PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 12, December 2008, Pages 4151–4156
S 0002-9939(08)09461-6
Article electronically published on July 23, 2008

A NOTE ON FINITE ABELIAN GERBES OVER TORIC DELIGNE-MUMFORD STACKS

YUNFENG JIANG

(Communicated by Ted Chinburg)

Abstract. Any toric Deligne-Mumford stack is a μ-gerbe over the underlying toric orbifold for a finite abelian group μ. In this paper we give a sufficient condition so that certain kinds of gerbes over a toric Deligne-Mumford stack are again toric Deligne-Mumford stacks.

1. Introduction

Let $\mathbf{\Sigma} := (N, \Sigma, \beta)$ be a stacky fan of $rank(N) = d$ as defined in [4]. If there are n one-dimensional cones in the fan Σ, then modelling the construction of toric varieties [5], [6], the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$ is a quotient stack, where $Z = \mathbb{C}^n - V$, the close subvariety $V \subset \mathbb{C}^n$ is determined by the ideal J_Σ generated by $\{\prod_{\rho_i \nsubseteq \sigma} z_i : \sigma \in \Sigma\}$ and G acts on Z through the map $\alpha : G \longrightarrow (\mathbb{C}^\times)^n$ in the following exact sequence determined by the stacky fan (see [4]):

$$1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1. \tag{1.1}$$

Let $\overline{G} = Im(\alpha)$. Then $[Z/\overline{G}]$ is the underlying toric orbifold $\mathcal{X}(\mathbf{\Sigma_{red}})$. The toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is a μ-gerbe over $\mathcal{X}(\mathbf{\Sigma_{red}})$.

Let $\mathcal{X}(\mathbf{\Sigma})$ be a toric Deligne-Mumford stack associated with the stacky fan $\mathbf{\Sigma}$. Let ν be a finite abelian group, and let $\mathcal{G}$ be a ν-gerbe over $\mathcal{X}(\mathbf{\Sigma})$. We give a sufficient condition so that $\mathcal{G}$ is also a toric Deligne-Mumford stack. We have the following theorem:

Theorem 1.1. *Let $\mathcal{X}(\mathbf{\Sigma})$ be a toric Deligne-Mumford stack with stacky fan $\mathbf{\Sigma}$. Then every ν-gerbe $\mathcal{G}$ over $\mathcal{X}(\mathbf{\Sigma})$ is induced by a central extension*

$$1 \longrightarrow \nu \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1;$$

i.e., we have a Cartesian diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{B}\widetilde{G} \arrow[d] \\ \mathcal{X}(\mathbf{\Sigma}) \arrow[r] & \mathcal{B}G. \end{tikzcd}$$

Received by the editors September 11, 2006, and, in revised form, May 8, 2007, June 10, 2007, October 11, 2007, and November 6, 2007.
2000 *Mathematics Subject Classification.* Primary 14A20.
Key words and phrases. Gerbes, toric Deligne-Mumford stacks.

In general, the ν-gerbe $\mathcal{G}$ is not a toric Deligne-Mumford stack. But if the central extension is abelian, then we have:

Corollary 1.2. *If the ν-gerbe $\mathcal{G}$ is induced from an abelian central extension, then it is a toric Deligne-Mumford stack.*

This small note is organized as follows. In Section 2 we construct the new toric Deligne-Mumford stack from an abelian central extension and prove the main results. In Section 3 we give an example of a ν-gerbe over a toric Deligne-Mumford stack.

In this paper, by an *orbifold* we mean a smooth Deligne-Mumford stack with trivial stabilizers at the generic points.

2. The proof of the main results

We refer the reader to [4] for the construction and notation of toric Deligne-Mumford stacks. For the general theory of stacks, see [2].

