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SPECTRAL RADIUS ALGEBRAS AND C_0 CONTRACTIONS

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ABSTRACT. We consider the spectral radius algebras associated to C_0 contractions. If A is such an operator we show that the spectral radius algebra $\mathcal{B}_A$ always properly contains the commutant of A.

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{L}(\mathcal{H})$, then a subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T if $T\mathcal{M} \subset \mathcal{M}$, and it is hyperinvariant for T if it is invariant for every operator in the commutant $\{T\}'$ of T. A nontrivial invariant subspace (n. i. s.) is one that is neither $\mathcal{H}$ nor the zero subspace. It was shown in [5] that one can associate the so-called spectral radius algebra $\mathcal{B}_A$ to each operator A. Such an algebra always contains $\{A\}'$, so, when it has an n. i. s. it represents a generalization of the concept of a hyperinvariant subspace. This is the case when A is compact (cf. [5]) or one type of normal operators (cf. [2]). Of course, it is important to establish that the inclusion $\{A\}' \subset \mathcal{B}_A$ is proper. Although this has been done for some classes of operators (cf. [2], [3], [5], [6]), the question is, as of this writing, still open.

This paper can be regarded as a sequel to [4] where an extensive investigation of Jordan blocks and C_0 contractions (to be defined below) was conducted. A prominent role in this study was played by the so-called *extended eigenvalues*. (A complex number λ is an extended eigenvalue of A if there is a nonzero operator X such that $AX = \lambda XA$.) As we will see, the presence of an eigenvalue or an extended eigenvalue is sufficient to guarantee that $\mathcal{B}_A \neq \{A\}'$. Unfortunately, not every Jordan block $S(\theta)$ has either of these. Nevertheless, we will demonstrate that the inclusion under consideration is proper when A belongs to the class C_0 (Theorems 3, 5, and 18). Our method utilizes the relationship between $S(\theta)$ and the shift S as well as the quasisimilarity model for C_0 contractions.

We briefly review the relevant facts and notation. A contraction A is completely nonunitary if there is no invariant subspace $\mathcal{M}$ for A such that $A|\mathcal{M}$ is a unitary operator. A completely nonunitary contraction A is said to be of class C_0 if there exists a nonzero function $h \in H^\infty$ such that $h(A) = 0$. The inner function v such that $vH^\infty = \{u \in H^\infty : u(A) = 0\}$ is the minimal function of A and is denoted by m_A. A very important subclass of C_0 contractions are the Jordan blocks. Throughout the paper we will use S to denote the forward unilateral shift of multiplicity 1, and $\{e_n\}_{n=0}^\infty$ the orthonormal basis such that $Se_n = e_{n+1}$, $n \geq 0$. One knows that S can be viewed as multiplication by z on the Hardy space H^2.

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From this viewpoint, every invariant subspace of S is of the form θH^2 for some inner function θ. The compression of S to $H^2 \ominus \theta H^2$ is called a Jordan block and denoted by $S(\theta)$. Also, if θ is an inner function, then there exists a Blaschke product b, a singular inner function s, and a constant γ, $|\gamma| = 1$, such that $\theta = \gamma bs$. We refer to this as the canonical factorization of θ. Furthermore, there is a finite, positive, singular measure μ on the circle $\mathbb{T}$ such that

$$s(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right).$$

For more information one may consult [1].

Given an operator $A \in \mathcal{L}(\mathcal{H})$ with spectral radius r and $m \in \mathbb{N}$, we define $d_m = m/(1 + rm)$ and $R_m = \left(\sum_{n=0}^{\infty} d_m^{2n} A^{*\,n} A^n\right)^{1/2}$. The spectral radius algebra $\mathcal{B}_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ such that $\sup_{m\in\mathbb{N}} \|R_m T R_m^{-1}\| < \infty$. The following result from [5] summarizes some of the important properties of $\mathcal{B}_A$ (cf. [5, Proposition 2.3, Corollary 2.4]).

Proposition 1. *Let A be an operator in $\mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{B}_A$ if and only if there exists $M > 0$ such that, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$, $\sum_{n\geq 0} d_m^{2n} \|A^n T x\|^2 \leq M \sum_{n\geq 0} d_m^{2n} \|A^n x\|^2$. When $AT = \lambda TA$, $|\lambda| \leq 1$, and in particular if $AT = TA$, then $T \in \mathcal{B}_A$.*

Proposition 1 has a consequence whose verification is straightforward and we leave it to the reader.

