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A NOTE ON THE $2k$-TH POWER MEAN OF CHARACTER SUMS OVER THE QUARTER INTERVAL

ZHEFENG XU

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Abstract. We obtain an asymptotic formula for the $2k$-th power mean of odd primitive character sums over the interval $[1, \frac{q}{4})$.

1. Introduction

Let $q \geq 3$ be an integer, χ be a Dirichlet character modulo q. The various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a)$$

were investigated by many authors; see [1], [2], [3], [4]. D.A. Burgess [5] obtained the mean value estimate

$$\sideset{}{^*}\sum_{\chi \bmod q} \sum_{n=1}^{p} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 \leq 8\tau^7(q) q^2 h^2,$$

where $\sideset{}{^*}\sum_{\chi \bmod q}$ denotes the sum over all primitive characters modulo q and $\tau(n)$ is the divisor function.

The author and Zhang [6] studied the $2k$-th power mean of the even primitive character sums over the quarter interval $[1, \frac{q}{4})$ by using mean value theorems of Dirichlet L-functions, and obtained the asymptotic formula as follows:

$$\sideset{}{^*}\sum_{\chi(-1)=1} \left| \sum_{a<\frac{q}{4}} \chi(a) \right|^{2k}$$
$$= \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O\left(q^{k+\epsilon}\right),$$

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where $\sideset{}{^*}\sum_{\chi(-1)=1}$ denotes the sum over all primitive characters modulo q such that $\chi(-1)=1$, ϵ is any fixed positive number, $J(q)$ denotes the number of all primitive characters modulo q, $\prod_{p|q}$ denotes the product over all prime divisors p of q, and $C_m^n = \frac{m!}{n!(m-n)!}$. Because Lemma 3 of reference [8] is not correct, which will be explained in our Lemma 2.6, this result is also incorrect. The last factor in the main term $\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)$ should be $A(0,k,p,2)$, as defined in our theorem below. However, this result is correct in the case $k=2$.

In this paper we study odd primitive character sums over the interval $[1,\frac{q}{4})$ by transforming them to L-functions, and we obtain an asymptotic formula for the $2k$-th power mean using the same method as in [6].

Theorem 1.1. *Let $q \geq 5$ be an odd integer. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi(-1)=-1}\left|\sum_{a<\frac{q}{4}}\chi(a)\right|^{2k}$$
$$= C(k)q^kJ(q)\zeta^{2k-1}(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}A(0,k,p,2)+O_k(q^{k+\epsilon}),$$

where

$$C(k)=\frac{1}{\pi^{2k}}\sum_{i=0}^{k}C_k^i(-2)^{k-i}\sum_{j=0}^{i}6^j\sum_{s=0}^{k-i}\sum_{t=0}^{i-j}\frac{A(|3i+4s-j-2t-2k|,k,2,2)}{2^{|3i+4s-j-2t-2k|+2k+1}},$$

$$A(m,k,p,s)=\sum_{i=0}^{2k-2}\frac{1}{p^{is}}\sum_{j=0}^{i}(-1)^jC_{2k-1}^jC_{k+m+i-j-1}^{m+i-j}C_{k+i-j-1}^{i-j}$$

and $\zeta(s)$ denotes the Riemann zeta function.

Taking $k=2$ in our theorem and noting that

$$\prod_{p|q}\left(1-\frac{1}{p^2}\right)^3\prod_{p\nmid 2q}\left(1+\frac{1}{p^2}\right)$$
$$=\prod_{p}\left(1+\frac{1}{p^2}\right)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^3\prod_{p|2q}\frac{1}{1+\frac{1}{p^2}}$$
$$=\frac{4}{5}\frac{\zeta(2)}{\zeta(4)}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)},$$

$\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$, we get

Corollary 1.2. *Let $q \geq 5$ be an odd integer. Then*

$$\sideset{}{^*}\sum_{\chi(-1)=-1}\left|\sum_{a<\frac{q}{4}}\chi(a)\right|^4=\frac{9}{128}q^2J(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)}+O\left(q^{2+\epsilon}\right).$$

Combining this with Corollary 2 of [6], we have the following

Corollary 1.3. *Let $q \geq 5$ be an odd integer. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a<\frac{q}{4}} \chi(a) \right|^4 = \frac{21}{256} q^2 J(q) \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\epsilon}\right).$$