Let $\mathbf{\Sigma} := (N, \Sigma, \beta)$ be a stacky fan. From Proposition 2.2 in [4], we have the following exact sequences:

$$0 \longrightarrow DG(\beta)^* \longrightarrow \mathbb{Z}^n \xrightarrow{\beta} N \longrightarrow Coker(\beta) \longrightarrow 0,$$

$$0 \longrightarrow N^* \longrightarrow \mathbb{Z}^n \xrightarrow{\beta^\vee} DG(\beta) \longrightarrow Coker(\beta^\vee) \longrightarrow 0,$$

where $\beta^\vee$ is the Gale dual of β. As a $\mathbb{Z}$-module, $\mathbb{C}^\times$ is divisible, so it is an injective $\mathbb{Z}$-module and hence the functor $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ is exact. We get the exact sequence:

$$\begin{aligned} 1 \longrightarrow Hom_{\mathbb{Z}}(Coker(\beta^\vee), \mathbb{C}^\times) \longrightarrow Hom_{\mathbb{Z}}(DG(\beta), \mathbb{C}^\times) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{C}^\times) \\ \longrightarrow Hom_{\mathbb{Z}}(N^*, \mathbb{C}^\times) \longrightarrow 1. \end{aligned}$$

Letting $\mu := Hom_{\mathbb{Z}}(Coker(\beta^\vee), \mathbb{C}^\times)$, we have the exact sequence (1.1). Let $\Sigma(1) = n$ be the set of one-dimensional cones in Σ and $V \subset \mathbb{C}^n$ the closed subvariety defined by the ideal generated by

$$J_\Sigma = \left\langle \prod_{\rho_i \nsubseteq \sigma} z_i : \sigma \in \Sigma \right\rangle.$$

Let $Z := \mathbb{C}^n \setminus V$. From [5], the complex codimension of V in $\mathbb{C}^n$ is at least 2. The toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$ is the quotient stack where the action of G is through the map α in (1.1).

Lemma 2.1. *If $Codim_{\mathbb{C}}(V, \mathbb{C}^n) \geq 2$, then $H^1(Z, \nu) = H^2(Z, \nu) = 0$, where ν is a finite abelian group.*

Proof. Consider the following exact sequence:

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^0_V(\mathbb{C}^n, \nu) \longrightarrow H^0(\mathbb{C}^n, \nu) \longrightarrow H^0(Z, \nu) \longrightarrow \\ \longrightarrow \quad & H^1_V(\mathbb{C}^n, \nu) \longrightarrow H^1(\mathbb{C}^n, \nu) \longrightarrow H^1(Z, \nu) \longrightarrow \\ \longrightarrow \quad & H^2_V(\mathbb{C}^n, \nu) \longrightarrow \cdots . \end{aligned}$$

Since $Codim_{\mathbb{C}}(V, \mathbb{C}^n) \geq 2$, so the real codimension is at least 4 and $H^i_V(\mathbb{C}^n, \nu) = 0$ for $i = 1, 2, 3$, so from the exact sequence and $H^i(\mathbb{C}^n, \nu) = 0$ for all $i > 0$ we prove the lemma. $\square$

2.1. **The Proof of Theorem 1.1.** Consider the following diagram:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & pt \arrow[d] \\ {[Z/G]} \arrow[r, "\pi"] & \mathcal{B}G \end{tikzcd}$$

which is Cartesian. Consider the Leray spectral sequence for the fibration π:

$$H^p(\mathcal{B}G, R^q\pi_*\nu) \Longrightarrow H^{p+q}([Z/G],\nu).$$

We compute

$$H^2([Z/G],\nu) = \bigoplus_{p+q=2} H^p(\mathcal{B}G, R^q\pi_*\nu).$$

First we have that $R^q\pi_*\nu = [H^q(Z,\nu)/G]$. There are three cases.