Corollary 2. *If $Au = \lambda u$ for some $|\lambda| < r(A)$, then, for any $v \in \mathcal{H}$, the rank one operator $u \otimes v$ belongs to $\mathcal{B}_A$. Furthermore, if $A^* v \neq \overline{\lambda} v$, then $u \otimes v$ does not commute with A. In particular, if A has nontrivial kernel, then $\mathcal{B}_A \neq \{A\}'$.*

Now we can prove our first result about the inclusion $\{A\}' \subset \mathcal{B}_A$.

Theorem 3. *Let $A \in C_0$ and suppose that m_A is neither a singular inner function nor a Blaschke product with all zeros of the same modulus. Then $\mathcal{B}_A \neq \{A\}'$.*

Proof. The assumption is that $m_A = \gamma bs$ and that $b(\alpha) = 0$ with $|\alpha| < r(A)$. This implies that α is an eigenvalue of A, whence the result follows from Corollary 2. □

Proposition 1 shows that, in order to establish that an operator $T \in \mathcal{B}_A$, it suffices to show that it satisfies $AT = \lambda TA$ for some $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. The following result (cf. [4, Theorem 4.5]) provides a condition under which a C_0 contraction has an extended eigenvalue. Here we use the convention that, if μ is a measure and $\lambda \in \mathbb{T}$, then $\mu_\lambda(E) = \mu(\lambda E)$.

Theorem 4. *Let A be a C_0 contraction and $|\lambda| = 1$. The equation $AX = \lambda XA$ has a solution $X \neq 0$ if and only if either the measures μ and μ_λ are not mutually singular or if $\alpha = \lambda\beta$ for some zeros α, β of m_A.*

Theorem 3 allows us to consider the situation when m_A is either a singular inner function or a Blaschke product with all zeros of the same modulus. Now we make a further reduction based on Theorem 4.

Theorem 5. *Let $A \in C_0$ and suppose that m_A is either a singular inner function such that the support of μ contains more than one point or that m_A is a Blaschke product with at least 2 different zeros of the same modulus. Then $\mathcal{B}_A \neq \{A\}'$.*

Proof. It is easy to see that, under the hypotheses of the theorem, there exists $\lambda \in \mathbb{T}$, $\lambda \neq 1$, such that the conditions of Theorem 4 are satisfied. Consequently, there exists $X \neq 0$ satisfying $AX = \lambda XA$, so Proposition 1 implies that $X \in \mathcal{B}_A$. If X commutes with A, taking into account that $\lambda \neq 1$, it is not hard to see that $AX = 0$. Since A is injective this leads to a contradiction, so $X \in \mathcal{B}_A \setminus \{A\}'$. □

In view of Theorem 3 and Theorem 5 there remain two cases to consider. In the first m_A is a Blaschke product with only one zero, and in the second m_A is a singular inner function with the singular measure supported at one point. We will first explore both of these possibilities in the special case when $A = S(\theta)$ and, of course, $m_A = \theta$. In the former, we will need a fact established in [3].

Theorem 6. *Let A be an operator acting on a finite dimensional space. Then $\mathcal{B}_A \neq \{A\}'$.*

Corollary 7. *If θ is a Blaschke product with only one zero, then $S(\theta)$ acts on a finite dimensional Hilbert space. Consequently, $B_{S(\theta)} \neq \{S(\theta)\}'$*

The case when θ is a singular inner function with the singular measure μ supported at one point λ on the circle is much more complicated. One knows (cf. [1, p. 22]) that in this case

$$\theta(z) = \gamma \exp\left\{\frac{z+\lambda}{z-\lambda}\, p\right\}, \tag{1}$$

with $|\gamma| = 1$ and $p = \mu(\{\lambda\})$.

We will exploit the relationship between $S(\theta)$ and the unilateral shift S. In the rest of the paper, unless specifically noted, R_m will always mean $R_m(S^*)$; i.e., it is associated to S^*. We start with a computational result. We leave its verification to the reader.

Lemma 8. *For all $m \in \mathbb{N}$ and all $i \geq 0$, $R_m e_i = \alpha_{m,i} e_i$, where $\alpha_{m,i}$ are complex numbers satisfying $|\alpha_{m,i}| \leq \sqrt{i+1}$, $m \in \mathbb{N}$, $i \geq 0$.*

Before we proceed, we notice that, as a consequence of Lemma 8, $\mathcal{B}_{S^*}$ is quite different from $\mathcal{B}_S = \mathcal{L}(\mathcal{H})$.

