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IMPROVEMENTS OF LOWER BOUNDS FOR THE LEAST COMMON MULTIPLE OF FINITE ARITHMETIC PROGRESSIONS

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Abstract. Let u_0, r, α and n be positive integers such that $(u_0, r) = 1$. Let $u_k = u_0 + kr$ for $1 \leq k \leq n$. We prove that $L_n := \mathrm{lcm}\{u_0, u_1, \cdots, u_n\} \geq u_0 r^\alpha (r+1)^n$ if $n > r^\alpha$. This improves the lower bound of L_n obtained previously by Farhi, Hong and Feng.

1. Introduction

Arithmetic progression is an important topic in number theory. The renowned Dirichlet theorem tells us that any arithmetic progression with the first term and the common difference coprime contains infinitely many primes; see, for example, [1], [7] or [10]. Recently, Green and Tao [5] proved a significant theorem saying that the set of primes contains arbitrarily long arithmetic progressions. On the other hand, Bachman and Kessler [2] and Myerson and Sander [11] investigated the divisibility properties of $\mathrm{lcm}\{1, \cdots, n\}$ (i.e., the least common multiple of all elements in the set $\{1, ..., n\}$) while Hong and Loewy [9] studied asymptotic behavior of eigenvalues of Smith matrices defined on arithmetic progressions.

The bounds for the least common multiple of finite arithmetic progressions have received lots of attention. Hanson [6] and Nair [12] derived the upper bound and lower bound of $\mathrm{lcm}\{1, \cdots, n\}$ respectively. Farhi [3], [4] obtained non-trivial lower bounds for the least common multiple of some finite arithmetic progressions. In what follows we always let u_0, r and n be positive integers such that u_0 and r are coprime. Let $u_k = u_0 + kr$ for $1 \leq k \leq n$ and define $L_n := \mathrm{lcm}\{u_0, u_1, \cdots, u_n\}$. Farhi [3], [4] conjectured that for $n \geq 1$, we have $L_n \geq u_0(r+1)^n$. Note that this conjecture extends Nair's result [12] from the set $\{1, \cdots, n\}$ to a general arithmetic progression with n terms. Hong and Feng [8] confirmed Farhi's conjecture. Meanwhile, Hong and Feng [8] obtained an improved lower bound under certain conditions. In fact, they proved that $L_n \geq u_0 r(r+1)^n$ if $n > r$.

In this paper, our main interest is the lower bounds for the least common multiple of finite arithmetic progressions. We improve the above lower bound for L_n if n is large enough. We have the following result.

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Theorem 1.1. *Let $\alpha \geq 1$ be an integer. If $n > r^\alpha$, then $L_n \geq u_0 r^\alpha (r+1)^n$.*

The proof of Theorem 1.1 will be given in the third section. Throughout this paper, we let $[x]$ denote the integer part of a given real number x, as usual. We say that a real number x is a multiple of a non-zero real number y if the quotient $\frac{x}{y}$ is an integer.

2. Preliminaries

In the present section, we state some notations and known results which will be needed in the proof of Theorem 1.1. The following lemma was first stated in [3]. For an alternative proof, see [8].

Lemma 2.1 ([3], [4], [8])**.** *For any positive integer n, L_n is a multiple of $\frac{u_0 u_1 \cdots u_n}{n!}$.*

For an integer $0 \leq k \leq n$, define $C_{n,k} := \frac{u_k \cdots u_n}{(n-k)!}$ and $L_{n,k} := \operatorname{lcm}\{u_k, \cdots, u_n\}$. Then $L_n = L_{n,0}$. By Lemma 2.1 we have

$$L_{n,k} = A_{n,k} \frac{u_k u_{k+1} \cdots u_n}{(n-k)!} = A_{n,k} C_{n,k}, \tag{2.1}$$

where $A_{n,k} \geq 1$ is an integer. Define

$$k_n := \max\left\{0, \left[\frac{n-u_0}{r+1}\right] + 1\right\}.$$

It is proved in [8] that for any $0 \leq k \leq n$, $C_{n,k} \leq C_{n,k_n}$. Evidently L_n is a multiple of $L_{n,k}$ for all $0 \leq k \leq n$. Therefore for all $0 \leq k \leq n$, by (2.1) we have $L_n \geq L_{n,k} \geq C_{n,k}$. In particular, $L_n \geq C_{n,k_n}$. Thus we have the following result which also implies the truth of Farhi's conjecture.

Lemma 2.2 ([8])**.** *Let $C_{n,k}$ and k_n be defined as above. Then $C_{n,k_n} \geq u_0(r+1)^n$. Consequently, we have $L_n \geq u_0(r+1)^n$.*

3. Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By [8], we know that Theorem 1.1 is true if $\alpha = 1$ or $r = 1$. It remains to consider the case where $\alpha \geq 2$ and $r \geq 2$.

In the following we let $\alpha \geq 2$ and $r \geq 2$. We first treat the special case where $\alpha = r = 2, u_0 = 1$ and $n = 5$. For this special case, we have

$$L_n = \operatorname{lcm}\{1, 3, 5, 7, 9, 11\} = 3465 \quad \text{and} \quad u_0 r^\alpha (r+1)^n = 972.$$

Thus we have $L_n \geq u_0 r^\alpha (r+1)^n$. That is, Theorem 1.1 holds for the special case.