The number $n > 1$ is called square-full if in the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ we have $\alpha_i \geq 2$ for all $i = 1, 2, \cdots, r$. Noting that $J(q) = \frac{\phi^2(q)}{q}$ if q is a square-full number, we have

Corollary 1.4. *Let $q \geq 5$ be a square-full number with $2 \nmid q$. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a<\frac{q}{4}} \chi(a) \right|^4 = \frac{21}{256} q \phi^2(q) \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\epsilon}\right).$$

Corollary 1.5. *Let $p \geq 5$ be a prime. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} \left| \sum_{a<\frac{p}{4}} \chi(a) \right|^4 = \frac{21}{256} p^3 + O\left(p^{2+\epsilon}\right).$$

Remark 1.6. From our theorem and the theorem in [6], we find that odd and even primitive characters have a very different contribution to the higher moment of character sums over the quarter interval. For the case $k = 2$, from Corollaries 1.2 and 1.3, we see that the contribution of odd primitive characters is six times that of the even ones, although (see Remark 2.4 in the next section)

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = 0$$

for some odd primitive characters, while

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) \neq 0$$

for all even primitive characters.

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 2.1. *Let χ be a primitive character modulo m with $\chi(-1) = -1$. Then we have*

$$\frac{1}{m} \sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi} \tau(\chi) L(1, \bar{\chi}),$$

where $\tau(\chi) = \sum_{a=1}^{m} \chi(a) e\left(\frac{a}{q}\right)$ is the Gauss sum, $e(y) = e^{2\pi i y}$, and $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. This can be easily deduced from Theorem 12.11 and Theorem 12.20 of [7]. □

Lemma 2.2. *Let $q \geq 3$ be an odd number. For any nonprincipal character $\chi \bmod q$, we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)} \sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

Proof. From the properties of Dirichlet characters, we have

$$\begin{aligned}\sum_{a=1}^{q} 2a\chi(2a) &= \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=\frac{q+1}{2}}^{q} 2a\chi(2a) \\ &= \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=1}^{\frac{q+1}{2}} (2a-1)\chi(q+2a-1) + q\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1) \\ &= \sum_{a=1}^{q} a\chi(a) + q\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1).\end{aligned}$$

Noting that

$$\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1) + \sum_{a=1}^{\frac{q-1}{2}} \chi(2a) = \sum_{a=1}^{q} \chi(a) = 0,$$

we can write

$$(1-2\chi(2)) \sum_{a=1}^{q} a\chi(a) = \sum_{a=1}^{q} a\chi(a) - \sum_{a=1}^{q} 2a\chi(2a) = q\sum_{a=1}^{\frac{q-1}{2}} \chi(2a) = \chi(2)q\sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

That is,

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)} \sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

This proves Lemma 2.2. □

Lemma 2.3. *Let q be an odd number and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = -1$. Then we have*

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = \frac{2+\bar{\chi}(2)-\bar{\chi}(4)}{2i\pi} \tau(\chi) L(1, \bar{\chi}).$$

Remark 2.4. For even primitive characters χ, by Lemma 2 of [6],

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = -\frac{i\bar{\chi}(4)}{2\pi} \tau(\chi\chi_4) L(1, \bar{\chi}\chi_4),$$

where χ_4 is the primitive character modulo 4. Since $\tau(\chi\chi_4)L(1, \bar{\chi}\chi_4) \neq 0$, we have

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) \neq 0$$

for all even primitive characters. For odd primitive characters, from the identity in Lemma 2.3, we know that

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = 0$$

if and only if $\chi(2) = -1$. For example, if p is a prime, then the real character sum

$$\sum_{a=1}^{\left[\frac{p-1}{4}\right]} \left(\frac{a}{p}\right) = 0$$

if and only if $p \equiv 3 \pmod 8$. In this case, the number of quadratic residues and quadratic non-residues are equal in $\left[1, \frac{p-1}{4}\right]$.