Theorem 9. *$\mathcal{B}_{S^*}$ is weakly dense in, but properly contained in, $\mathcal{L}(\mathcal{H})$.*

Proof. Since S^* is not a multiple of an isometry, the fact that $\mathcal{B}_{S^*} \neq \mathcal{L}(\mathcal{H})$ follows from [2, Theorem 2.7]. In order to establish that $\mathcal{B}_{S^*}$ is weakly dense in $\mathcal{L}(\mathcal{H})$ it suffices to show that, for any $i, j \geq 0$, the rank one operator $e_i \otimes e_j$ belongs to $\mathcal{B}_{S^*}$. Since $\|R_m e_i\| \leq \sqrt{i+1}$ and R_m^{-1} is a contraction, the result follows easily. □

Lemma 8 shows that $\sup_m \|R_m f\| < \infty$ when $f = e_i$. This can be easily extended to a larger class of functions.

Corollary 10. *If $f \in H^2$ and $f' \in H^2$, then $\sup_m \|R_m f\| < \infty$.*

Our next goal is to prove that the function f in Corollary 10 can be found in a specific subspace. Let J be the antiderivative operator on H^2, i.e., an operator such that, for all $f \in H^2$, $(Jf)' = f$ and $(Jf)(0) = 0$. We will show that there is a function $g \in H^2$ such that $f = J^* g \in H^2 \ominus \theta H^2$. Clearly, $J^* g \perp \theta H^2$ if and only if $g \perp J(\theta H^2)$, so it suffices to establish that $J(\theta H^2)$ is a subspace of H^2 whose codimension is infinite.

Theorem 11. *If θ is a singular inner function such that the associated singular measure is supported at one point, and if J is the antiderivative operator as above, then the codimension of $J(\theta H^2)$ in H^2 is infinite.*

Proof. Suppose that θ is as in (1), and that the codimension of $J(\theta H^2)$ is finite. Let $p_0 = 1$, and $p_n = z^{n-1}(2z - 2\lambda - 2p\lambda) + (n-1)z^{n-2}(z-\lambda)^2$ for $n \geq 1$. It is easy to see that the span of polynomials $\{p_n\}_{n=0}^{\infty}$ is dense in H^2. Therefore, the span of $\{J(\theta p_n)\}_{n=0}^{\infty}$ has finite codimension.

On the other hand, a calculation shows that $-2\lambda p\theta = (z-\lambda)^2\theta'$, so for all $f \in H^2$, $-2\lambda p J(\theta f) = J((z-\lambda)^2\theta' f) = (z-\lambda)^2\theta f - J[(z-\lambda)^2\theta f' + 2(z-\lambda)\theta f]$. It follows that

$$J[(2z - 2\lambda - 2p\lambda)\theta f + (z-\lambda)^2\theta f'] = (z-\lambda)^2\theta f.$$

By choosing $f = z^{n-1}$, $n \in \mathbb{N}$, we obtain that $J(\theta p_n) = (z-\lambda)^2 z^{n-1}\theta$ for all $n \in \mathbb{N}$. Consequently, $J(\theta p_n) \in \theta H^2$ and the span of $\{J(\theta p_n)\}_{n=1}^{\infty}$ is a subspace of θH^2. Since the latter has infinite codimension the result follows. □

From this theorem we obtain an important consequence.

Corollary 12. *There exists a nonzero function $f \in H^2 \ominus \theta H^2$ such that $f' \in H^2$ and, consequently, such that* $\sup_m \|R_m f\| < \infty$.

Proof. By Theorem 11 there exists a nonconstant function $g \in H^2 \ominus J(\theta H^2)$. Let $f = J^* g$. It is easy to see that $f \in H^2 \ominus \theta H^2$ and $f \neq 0$. Furthermore, if $g = \sum g_n z^n$, a straightforward calculation shows that $(J^*g)' = \sum_{n\geq 0}(n+1)/(n+2)g_{n+2}z^n$. Thus, $f' \in H^2$ and an application of Corollary 10 completes the proof. □

The significance of the membership of f (in Corollary 12) in $H^2 \ominus \theta H^2$ lies in the fact that we can now deduce an analogous result for $R_m(S(\theta)^*)$.

Corollary 13. *Suppose that θ is a singular inner function as in* (1). *There exists a nonzero function $u \in H^2$ such that* $\sup_m \|R_m(S(\theta)^*)u\| < \infty$.

Proof. Notice that both $S(\theta)^*$ and S^* have the spectral radius 1, so $d_m(S(\theta)^*) = d_m(S^*) = m/(m+1)$. Therefore, relative to the decomposition $H^2 = \theta H^2 \oplus (H^2 \ominus \theta H^2)$, taking into account that $H^2 \ominus \theta H^2$ is invariant for S^*,

$$R_m^2(S^*) = \begin{pmatrix} \star & \star \\ \star & R_m^2(S(\theta)^*) \end{pmatrix}.$$

Let f be the function provided by Corollary 12. Clearly, we can write $f = 0 \oplus u$ relative to the same decomposition, and it is easy to see that $\|R_m(S(\theta)^*)u\| = \|R_m(S^*)f\|$. □

Corollary 13 describes a property of $S(\theta)^*$. Since we are more interested in $S(\theta)$ it is useful to recall [1, Corollary 3.1.7]. We use the notation $\tilde{\theta}(z) = \overline{\theta(\overline{z})}$.

