z \in \Delta,
\]

and, consequently, by the same theorem (see [4]), \( g \) is indeed a holomorphic generator on \( \Delta \). (We remark in passing that this also follows from the characterization of generators in terms of their \( \rho \)-monotonicity [18] [16].)
We claim that under our assumptions, \( g \equiv 0 \) on \( \Delta \). Indeed,
\[
g(z) = (U_y^* M_y'(m(z\tau))) f(m(z\tau)), \tau) = \langle M_y'(m(z\tau)) f(m(z\tau)), U_y \tau \rangle
\]
(7)
and, consequently,
\[
\frac{g(z)}{|z - 1|^3} = \frac{1}{|z - 1|^3} \frac{\langle f(m(z\tau)), [M_y'(m(z\tau))]^* U_y \tau \rangle}{\|m(z\tau) - \tau\|_3^3} \left\langle \frac{f(m(z\tau))}{\|m(z\tau) - \tau\|_3^3}, [M_y'(m(z\tau))]^* U_y \tau \right\rangle.
\]
(8)
Note that each automorphism \( h \) of \( \mathbb{B} \) is the restriction to \( \mathbb{B} \) of a holomorphic mapping defined either on the larger ball \( B(0, R) \) centered at zero of radius \( R = \frac{1}{\|h^{-1}(0)\|} \) if \( h(0) \neq 0 \) or on all of \( H \) if \( h \) fixes the origin. So, \( M_y \) and \( m \) are, in fact, holomorphic mappings defined either on the open ball \( B(0, R) \) of radius \( R = \frac{1}{\|y\|} > 1 \) if \( y \neq 0 \) or on \( H \) if \( y = 0 \). Hence the first factor on the right-hand side of equality 8 has a finite limit as \( z \to 1 \), and so has the second factor of the inner product.

Now we show that the first factor of the last inner product in 8 tends to zero as \( z \to 1 \) nontangentially in \( \Delta \).

For \( z \) close enough to 1 in the nontangential approach region
\[
\Gamma_k = \left\{ z \in \Delta : \frac{|z - 1|}{1 - |z|} < k \right\}, \quad k > 1,
\]
m\( (z\tau) \) belongs to the Korányi region \( D_\alpha(\tau) \) whenever \( \alpha > k \). Indeed, it can be shown by direct calculations that the function \( m \) satisfies the equality
\[
\frac{1 - \langle m(z\tau), \tau \rangle}{1 - \|m(z\tau)\|_3^2} = L \frac{1 - |z|}{1 - |z|}^2, \quad z \in \Delta,
\]
where
\[
L := \frac{d}{dz} \frac{m(z\tau)}{|m(z\tau)|} \bigg|_{z=1} = \frac{1 - \langle y, \tau \rangle}{1 - \langle U_y \tau, y \rangle} = \frac{1 - |y\tau|}{1 - \|y\|_2^2} > 0.
\]
Consequently, we have for \( z \in \Gamma_k \),
\[
\left| \begin{array}{c}
1 - \langle m(z\tau), \tau \rangle \\
1 - \|m(z\tau)\|_3^2
\end{array} \right| = L \left| \begin{array}{c}
1 - |z| \\
1 - |z|_2^2
\end{array} \right| \leq L k \left| \begin{array}{c}
1 - |z| \\
1 - \langle m(z\tau), \tau \rangle
\end{array} \right|.
\]
Since \( \lim_{z \to 1} \frac{1 - \langle m(z\tau), \tau \rangle}{1 - |z|} = L \), it follows that if \( z \in \Gamma_k \) is close enough to 1, then \( m(z\tau) \) is in \( D_\alpha(\tau) \) \( (\alpha > k) \). Hence, by hypothesis 8 of the theorem,
\[
\angle lim_{z \to 1} \frac{g(z)}{|z - 1|^3} = 0.
\]
Therefore equality 8 implies that \( \angle lim_{z \to 1} \frac{g(z)}{|z - 1|^3} = 0 \), and by Proposition 2, \( g \equiv 0 \) on \( \Delta \). So, by 7,
\[
\langle f(m(z\tau)), [M_y'(m(z\tau))]^* U_y \tau \rangle = 0 \quad \text{for all} \quad z \in \Delta.
\]
In particular, this equality holds for \( z = 0 \); i.e.,
\[
\langle f(y), [M_y'(y)]^* U_y \tau \rangle = 0 \quad \text{for each} \quad y \in \mathbb{B}.
\]
(9)
By direct calculations, one obtains that
\[ M'(x)h = \frac{1}{(1-\langle x, y \rangle)^2} \left[ -(1-\langle x, y \rangle)(P_y + sQ_y)h + \langle h, y \rangle(y - P_y x - sQ_y x) \right]. \]
Hence,
\[ M'(y)h = -\frac{1}{1-\|y\|^2}(P_y + sQ_y)h, \]
and equality (9) is equivalent to
\[ \langle f(y), (P_y + sQ_y)U_y \rangle = 0. \]
Substituting
\[ U_y = \frac{y - P_y y - sQ_y y}{1-\langle \tau, y \rangle} \]
in this equality, we obtain
\[ \langle f(y), y - \tau + \|y\|^2 \tau - \langle \tau, y \rangle y \rangle = 0 \quad \text{for all} \quad y \in B. \]
Let \( y = y_1 \tau + \tilde{y} \), where \( y_1 = \langle y, \tau \rangle \) and \( \langle \tilde{y}, \tau \rangle = 0 \).
Similarly, \( f(y) = f_1(y) \tau + \tilde{f}(y) \) with \( f_1(y) = \langle f(y), \tau \rangle \) and \( \tilde{f}(y), \tau \rangle = 0 \) for all \( y \in B \).
Using this notation, we have
\[ \langle f_1(y) \tau, y_1 \tau - \tau + \|y\|^2 \tau - |y_1|^2 \tau \rangle = -(\tilde{f}(y), \tilde{y} - \overline{y}_1 \tilde{y}) \]
and
\[ (1 - \overline{y}_1 - \|y\|^2) f_1(y) = (1 - y_1) \langle \tilde{f}(y), \tilde{y} \rangle. \]
Differentiating this equality with respect to \( \overline{y}_1 \), we conclude that it can hold only if \( f_1(y) = 0 \) and
\[ \langle \tilde{f}(y), \tilde{y} \rangle = 0 \quad \text{for all} \quad y \in B. \]