Proof of Lemma 2.3. We separate q into two cases: $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$. First, we suppose $q \equiv 1 \pmod 4$. From the properties of the Dirichlet character modulo q, we can write

$$\begin{aligned}
4\chi(4)\sum_{a=1}^{q-1} a\chi(a) &= \sum_{a=1}^{\frac{q-1}{4}} 4a\chi(4a) + \sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} 4a\chi(4a) + \sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} 4a\chi(4a) + \sum_{a=\frac{3q+1}{4}}^{q-1} 4a\chi(4a) \\
&= \sum_{a=1}^{\frac{q-1}{4}} 4a\chi(4a) + \sum_{a=1}^{\frac{q-1}{4}} (4a+q-1)\chi(4a-1) \\
&\quad + \sum_{a=1}^{\frac{q-1}{4}} (4a+2q-2)\chi(4a-2) + \sum_{a=1}^{\frac{q-1}{4}} (4a+3q-3)\chi(4a-3) \\
&= \sum_{a=1}^{q-1} a\chi(a) + \chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a-\bar{4}) \\
&\quad + 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a-2\cdot\bar{4}) + 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a-3\cdot\bar{4}). \qquad (2.1)
\end{aligned}$$

Note that $\bar{4} \equiv \frac{3q+1}{4} \pmod q$ if $q \equiv 1 \pmod 4$. So from (2.1), we have

$$\begin{aligned}
4\chi(4)\sum_{a=1}^{q-1} a\chi(a) &= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} \chi(a) \\
&\quad - 2\chi(4)q\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) - 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) - 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=1}^{\frac{q-1}{2}} \chi(a) - 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a), \qquad (2.2)
\end{aligned}$$

where we used the fact $\chi(-1) = -1$ and

$$\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) = -\sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} \chi(a).$$

Now, from (2.2) and Lemma 2.2, we get

$$4\chi(4)\sum_{a=1}^{q-1} a\chi(a) = \sum_{a=1}^{q-1} a\chi(a) - (\chi(2) - 2\chi(4))\sum_{a=1}^{q-1} a\chi(a) - 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a).$$

That is,

$$\sum_{a=1}^{\frac{q-1}{4}} \chi(a) = \frac{\bar{\chi}(4) - \bar{\chi}(2) - 2}{2q}\sum_{a=1}^{q-1} a\chi(a) = \frac{\bar{\chi}(4) - \bar{\chi}(2) - 2}{2q}\sum_{a=1}^{q} a\chi(a).$$

Then from Lemma 2.1, we have

$$\sum_{a=1}^{\frac{q-1}{4}} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}). \tag{2.3}$$

This proves Lemma 2.3 in the case of $q \equiv 1 \pmod 4$. By the same method, we can also prove

$$\sum_{a=1}^{\frac{q-3}{4}} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}) \tag{2.4}$$

if $q \equiv 3 \pmod 4$. Combining (2.3) and (2.4), we can immediately get

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}).$$

This completes the proof of Lemma 2.3. □

Lemma 2.5. *Let $f(x)$ be a polynomial of degree k with leading coefficient a_0, and define the difference operator $\triangle$ by $(\triangle f)(x) = f(x) - f(x-1)$. Then we have*

$$\triangle^k f(x) = k!a_0, \qquad \triangle^l f(x) = 0 \qquad (l \geq k+1).$$

Proof. This can be easily deduced by the definition of the difference operator and mathematical induction. □

Lemma 2.6. *Let q be an integer with $q > 2$ and let $\tau_k(n)$ denote the k-th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \cdots n_k = n$ in positive integers $n_1, n_2, \cdots, n_k$). Then for any complex variable s with $Re(s) > 1$, we have the identity*

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^s} = \zeta^{2k-1}(s)\prod_{p|q}\left(1 - \frac{1}{p^s}\right)^{2k-1}\prod_{p\nmid q} A(0, k, p, s).$$

Proof. This is Lemma 3 of [8]. But the result in [8] is not correct. From the Euler product formula and noting that (see formula 6.4.12 of [9])

$$\tau_k(n) = \prod_{j=1}^{r} C_{k+\alpha_j-1}^{\alpha_j}$$

if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the factorization of n into prime powers, we have

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^s} = \prod_{p \nmid q} S(k,p,s),$$

where

$$S(k,p,s) = 1 + \frac{(C_k^1)^2}{p^s} + \frac{(C_{k+1}^2)^2}{p^s} + \cdots + \frac{(C_{k+i-1}^i)^2}{p^s} + \cdots .$$