Theorem 14. *For every inner function θ the adjoint $S(\theta)^*$ is unitarily equivalent to $S(\tilde{\theta})$.*

Theorem 14 allows us to move the focus of our investigation from the inclusion $\{A\}' \subset \mathcal{B}_A$ to $\{B\}' \subset \mathcal{B}_B$, where B is unitarily equivalent to A. There is no loss of generality in doing so since the unitary equivalence between A and B gives rise to an algebra isomorphsim ϕ such that $\phi(\{A\}') = \{B\}'$ and $\phi(\mathcal{B}_A) = \mathcal{B}_B$ (cf. [2, Theorem 2.4]). Finally, we can prove our main result concerning the Jordan blocks.

Theorem 15. $B_{S(\theta)} \neq \{S(\theta)\}'$

Proof. Combining Theorem 3, Theorem 5, and Corollary 7 we see that the only case to consider is when θ is given by (1). By Theorem 14, it suffices to show that $\mathcal{B}_{S(\theta)^*} \neq \{S(\theta)^*\}'$. If u is the function supplied by Corollary 13, then $u \otimes v \in B(S(\theta)^*)$ for any $v \in H^2$. However, $u \otimes v$ commutes with $S(\theta)^*$ if and only if v is an eigenvector for $S(\theta)$. $\square$

Next, we return to contraction operators of class C_0. In order to obtain a better insight into their structure we rely on the following result (cf. [1, Theorem 3.5.1]).

Theorem 16. *Let A be a C_0 contraction. Then A is quasisimilar to an infinite direct sum of Jordan blocks $\bigoplus_i S(\theta_i)$.*

Since Theorem 16 relates quasisimilar operators we need to establish the relationship between their respective spectral radius algebras as well as between their commutants.

Lemma 17. *Suppose that A and B are quasisimilar C_0 contractions and let Y, Z be quasi-affinities such that $AY = YB$ and $ZA = BZ$. If $T \in \mathcal{B}_B$, then $YTZ \in \mathcal{B}_A$. Also $T \in \{B\}'$ if and only if $YTZ \in \{A\}'$.*

Proof. A and B have essentially the same quasisimilarity model, so they share the same spectral radius. In particular, $d_m(A) = d_m(B)$ and we will denote both by d_m. A calculation shows that $A^nYTZ = YB^nTZ$, so, for all $x \in \mathcal{H}$, $\sum d_m^{2n}\|A^nYTZx\|^2 \leq \|Y\|^2 \sum d_m^{2n}\|B^nTZx\|^2$. Now, if $T \in \mathcal{B}_B$, Proposition 1 shows that there exists $M > 0$ such that, for all $x \in \mathcal{H}$, the last expression is dominated by $M^2\|Y\|^2 \sum d_m^{2n}\|B^nZx\|^2 = M^2\|Y\|^2 \sum d_m^{2n}\|ZA^nx\|^2 \leq M^2\|Y\|^2\|Z\|^2 \sum d_m^{2n}\|A^nx\|^2$. Consequently, $YTZ \in \mathcal{B}_A$. The other assertion is even easier: $AYTZ = YBTZ = YTBZ = YTZA$. $\square$

Now we can make the final step in our analysis.

Theorem 18. *Let A be a C_0 contraction and suppose that m_A is either a Blaschke product with only one zero or a singular inner function such that the support of μ consists of a single point $\lambda \in \mathbb{T}$. Then $\mathcal{B}_A \neq \{A\}'$.*

Proof. By Theorem 16, A is quasisimilar to a direct sum $S(\Theta) = \bigoplus_i S(\theta_i)$, where each inner function θ_i is either a Blaschke product with only one zero or a singular inner function of the form (1) for some $p > 0$. Therefore (cf. [1, Theorem 2.4.11]), $r(A) = r(S(\Theta)) = r(S(\theta_1))$. By Theorem 15, there is $X \in \mathcal{B}_{S(\theta_1)} \setminus \{S(\theta_1)\}'$ and it is easy to see that the operator $X \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{B}_{S(\Theta)} \setminus \{S(\Theta)\}'$. Now the result follows from Lemma 17. $\square$

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