(1) When $p=2, q=0$, $R^0\pi_*\nu = \nu$ because Z is connected, so

$$H^p(\mathcal{B}G, R^q\pi_*\nu) = H^2(\mathcal{B}G,\nu).$$

(2) When $p=1, q=1$, $R^1\pi_*\nu = [H^1(Z,\nu)/G]$, so

$$H^p(\mathcal{B}G, R^q\pi_*\nu) = H^1(\mathcal{B}G, H^1(Z,\nu)),$$

and by Lemma 2.1, $H^1(Z,\nu)=0$, so we have $H^p(\mathcal{B}G, R^q\pi_*\nu)=0$.

(3) When $p=0, q=2$, $R^2\pi_*\nu = [H^2(Z,\nu)/G]$, so

$$H^p(\mathcal{B}G, R^q\pi_*\nu) = H^0(\mathcal{B}G, H^2(Z,\nu));$$

also from Lemma 2.1, $H^2(Z,\nu)=0$, and so we have $H^p(\mathcal{B}G, R^q\pi_*\nu)=0$.

So we get that

$$H^2([Z/G],\nu) \cong H^2(\mathcal{B}G,\nu).$$

Since for the finite abelian group ν, the ν-gerbes are classified by the second cohomology group with coefficient in the group ν, and Theorem 1.1 is proved. □

2.2. **The Proof of Corollary 1.2.** Let $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$. The ν-gerbe $\mathcal{G}$ over $[Z/G]$ is induced from a ν-gerbe $\mathcal{B}\widetilde{G}$ over $\mathcal{B}G$ in the following central extension:

$$1 \longrightarrow \nu \longrightarrow \widetilde{G} \xrightarrow{\varphi} G \longrightarrow 1,$$

where $\widetilde{G}$ is an abelian group. So the pullback gerbe over Z under the map $Z \longrightarrow [Z/G]$ is trivial. So we have

$$\mathcal{G} = \mathcal{B}\widetilde{G} \times_{\mathcal{B}G} [Z/G] = [Z/\widetilde{G}].$$

The stack $[Z/\widetilde{G}]$ is this ν-gerbe $\mathcal{G}$ over $[Z/G]$. Consider the commutative diagram:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\varphi"] \arrow[d, "\widetilde{\alpha}"'] & G \arrow[d, "\alpha"] \\ (\mathbb{C}^\times)^n \arrow[r, "\cong"] & (\mathbb{C}^\times)^n \end{tikzcd} \tag{2.1}$$

where α is the map in (1.1). So we have the following exact sequences:

$$1 \longrightarrow \nu \longrightarrow ker(\widetilde{\alpha}) \longrightarrow \mu \longrightarrow 1$$

and

$$1 \longrightarrow ker(\widetilde{\alpha}) \longrightarrow \widetilde{G} \xrightarrow{\widetilde{\alpha}} (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1,$$

where T is the torus of the simplicial toric variety $X(\Sigma)$. Since the abelian groups $\widetilde{G}$, G and $(\mathbb{C}^\times)^n$ are all locally compact topological groups, taking Pontryagin duality and the Gale dual, we have the following diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta^\vee} & DG(\beta) & \longrightarrow & Coker(\beta^\vee) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle p_\varphi} & & \downarrow & & \\
0 & \longrightarrow & \widetilde{N}^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{(\widetilde{\beta})^\vee} & DG(\widetilde{\beta}) & \longrightarrow & Coker((\widetilde{\beta})^\vee) & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & DG(\widetilde{\beta})^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\widetilde{\beta}} & \widetilde{N} & \longrightarrow & Coker(\widetilde{\beta}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & DG(\beta)^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta} & N & \longrightarrow & Coker(\beta) & \longrightarrow & 0,
\end{array}$$

where p_φ is induced by φ in (2.1) under the Pontryagin duality. Suppose $\widetilde{\beta} : \mathbb{Z}^n \longrightarrow \widetilde{N}$ is given by $\{\widetilde{b}_1, \cdots, \widetilde{b}_n\}$, then $\widetilde{\mathbf{\Sigma}} := (\widetilde{N}, \Sigma, \widetilde{\beta})$ is a new stacky fan. The toric Deligne-Mumford stack $\mathcal{X}(\widetilde{\mathbf{\Sigma}}) = [Z/\widetilde{G}]$ is the ν-gerbe $\mathcal{G}$ over $\mathcal{X}(\mathbf{\Sigma})$. $\square$

Remark 2.2. From Proposition 4.6 in [3], any Deligne-Mumford stack is a ν-gerbe over an orbifold for a finite group ν. Our results are the toric case of that general result.

