A VERSION OF FABRY’S THEOREM FOR POWER SERIES WITH REGULARLY VARYING COEFFICIENTS

ALEXANDRE EREMENKO

(Communicated by Mario Bonk)

Abstract. For real power series whose non-zero coefficients satisfy $|a_m|^{1/m} \to 1$, we prove a stronger version of Fabry’s theorem relating the frequency of sign changes in the coefficients and analytic continuation of the sum of the power series.

For a set $\Lambda$ of non-negative integers, we consider the counting function $n(x, \Lambda) = \#\Lambda \cap [0, x]$. We say that $\Lambda$ is measurable if the limit
$$\lim_{x \to +\infty} \frac{n(x, \Lambda)}{x}$$
exists, and we call this limit the density of $\Lambda$.

Let $S = \{a_m\}$ be a sequence of real numbers. We say that a sign change occurs at the place $m$ if there exists $k < m$ such that $a_m a_k < 0$ while $a_j = 0$ for $k < j < m$.

**Theorem A.** Let $\Delta$ be a number in $[0, 1]$. The following two properties of a set $\Lambda$ of positive integers are equivalent:

(i) Every power series
$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$
of radius of convergence 1, with real coefficients and such that the changes of sign of $\{a_m\}$ occur only for $m \in \Lambda$, has a singularity on the arc
$$I_\Delta = \{ e^{i\theta} : |\theta| \leq \pi \Delta \}.$$

(ii) For every $\Delta' > \Delta$ there exists a measurable set $\Lambda' \subset \mathbb{N}$ of density $\Delta'$ such that $\Lambda \subset \Lambda'$.

Implication (ii) $\longrightarrow$ (i) is a consequence of Fabry’s General Theorem [5, 3], as restated by Pólya. For the implication (i) $\longrightarrow$ (ii) see [9]. Fabry’s General Theorem takes into account not only the sign changes of coefficients but also the absolute values of coefficients. It has a rather complicated statement, and the sufficient condition of the existence of a singularity given by this theorem is not the best.

Received by the editors November 19, 2007.
2000 Mathematics Subject Classification. Primary 30B10, 30B40.
The author was supported by NSF grant DMS-0555279.
possible. The best possible condition in Fabry’s General Theorem is unknown; see, for example, the discussion in [4].

Alan Sokal (private communication) asked what happens if we assume that the power series (1) satisfies the additional regularity condition:

\[
\lim_{m \in P, m \to \infty} |a_m|^{1/m} = 1,
\]

where \( P = \{ m : a_m \neq 0 \} \). This condition holds for most interesting generating functions. The answer is somewhat surprising:

**Theorem 1.** Let \( \Delta \) be a number in \([0, 1]\). The following two properties of a set \( \Lambda \) of positive integers are equivalent:

a) Every power series (1) satisfying (2), with real coefficients and such that the changes of sign of the coefficients \( a_m \) occur only for \( m \in \Lambda \), has a singularity on the arc \( I_\Delta \).

b) All measurable subsets \( \Lambda' \subset \Lambda \) have densities at most \( \Delta \).

We recall that the minimum density

\[
D_2(\Lambda) = \lim_{r \to 0+} \liminf_{x \to +\infty} \frac{n((r+1)x, \Lambda) - n(x, \Lambda)}{rx}
\]

can be alternatively defined as the sup of the limits

\[
\lim_{x \to +\infty} \frac{n(x, \Lambda')}{x}
\]

over all measurable sets \( \Lambda' \subset \Lambda \).

Similarly the maximum density of \( \Lambda \) is

\[
\mathcal{D}_2(\Lambda) = \lim_{r \to 0+} \limsup_{x \to +\infty} \frac{n((r+1)x, \Lambda) - n(x, \Lambda)}{rx},
\]

and it equals the inf of the limits (3) over all measurable sequences of non-negative integers \( \Lambda' \) containing \( \Lambda \).

For all these properties of minimum and maximum densities, see [12].

Thus condition (ii) is equivalent to \( D_2(\Lambda) \leq \Delta \), while condition b) is equivalent to \( \mathcal{D}_2(\Lambda) \leq \Delta \).

Here is a gap version of Theorem 1:

**Theorem 2.** The following two properties of a set \( \Lambda \) of positive integers are equivalent:

A. Every power series

\[
\sum_{m \in \Lambda} a_m z^m
\]

satisfying (2) has a singularity on \( I_\Delta \).

A’. Every power series (1) satisfying (2) has a singularity on every closed arc of length \( 2\pi \Delta \) of the unit circle.

B. \( D_2(\Lambda) \leq \Delta \).

The equivalence between A and A’ is immediate, as all assumptions of the statement A are invariant with respect to the change of the variable \( z \mapsto \lambda z, \ |\lambda| = 1 \).

**Proof of Theorem 1.** b) --- a).
A VERSION OF FABRY'S THEOREM FOR POWER SERIES

Proving this by contradiction, we assume that \( D_\Delta(\Lambda) \leq \Delta \), and that there exists a function \( f \) of the form (1) with the property (2) which has an analytic continuation to \( I_\Delta \), and such that the sign changes occur only for \( m \in \Lambda \).