Now we calculate $S(k,p,s)$. For convenience, let $f(i) = (C_{k+i-1}^i)^2$. From the definition of the difference operator $\triangle$ in Lemma 2.5, we have

$$S(k,p,s)\left(1 - \frac{1}{p^s}\right) = 1 + \frac{\triangle f(1)}{p^s} + \frac{\triangle f(2)}{p^{2s}} + \cdots + \frac{\triangle f(i)}{p^{is}} + \cdots ,$$

$$S(k,p,s)\left(1 - \frac{1}{p^s}\right)^2 = 1 + \frac{\triangle f(1) - f(0)}{p^s} + \frac{\triangle^2 f(2)}{p^{2s}} + \cdots + \frac{\triangle^2 f(i)}{p^{is}} + \cdots .$$

So, from the binomial expansion theorem and comparing the coefficients of p^{is} for $1 \le i \le 2k-2$, we have

$$S(k,p,s)\left(1 - \frac{1}{p^s}\right)^{2k-1} = S'(k,p,s) + \frac{\triangle^{2k-1} f(2k-1)}{p^{(2k-1)s}} + \frac{\triangle^{2k} f(2k)}{p^{2ks}} + \cdots ;$$

here

$$S'(k,p,s) = \sum_{i=0}^{2k-2} \frac{1}{p^{is}} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j f(i-j).$$

Noting that $f(i)$ is a polynomial in i of degree $2k-2$, so from Lemma 2.5 we know that $\triangle^{2k-1} f(2k-1) = \triangle^{2k} f(2k) = \cdots = 0$. This yields

$$\begin{aligned} S(k,p,s) &= \left(1 - \frac{1}{p^s}\right)^{1-2k} \sum_{i=0}^{2k-2} \frac{1}{p^{is}} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j (C_{k+i-j-1}^{i-j})^2 \\ &= \left(1 - \frac{1}{p^s}\right)^{1-2k} A(0,k,p,s). \end{aligned}$$

Hence, Lemma 2.6 follows immediately from the fact $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$. $\square$

Lemma 2.7. *Let q be any odd integer with $q > 2$ and $m \ge 0$ a fixed integer. Then for any complex variable s with $Re(s) > 1$, we have the identity*

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_k(2^m n)\tau_k(n)}{n^s} = A(m,k,2,s)\zeta^{2k-1}(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right)^{2k-1} \prod_{p \nmid 2q} A(0,k,p,s).$$

This lemma is a generalization of Lemma 2.6, which is just the case $m = 0$.

Proof. Noting that $\tau_k(n)$ is a multiplicative function, we can write

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_k(2^m n)\tau_k(n)}{n^s} &= \tau_k(2^m)\sum_{\substack{n=1\\(n,q)=1\\2\nmid n}}^{\infty}\frac{\tau_k^2(n)}{n^s}+\sum_{\substack{n=1\\(n,q)=1\\2\mid n}}^{\infty}\frac{\tau_k(2^m n)\tau_k(n)}{n^s}\\ &= \tau_k(2^m)\sum_{\substack{n=1\\(n,q)=1\\2\nmid n}}^{\infty}\frac{\tau_k^2(n)}{n^s}+\sum_{\substack{r=1\\(r,q)=1\\2\nmid r}}^{\infty}\sum_{j=1}^{\infty}\frac{\tau_k(2^{m+j})\tau_k(2^j)\tau_k^2(r)}{(r\cdot 2^j)^s}\\ &= \left(\sum_{j=0}^{\infty}\frac{\tau_k(2^{m+j})\tau_k(2^j)}{2^{sj}}\right)\sum_{\substack{n=1\\(n,q)=1\\2\nmid n}}^{\infty}\frac{\tau_k^2(n)}{n^s}.\end{aligned}$$

For $n>1$, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. Then we have

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_k(2^m n)\tau_k(n)}{n^s}=\left(\sum_{j=0}^{\infty}\frac{C_{k+m+j-1}^{m+j}C_{k+j-1}^{j}}{2^{sj}}\right)\sum_{\substack{n=1\\(n,q)=1\\2\nmid n}}^{\infty}\frac{\tau_k^2(n)}{n^s}. \tag{2.5}$$

Let

$$S=\sum_{j=0}^{\infty}\frac{C_{k+m+j-1}^{m+j}C_{k+j-1}^{j}}{2^{sj}}.$$

Now using the same method as calculating $S(k,p,s)$ in Lemma 2.6, we can also get

$$\begin{aligned}S &= \left(1-\frac{1}{2^s}\right)^{1-2k}\sum_{i=0}^{2k-2}\frac{1}{2^{is}}\sum_{j=0}^{i}(-1)^j C_{2k-1}^{j}C_{k+m+i-j-1}^{m+i-j}C_{k+i-j-1}^{i-j}\\ &= \left(1-\frac{1}{2^s}\right)^{1-2k}A(m,k,2,s). \qquad (2.6)\end{aligned}$$