Now let \( \sigma \) be an arbitrary unit vector orthogonal to \( \tau \), i.e., \( \langle \sigma, \tau \rangle = 0 \). Suppose that \( \tilde{y} = y_2 \sigma + u \), where \( y_2 = \langle \tilde{y}, \sigma \rangle \) and \( \langle u, \sigma \rangle = 0 \).
Similarly, \( \tilde{f}(y) = f_2(y) \sigma + v(y) \) with \( f_2(y) = \langle \tilde{f}(y), \sigma \rangle \) and \( \langle v(y), \sigma \rangle = 0 \) for all \( y \in B \). Then by (10),
\[ f_2(y) \overline{y}_2 = -(v(y), u). \]
Differentiating this equality with respect to \( \overline{y}_2 \), we obtain \( f_2(y) = 0 \). Hence, \( f \equiv 0 \) on \( B \).

Following L. A. Harris \([13]\), we define the numerical range of each \( h \in \text{Hol}(B, H) \) which has a norm continuous extension to \( B \) by
\[ V(h) := \{ \langle h(x), x \rangle : \|x\| = 1 \}. \]
For an arbitrary holomorphic mapping \( h \in \text{Hol}(B, H) \) and for each \( s \in (0, 1) \), we define the mapping \( h_s : sB \to H \) by
\[ h_s := h(sx), \quad \|x\| < \frac{1}{s}, \]
and set
\[ L(h) := \lim_{s \to 1^-} \sup \text{Re}(V(h_s)). \]
It is known (Theorem 1 in [14]) that the mapping \( I - h \) is a generator if and only if \( L(h) \leq 1 \). So the following corollary is an immediate consequence of our theorem.

**Corollary.** Let \( h \in \text{Hol}(\mathbb{B}, H) \) with \( L(h) \leq 1 \). If for some \( \tau \in \partial \mathbb{B} \), the \( K \)-limit

\[
K \cdot \lim_{x \to \tau} \frac{h(x) - x}{\|x - \tau\|^3} = 0,
\]

then \( h \equiv I \) on \( \mathbb{B} \).

Since obviously \( L(h) \leq 1 \) for all self-mappings of \( \mathbb{B} \), this corollary properly contains Proposition 3.

**References**


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