Without loss of generality we assume that \( a_0 = 1 \) and \( \Delta < 1 \).

**Lemma 1.** For a function \( f \) as in (1) to have an immediate analytic continuation from the unit disc to the arc \( I_\Delta \) it is necessary and sufficient that there exists an entire function \( F \) of exponential type with the properties

\[
a_m = (-1)^m F(m), \quad \text{for all } m \geq 0,
\]

and

\[
\limsup_{t \to \infty} \frac{\log |F(te^{i\theta})|}{t} \leq \pi b |\sin \theta|, \quad |\theta| < \alpha,
\]

with some \( b < 1 - \Delta \) and some \( \alpha \in (0, \pi) \).

This result can be found in [1]; see also [2, 4].

Consider the sequence of subharmonic functions

\[
u_m(z) = \frac{1}{m} \log |F(mz)|, \quad m = 1, 2, 3, \ldots.
\]

This sequence is uniformly bounded from above on every compact subset of the plane, because \( F \) is of exponential type. Moreover, \( u_m(0) = 0 \) because of our assumption that \( a_0 = F(0) = 1 \). The Compactness Principle [8, Th. 4.1.9] implies that from every sequence of integers \( m \) one can choose a subsequence such that the limit \( u = \lim u_m \) exists. This limit is a subharmonic function in the plane that satisfies, in view of (6),

\[
lu(re^{i\theta}) \leq \pi br |\sin \theta|, \quad |\theta| < \alpha,
\]

with some \( b \) satisfying

\[0 < b < 1 - \Delta.\]

We use the following result of Pólya [11, footnote 18, p. 703]:

**Lemma 2.** Let \( f \) be a power series (1) of radius of convergence 1. Let \( \{a_{mk}\} \) be a subsequence of coefficients with the property

\[
\lim_{k \to \infty} |a_{mk}|^{1/m_k} = 1,
\]

and assume that for some \( r > 0 \) the number of non-zero coefficients \( a_j \) on the interval \( m_k \leq j \leq (1 + r)m_k \) is \( o(m_kr) \) as \( k \to \infty \). Then \( f \) has no analytic continuation to any point of the unit circle.

Lemma 2 also follows from the results of [1] or [4].

Now we show that (2) implies the following:

**Lemma 3.** Every limit function has the property \( u(x) = 0 \) for \( x \geq 0 \).

**Proof of Lemma 3.** Let \( U = \{x : x \geq 0, u(x) < 0\} \). This set is open because \( u \) is upper semi-continuous. Take any closed interval \( J = [c, d] \subset U \). Then \( u(x) \leq -\epsilon, \; x \in J \), with some \( \epsilon > 0 \). Let \( \{m_k\} \) be the sequence of integers such that \( u_{m_k} \to u \). Then from the definition of \( u_m \) we see that

\[
\log |F(m_k x)| \leq -m_k \epsilon/2 \quad \text{for } x \in J
\]

and for all large \( k \). Together with (3) and (2) this implies that \( a_j = 0 \) for all \( j \in m_k J \). Let \( a_{m'_k} \) be the last non-zero coefficient before \( cm_k \). Applying Lemma 2
to the sequence \( \{ m'_k \} \) we conclude that \( f \) has no analytic continuation from the
unit disc. This is a contradiction, which proves Lemma 3. □

Now we use the following general fact:

**Grishin’s Lemma.** Let \( u \leq v \) be two subharmonic functions, and let \( \mu \) and \( \nu \) be
their respective Riesz measures. Let \( E \) be a Borel set such that \( u(z) = v(z) > -\infty \)
for \( z \in E \). Then the restrictions of the Riesz measures on \( E \) satisfy

\[
\mu|_E \leq \nu|_E.
\]

The references are [13, 7, 6].

In view of Lemma 2, we can apply Grishin’s Lemma to \( u \) and \( v(z) = \pi b|\text{Im } z| \)
and \( E = [0, \infty) \subset \mathbb{R} \). We obtain that the Riesz measure \( d\mu \) of any limit function \( u \)
of the sequence \( \{ u_k \} \) satisfies

\[
d\mu|_{[0, \infty)} \leq b \, dx.
\]

Now we go back to our coefficients and function \( F \). By our assumption, the sign
changes occur on a sequence \( \Lambda \) whose minimum density is at most \( \Delta \). Choose a
number \( a \) such that \( b < a < 1 - \Delta \). By the first definition of the minimum density,
there exist \( r > 0 \) and a sequence \( x_k \to \infty \) such that

\[
n((1 + r)x_k, \Lambda) - n(x_k, \Lambda) \leq (1 - a)r x_k.
\]

**Lemma 4.** Let \( (a_0, a_1, \ldots, a_N) \) be a sequence of real numbers, and let \( f \) be a real
analytic function on the closed interval \([0, N]\), such that \( f(n) = (-1)^n a_n \). Then
the number of zeros of \( f \) on \([0, N]\), counting multiplicities, is at least \( N \) minus the
number of sign changes of the sequence \( \{ a_n \} \).