In particular, given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, let $\mathbf{\Sigma_{red}} = (\overline{N}, \Sigma, \overline{\beta})$ be the *reduced* stacky fan, where $\overline{N}$ is the abelian group N modulo torsion, and $\overline{\beta} : \mathbb{Z}^n \longrightarrow \overline{N}$ is given by $\{\overline{b}_1, \cdots, \overline{b}_n\}$, which are the images of $\{b_1, \cdots, b_n\}$ under the natural projection $N \longrightarrow \overline{N}$. Then the toric orbifold $\mathcal{X}(\mathbf{\Sigma_{red}}) = [Z/\overline{G}]$. From (1.1), let $\overline{G} = Im(\alpha)$. Then we have the following exact sequences:

$$1 \longrightarrow \overline{G} \longrightarrow (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1,$$
$$1 \longrightarrow \mu \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.$$

So G is an abelian central extension of $\overline{G}$ by μ. $\mathcal{X}(\mathbf{\Sigma})$ is a μ-gerbe over the toric orbifold $\mathcal{X}(\mathbf{\Sigma_{red}})$. Any μ-gerbe over the toric orbifold coming from an abelian central extension is a toric Deligne-Mumford stack. This is a special case of the main results and is the toric case of rigidification construction in [1].

Remark 2.3. From the proof of Corollary 1.2 we see that if a ν-gerbe over $\mathcal{X}(\mathbf{\Sigma})$ comes from a gerbe over $\mathcal{B}G$ and the central extension is abelian, then we can construct a new toric Deligne-Mumford stack.

3. An example

Example 3.1. Let Σ be the complete fan of the projective line, $N = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and $\beta : \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ be given by the vectors $\{b_1 = (1,0), b_2 = (-1,1)\}$. Then $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : \mathbb{Z}^2 \longrightarrow DG(\beta) = \mathbb{Z}$ is given by the matrix [3,3]. So we get the following exact sequence:

$$1 \longrightarrow \mu_3 \longrightarrow \mathbb{C}^\times \xrightarrow{[3,3]^t} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1. \tag{3.1}$$

The toric Deligne-Mumford stack is $\mathcal{X}(\mathbf{\Sigma}) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$, where the action is given by $\lambda(x,y) = (\lambda^3 x, \lambda^3 y)$. So $\mathcal{X}(\mathbf{\Sigma})$ is the nontrivial μ_3-gerbe over $\mathbb{P}^1$ coming from the canonical line bundle over $\mathbb{P}^1$. Let $\mathcal{G} \longrightarrow \mathcal{X}(\mathbf{\Sigma})$ be a μ_2-gerbe such that it comes from the μ_2-gerbe over $\mathcal{B}\mathbb{C}^\times$ given by the central extension

$$1 \longrightarrow \mu_2 \longrightarrow \mathbb{C}^\times \xrightarrow{(\cdot)^2} \mathbb{C}^\times \longrightarrow 1. \tag{3.2}$$

From the sequences (3.1) and (3.2), we have

$$1 \longrightarrow \mu_3 \otimes \mu_2 \longrightarrow \mathbb{C}^\times \xrightarrow{[6,6]^t} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1.$$