Next, we deal with the remaining cases except for the above special case. We claim that $\alpha r \leq n - k_n$ except for the special case. Notice that $r^\alpha \geq \alpha r$ since $\alpha \geq 2$ and $r \geq 2$. In the following we verify the claim.

First let $n < u_0$. Then $k_n = 0$. Therefore $n - k_n = n > r^\alpha \geq \alpha r$ and the claim is proved for this case. Second let $n = u_0$. Thus $k_n = 1$ and $n - k_n = n - 1 \geq r^\alpha \geq \alpha r$, so the claim is proved for this case too. Now let $n > u_0$. Since $n > r^\alpha$, we have $n > \alpha r$. We divide the proof into the following three cases.

Case 1. $\alpha r < u_0 < n$. Then $k_n = \left[\frac{n-u_0}{r+1}\right] + 1$. So we have

$$\alpha r + k_n \le \alpha r + \frac{n-u_0}{r+1} + 1 \le u_0 + \frac{n-u_0}{r+1} = \frac{ru_0+n}{r+1} < n.$$

Thus $\alpha r < n - k_n$ as claimed.

Case 2. $u_0 \le \alpha r < 2\alpha r \le n$. Then $\frac{n}{2} \ge \alpha r$. Since $\alpha \ge 2$ and $r \ge 2$, we have $n > 4$. Hence

$$k_n \le \frac{n-u_0}{r+1} + 1 \le \frac{n-1}{r+1} + 1 \le \frac{n-1}{3} + 1 < \frac{n}{2}.$$

It follows that $n - k_n > n - \frac{n}{2} = \frac{n}{2} \ge \alpha r$ as claimed.

Case 3. $u_0 \le \alpha r < n < 2\alpha r$. Note that $r^\alpha < n$. Then we have $r^\alpha < 2\alpha r$. But $\alpha \ge 2$ and $r \ge 2$, so we get that $r = \alpha = 2$, or $r = 2$ and $\alpha = 3$, or $r = 3$ and $\alpha = 2$. Consider the following three subcases.

Subcase 3.1. $r = \alpha = 2$. Then we have $u_0 \le 4 < n < 8$. Hence $n = 5, 6$ or 7 and $u_0 = 1$ or 3 since u_0 and r are coprime. We need to exclude the exceptional case that $u_0 = 1$ and $n = 5$ for which we have $n - k_n = 3 < 4 = \alpha r$. So we must have $u_0 = 3$ if $n = 5$. For $u_0 = 3$ and $n = 5$, we have

$$n - k_n = 5 - \left[\frac{5-3}{3}\right] - 1 = 4 = \alpha r$$

as claimed. For $n = 6$ and 7, we have

$$n - k_n \ge 6 - \left[\frac{6-1}{3}\right] - 1 = 4 = \alpha r$$

and

$$n - k_n \ge 7 - \left[\frac{7-1}{3}\right] - 1 = 4 = \alpha r$$

respectively. Therefore the claim is proved for this subcase.

Subcase 3.2. $r = 2$ and $\alpha = 3$. Then $u_0 < 8 < n < 12$. It follows that

$$n - k_n \ge n - \frac{n-u_0}{r+1} - 1 = \frac{2n + (u_0 - 3)}{3} \ge \frac{2}{3}(n-1) \ge \frac{16}{3} > 5.$$

So we have found that $n - k_n \ge 6 = \alpha r$. The claim is proved for this subcase.

Subcase 3.3. $r = 3$ and $\alpha = 2$. Then $u_0 < 9 < n < 12$. We deduce that

$$k_n \le \frac{n-u_0}{r+1} + 1 \le \frac{n-1}{4} + 1 = \frac{n+3}{4}.$$

It follows that

$$n - k_n \ge n - \frac{n+3}{4} = \frac{3}{4}(n-1) \ge \frac{27}{4} > 6 = \alpha r$$

as claimed. The claim is proved for this subcase.

From the claim we deduce immediately that $r^\alpha | (n-k_n)!$. Let $(n-k_n)! = r^\alpha B_n$ with $B_n \ge 1$ an integer. Since $(r, u_0) = 1$, we have $(r^\alpha, u_{k_n} u_{k_n+1} \cdots u_n) = 1$. Then letting $k = k_n$ in (2.1) gives us

$$r^\alpha B_n L_{n,k_n} = A_{n,k_n} \cdot u_{k_n} u_{k_n+1} \cdots u_n. \tag{3.1}$$

It then follows from (3.1) that $r^\alpha | A_{n,k_n}$. Hence $A_{n,k_n} \ge r^\alpha$. Thus by (2.1) and Lemma 2.2 we get

$$L_{n,k_n} \ge r^\alpha C_{n,k_n} \ge u_0 r^\alpha (r+1)^n.$$

Therefore the conclusion follows immediately and the proof of Theorem 1.1 is complete. □

Remark 3.1. We must point out that the restrictive condition $n > r^\alpha$ is necessary. Otherwise, Theorem 1.1 may not be true. For instance, let $u_0 = r = 2, \alpha = 3$ and $n = r^\alpha = 8$. Then $L_n = \mathrm{lcm}\{2, 4, 6, 8, 10, 12, 14, 16, 18\} = 5040$. On the other hand, $u_0 r^\alpha (r+1)^n = 104976$. Therefore $L_n < u_0 r^\alpha (r+1)^n$.

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