**Proof.** Consider first an interval \((k, n)\) such that \( a_k a_n \neq 0 \) but \( a_j = 0 \) for \( k < j < n \).
We claim that \( f \) has at least

\[
n - k - \#(\text{sign changes in the pair } (a_k, a_n))
\]

zeros on the open interval \((k, n)\). Indeed, the number of zeros of \( f \) on this interval is
at least \( n - k - 1 \) in any case. This proves the claim if there is a sign change in the pair
\((a_k, a_n)\). If there is no sign change, that is, if \( a_n a_k > 0 \), then \( f(n)f(k) = (-1)^{n-k} \).
So the number of zeros of \( f \) on the interval \((n, k)\) is of the same parity as \( n - k \).

But \( f \) has at least \( n - k - 1 \) zeros on this interval; thus the total number of zeros
is at least \( n - k \). This proves our claim.

Now let \( a_k \) be the first and \( a_n \) the last non-zero term of our sequence. As the
interval \((k, n)\) is a disjoint union of the intervals to which the above claim applies,
we conclude that the number of zeros of \( f \) on \((k, n)\) is at least \((n - k) \) minus the
number of sign changes of our sequence. On the rest of the interval \([0, N]\) our
function has at least \( N - n + k \) zeros, so the total number of zeros is at least \( N
\) minus the number of sign changes. □

Let \( u \) be a limit function of the subsequence \( \{ u_{m_k} \} \) with \( m_k = [x_k] \). By Lemma 4,
the function \( F \) has at least \( ar x_k - 2 \) zeros on each interval \([x_k, (1 + r)x_k]\), which
implies that the Riesz measure \( \mu \) of \( u \) satisfies

\[
\mu([1, 1 + r]) \geq ar.
\]

This contradicts (9) and thus proves the implication \( b) \rightarrow a \).
Proof of Theorem 2, $B \rightarrow A$. This is essentially the same as the previous proof. Proving by contradiction, we assume that $B$ holds but that there exists a function $f$ of the form (4) with the property (2) which has an analytic continuation to $I_\Delta$. Applying Lemma 1, we obtain an entire function $F$ with properties (5) and (6). Then we consider subharmonic functions $u_m$ and the limit functions $u$ of this sequence. Using Lemmas 2, 3 and Grishin’s lemma, we obtain the inequality (9) for the Riesz measure $\mu$ of $u$, exactly as in the proof of Theorem 1.

Now we notice that condition $B$ of Theorem 2 means that the entire function $F$ has zeros at some sequence of integers of maximum density at least $1 - \Delta$. Denoting by $n(x)$ the number of zeros of $F$ on $[0, x]$ and choosing a number $a \in (b, 1 - \Delta)$, we obtain that there exist $r > 0$ and a sequence of integers $m_k \rightarrow \infty$ such that

$$n((1 + r)m_k) - n(m_k) \geq arm_k.$$

This implies that for the limit function $u$ of the sequence $u_{m_k}$, the Riesz measure $\mu$ satisfies $\mu([1, 1 + r]) \geq ar$, which contradicts (9). This contradiction proves implication $B \rightarrow A$ in Theorem 2.

Proof of implications $a) \rightarrow b)$ of Theorem 1 and $A \rightarrow B$ of Theorem 2. Suppose that a set $\Lambda$ of positive integers does not satisfy b), $B$. We will construct power series $f$ of the form (4) with real coefficients which has an immediate analytic continuation from the unit disc to the arc $I_\Delta$. This will simultaneously prove the implications $a) \rightarrow b)$ of Theorem 1 and $A \rightarrow B$ of Theorem 2, as the number of sign changes of any sequence does not exceed the number of its non-zero terms.

Let $\Lambda' \subset \Lambda$ be a measurable set of density $\Delta' > \Delta$. Let $S$ be the complement of $\Lambda'$ in the set of positive integers. Then $S$ is also measurable and has density $1 - \Delta'$.

Consider the infinite product

$$F(z) = \prod_{t \in S} \left(1 - \frac{z^2}{t^2}\right).$$

This is an entire function of exponential type with indicator $\pi(1 - \Delta')|\sin \theta|$, and furthermore,

$$(10) \quad \log |F(z)| \geq \pi(1 - \Delta')|\operatorname{Im} z| + o(|z|),$$

as $z \rightarrow \infty$ outside the set $\{z : \operatorname{dist}(z, S) \leq 1/4\}$. (See [10] Ch. II, Thm. 5] for this result.) Now we use the sufficiency part of Lemma 1 and define the coefficients of our power series by $a_m = (-1)^m F(m)$. Then we have all the needed properties; in particular (2) follows from (10).

Acknowledgments

The author thanks Alan Sokal for many illuminating conversations about Fabry’s theorem, Mario Bonk for spotting a mistake in the original version of this paper and the referee for his valuable remarks.

References


DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: eremenko@math.purdue.edu