The Pontryagin dual of $\mathbb{C}^\times \xrightarrow{[6,6]^t} (\mathbb{C}^\times)^2$ is $(\widetilde{\beta})^\vee : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$, which is given by the matrix $[6,6]$. Taking the Gale dual we have

$$\widetilde{\beta} : \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_6,$$

which is given by the vectors $\{\widetilde{b}_1 = (1,0), \widetilde{b}_2 = (-1,1)\}$. Let $\widetilde{\mathbf{\Sigma}} = (\widetilde{N}, \Sigma, \widetilde{\beta})$ be the new stacky fan. Then we have the toric Deligne-Mumford stack $\mathcal{X}(\widetilde{\mathbf{\Sigma}}) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$, where the action is given by $\lambda(x,y) = (\lambda^6 x, \lambda^6 y)$. So $\mathcal{X}(\widetilde{\mathbf{\Sigma}})$ is the canonical μ_6-gerbe over $\mathbb{P}^1$.

If the μ_2-gerbe over $\mathcal{B}\mathbb{C}^\times$ is given by the central extension

$$1 \longrightarrow \mu_2 \longrightarrow \mathbb{C}^\times \times \mu_2 \xrightarrow{\alpha} \mathbb{C}^\times \longrightarrow 1, \tag{3.3}$$

where α is given by the matrix $[1,0]$, then from (3.1) and (3.3), we have

$$1 \longrightarrow \mu_3 \otimes \mu_2 \longrightarrow \mathbb{C}^\times \times \mu_2 \xrightarrow{\varphi} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1,$$

where φ is given by the matrix $\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$. The Pontryagin dual of φ is: $(\widetilde{\beta}')^\vee : \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2$, which is given by the transpose of the above matrix. Taking the Gale dual we get

$$\widetilde{\beta}' : \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2,$$

which is given by the vectors $\{\widetilde{b}_1 = (1,0,0), \widetilde{b}_2 = (-1,1,0)\}$. So $\widetilde{\mathbf{\Sigma}}' = (\widetilde{N}', \Sigma, \widetilde{\beta}')$ is a stacky fan. The toric Deligne-Mumford stack is $\mathcal{X}(\widetilde{\mathbf{\Sigma}}') = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times \times \mu_2]$, where the action is $(\lambda_1, \lambda_2) \cdot (x,y) = (\lambda_1^3 x, \lambda_1^3 y)$. So $\mathcal{G}' = \mathcal{X}(\widetilde{\mathbf{\Sigma}}')$ is the trivial μ_2-gerbe over $\mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{X}(\widetilde{\mathbf{\Sigma}}) \ncong \mathcal{X}(\widetilde{\mathbf{\Sigma}}')$.

Acknowledgments

I would like to thank the referee for nice comments about the proof of the main results. I thank my advisor, Kai Behrend, for encouragement and Hsian-Hua Tseng for valuable discussions.

References

1. D. Abramovich, A. Corti and A. Vistoli, Twisted bundles and admissible covers, *Commun. Algebra* 31 (2003), no. 8, 3547-3618. MR2007376 (2005b:14049)
2. K. Behrend, D. Edidin, B. Fantechi, W. Fulton, L. Göttsche, and A. Kresch, *Introduction to stacks*, in preparation.
3. K. Behrend and B. Noohi, Uniformization of Deligne-Mumford curves, *J. Reine Angew. Math.* 599 (2006), 111–153. MR2279100 (2007k:14017)
4. L. Borisov, L. Chen and G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, *J. Amer. Math. Soc.* 18 (2005), no. 1, 193-215. MR2114820 (2006a:14091)

5. D. Cox, The homogeneous coordinate ring of a toric variety, *J. of Algebraic Geometry*, 4 (1995), 17-50. MR1299003 (95i:14046)
6. W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies 131, Princeton University Press, Princeton, NJ, 1993. MR1234037 (94g:14028)

Department of Mathematics, University of Utah, 155 South 1400 East JWB233, Salt Lake City, Utah 84112

E-mail address: `jiangyf@math.utah.edu`