Noting that

$$\sum_{\substack{n=1\\(n,q)=1\\2\nmid n}}^{\infty}\frac{\tau_k^2(n)}{n^s}=\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau_k^2(n)}{n^s}$$

if q is odd, we can easily get Lemma 2.7 from (2.5), (2.6) and Lemma 2.6. □

Lemma 2.8. *Let q be any odd integer with $q>2$, χ be the Dirichlet character modulo q and $m\geq 1$ be a fixed integer. Then we have the following asymptotic formulas:*

$$\sum_{\chi(-1)=-1}^{*}\chi(2^m)|L(1,\bar{\chi})|^{2k}=\frac{J(q)}{2^{m+1}}\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_k(2^m n)\tau_k(n)}{n^2}+O(q^{\epsilon})$$

and

$$\sideset{}{^*}\sum_{\chi(-1)=-1} \bar{\chi}(2^m)|L(1,\bar{\chi})|^{2k} = \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\tau_k(2^m n)\tau_k(n)}{n^2} + O(q^\epsilon).$$

Proof. By using the same method as in proving Lemma 5 of [6], we can get these formulas. □

3. Proof of the theorem

In this section we will complete the proof of the theorem. It is well known that $|\tau(\chi)| = \sqrt{q}$ if χ is a primitive character. So from Lemma 2.3, we can write

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi(-1)=-1} \left|\sum_{a<\frac{q}{4}} \chi(a)\right|^{2k} \\
&= \frac{q^k}{(2\pi)^{2k}} \sideset{}{^*}\sum_{\chi(-1)=-1} |2+\bar{\chi}(2)-\bar{\chi}(4)|^{2k}|L(1,\bar{\chi})|^{2k} \\
&= \frac{q^k}{(2\pi)^{2k}} \sideset{}{^*}\sum_{\chi(-1)=-1} [6+\bar{\chi}(2)+\chi(2)-2(\bar{\chi}(4)+\chi(4))]^k|L(1,\bar{\chi})|^{2k} \\
&= \frac{q^k}{(2\pi)^{2k}} \sideset{}{^*}\sum_{\chi(-1)=-1} \sum_{i=0}^{k} C_k^i(6+\bar{\chi}(2)+\chi(2))^i(-2)^{k-i}(\bar{\chi}(4)+\chi(4))^{k-i}|L(1,\bar{\chi})|^{2k} \\
&= \frac{q^k}{(2\pi)^{2k}} \sum_{i=0}^{k} C_k^i(-2)^{k-i} \sum_{j=0}^{i} 6^j \sum_{s=0}^{k-i}\sum_{t=0}^{i-j} \sideset{}{^*}\sum_{\chi(-1)=-1} \chi^{3i+4s-j-2t-2k}(2)|L(1,\bar{\chi})|^{2k},
\end{aligned}$$

where χ^{-1} has the same meaning as $\bar{\chi}$, the conjugate of the character χ. Then from Lemma 2.8 and Lemma 2.7, we easily get

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi(-1)=-1} \left|\sum_{a<\frac{q}{4}} \chi(a)\right|^{2k} \\
&= C(k)q^k J(q)\zeta^{2k-1}(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid 2q} A(0,k,p,2) + O_k(q^{k+\epsilon}),
\end{aligned}$$

where

$$C(k) = \frac{1}{\pi^{2k}} \sum_{i=0}^{k} C_k^i(-2)^{k-i} \sum_{j=0}^{i} 6^j \sum_{s=0}^{k-i}\sum_{t=0}^{i-j} \frac{A(|3i+4s-j-2t-2k|,k,2,2)}{2^{|3i+4s-j-2t-2k|+2k+1}},$$

$$A(m,k,p,s) = \sum_{i=0}^{2k-2} \frac{1}{p^{is}} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j C_{k+m+i-j-1}^{m+i-j} C_{k+i-j-1}^{i-j}.$$

This completes the proof of the theorem.

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Department of Mathematics, Northwest University, Xi'an, Shaanxi, People's Republic of China

E-mail address: zhefengxu@hotmail.com; zfxu@nwu.edu.cn