MINIMAL GENERATORS FOR SYMMETRIC IDEALS

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Abstract. Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates $X$ over a field $K$, and let $S_X$ be the symmetric group of $X$. The group $S_X$ acts naturally on $R$, and this in turn gives $R$ the structure of a module over the group ring $R[S_X]$. A recent theorem of Aschenbrenner and Hillar states that the module $R$ is Noetherian. We address whether submodules of $R$ can have any number of minimal generators, answering this question positively.

Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates $X$ over a field $K$. Write $S_X$ (resp. $S_N$) for the symmetric group of $X$ (resp. $\{1,\ldots, N\}$) and $R[S_X]$ for its (left) group ring, which acts naturally on $R$. A symmetric ideal $I \subseteq R$ is an $R[S_X]$-submodule of $R$.

Aschenbrenner and Hillar recently proved [1] that all symmetric ideals are finitely generated over $R[S_X]$. They were motivated by finiteness questions in chemistry [3] and algebraic statistics [2]. In proving the Noetherianity of $R$, it was shown that a symmetric ideal $I$ has a special, finite set of generators called a minimal Gröbner basis. However, the more basic question of whether $I$ is always cyclic (already asked by Josef Schicho [4]) was left unanswered in [1]. Our result addresses a generalization of this important issue.

Theorem 1. For every positive integer $n$, there are symmetric ideals of $R$ generated by $n$ polynomials which cannot have fewer than $n R[S_X]$-generators.

In what follows, we work with the set $X = \{x_1, x_2, x_3, \ldots\}$, although as remarked in [1], this is not really a restriction. In this case, $S_X$ is naturally identified with $S_\infty$, the permutations of the positive integers, and $\sigma x_i = x_{\sigma i}$ for $\sigma \in S_\infty$.

Let $M$ be a finite multiset of positive integers and let $i_1, \ldots, i_k$ be the list of its distinct elements, arranged so that $m(i_1) \geq \cdots \geq m(i_k)$, where $m(i_j)$ is the multiplicity of $i_j$ in $M$. The type of $M$ is the vector $\lambda(M) = (m(i_1), m(i_2), \ldots, m(i_k))$. For instance, the multiset $M = \{1, 1, 1, 2, 3, 3\}$ has type $\lambda(M) = (3, 2, 1)$. Multisets are in bijection with monomials of $R$. Given $M$, we can construct the monomial:

$$x_M^{\lambda(M)} = \prod_{j=1}^k x_{i_j}^{m(i_j)}.$$

Conversely, given a monomial, the associated multiset is the set of indices appearing in it, along with multiplicities. The action of $S_\infty$ on monomials coincides with the

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natural action of $\mathfrak{S}_\infty$ on multisets $M_i$, and this action preserves the type of a multiset (resp. monomial). We also note the following elementary fact.

**Lemma 2.** Let $\sigma \in \mathfrak{S}_\infty$ and $f \in R$. Then there exists a positive integer $N$ and $\tau \in \mathfrak{S}_N$ such that $\tau f = \sigma f$.

Theorem 1 is a direct corollary of the following result.

**Theorem 3.** Let $G = \{g_1, \ldots, g_n\}$ be a set of monomials of degree $d$ with distinct types and fix a matrix $C = (c_{ij}) \in K^{n \times n}$ of rank $r$. Then the submodule $I = \langle f_1, \ldots, f_n \rangle_{R[\mathfrak{S}_\infty]} \subseteq R$ generated by the $n$ polynomials, $f_j = \sum_{i=1}^n c_{ij}g_i$ ($j = 1, \ldots, n$), cannot have fewer than $r$ $R[\mathfrak{S}_\infty]$-generators.

**Proof.** Suppose that $p_1, \ldots, p_k$ are generators for $I$; we prove that $k \geq r$. Since each $p_i \in I$, it follows that each is a linear combination, over $R[\mathfrak{S}_\infty]$, of monomials in $G$. Therefore, each monomial occurring in $p_i$ has degree at least $d$, and, moreover, any degree $d$ monomial in $p_i$ has the same type as one of the monomials in $G$.

Write each of the monomials in $G$ in the form $g_i = x^\lambda_M$, for multisets $M_1, \ldots, M_n$ with corresponding distinct types $\lambda_1, \ldots, \lambda_n$, and express each generator $p_i$ as

$$p_i = \sum_{i=1}^n \sum_{\lambda(M) = \lambda_i} u_{i\lambda(M)}x^\lambda_M + q_i,$$

in which $u_{i\lambda(M)} \in K$ with only finitely many of them nonzero, each monomial in $q_i$ has degree larger than $d$, and the inner sum is over multisets $M$ with type $\lambda_i$.

Since each polynomial in $\{f_1, \ldots, f_n\}$ is a finite linear combination of the $p_i$, and since only finitely many integers are indices of monomials appearing in $p_1, \ldots, p_k$, we may pick $N$ large enough so that all of these linear combinations can be expressed with coefficients in the subring $R[\mathfrak{S}_N]$ (cf. Lemma 2). Therefore, we have

$$f_j = \sum_{i=1}^k \sum_{\sigma \in \mathfrak{S}_N} s_{ij \sigma}p_i$$

for some polynomials $s_{ij \sigma} \in R$. Substituting (1) into (2) gives us that

$$f_j = \sum_{i=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M) = \lambda_i} v_{ij \sigma}u_{i\lambda(M)}x^\lambda_M + h_j,$$

in which each monomial appearing in $h_j \in R$ has degree greater than $d$ and $v_{ij \sigma}$ is the constant term of $s_{ij \sigma}$. Since each $f_j$ has degree $d$, we have that $h_j = 0$. Thus,

$$\sum_{i=1}^n c_{ij}x^\lambda_M = \sum_{i=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M) = \lambda_i} v_{ij \sigma}u_{i\lambda(M)}x^\lambda_M.$$

Next, for a fixed $i$, take the sum on each side in this last equation of the coefficients of monomials with the type $\lambda_i$. This produces the $n^2$ equations

$$c_{ij} = \sum_{i=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M) = \lambda_i} v_{ij \sigma}u_{i\lambda(M)} = \sum_{i=1}^k \left( \sum_{\lambda(M) = \lambda_i} u_{i\lambda(M)} \right) \left( \sum_{\sigma \in \mathfrak{S}_N} v_{ij \sigma} \right) = \sum_{i=1}^k U_i V_{ij},$$

in which $U_i = \sum_{\lambda(M) = \lambda_i} u_{i\lambda(M)}$ and $V_{ij} = \sum_{\sigma \in \mathfrak{S}_N} v_{ij \sigma}$. Set $U$ to be the $n \times k$ matrix $(U_{id})$ and similarly let $V$ denote the $k \times n$ matrix $(V_{ij})$. These $n^2$ equations are
represented by the equation $C = UV$, leading to the following chain of inequalities:

$$r = \text{rank}(C) = \text{rank}(UV) \leq \min\{\text{rank}(U), \text{rank}(V)\} \leq \min\{n, k\} \leq k.$$ 

Therefore, we have $k \geq r$, and this completes the proof. □

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