

PARAMETRIC DECOMPOSITION OF POWERS OF PARAMETER IDEALS AND SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. Let M be a finitely generated module of dimension d over a Noetherian local ring (R, \mathfrak{m}) and \mathfrak{q} an ideal generated by a system of parameters $\underline{x} = (x_1, \dots, x_d)$ of M . For each positive integer n , set

$$\Lambda_{d,n} = \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \mid \alpha_i \geq 1, 1 \leq i \leq d \text{ and } \sum_{i=1}^d \alpha_i = d + n - 1\}$$

and $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$ for each $\alpha \in \Lambda_{d,n}$. Then we prove in this note that M is a sequentially Cohen-Macaulay module if and only if there exists a good system of parameters \underline{x} such that the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$ holds true for all $n \geq 1$. As an application, we show that the sequentially Cohen-Macaulayness of a module can be characterized by a very special expression of the Hilbert-Samuel polynomial of a good parameter ideal.

1. INTRODUCTION

Throughout this paper we denote by R a commutative Noetherian local ring with the maximal ideal \mathfrak{m} and by M a finitely generated R -module with $\dim M = d$. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of the module M and $\mathfrak{q} = (x_1, \dots, x_d)$ the parameter ideal of M generated by \underline{x} . For each integer $n \geq 1$, we set

$$\Lambda_{d,n} = \{(\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq d \text{ and } \sum_{i=1}^d \alpha_i = d + n - 1\}.$$

Let $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \dots, \alpha_d) \in \Lambda_{d,n}$. We say that the system \underline{x} of parameters has the *property of parametric decomposition* if the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$ holds true for all $n \geq 1$. The main purpose of this paper is to study the question of when a given system of parameters of M has the property of parametric decomposition. Notice that Heinzer, Ratliff and Shah [HRS, Theorem 2.4] proved that an R -regular sequence always has the property of parametric decomposition. Later, Goto and Shimoda [GS1, Theorem 1.1] showed that the converse is also true

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when each element of the sequence is a non-zerodivisor in R . Moreover, they gave in [GS2, Theorem 1.1] a characterization of R with $\dim R \geq 2$, in which every system of parameters of R has the property of parametric decomposition. In order to generalize this result of Goto and Shimoda, let us recall some notions which were defined in [CC]. A filtration $\mathcal{D} : H_{\mathfrak{m}}^0(M) = D_0 \subset D_1 \subset \dots \subset D_t = M$ of submodules of M is said to be a *dimension filtration* if D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \dots, 1$. If D_i/D_{i-1} is Cohen-Macaulay for all $1 \leq i \leq t$, M is called a *sequentially Cohen-Macaulay module*. A system $\underline{x} = x_1, \dots, x_d$ of parameters of M is called a *good system of parameters* of M if $D_i \cap (x_{d_i+1}, \dots, x_d)M = 0$ for all $0 \leq i \leq t-1$, where $d_i = \dim D_i$. Now, we restrict our interest in the above question to the set of all good systems of parameters of M . It turns out that the property of parametric decomposition of a good system of parameters can be characterized by the sequentially Cohen-Macaulayness of the module. The following theorem is the main result of this paper.

Theorem 1.1. *The following statements are equivalent:*

- (i) M is a sequentially Cohen-Macaulay module.
- (ii) Every good system of parameters of M has the property of parametric decomposition.
- (iii) There exists a good system of parameters of M having the property of parametric decomposition.

As a consequence of Theorem 1.1 we again obtain the main result of Goto-Shimoda [GS2, Theorem 1.1]. It should be noted here that Theorem 1.1 of [GS2] was stated for local rings, but its proof still works in the module case.

Corollary 1.2. *Let $\dim M \geq 2$ and $H_{\mathfrak{m}}^0(M)$ the 0^{th} local cohomology module of M with respect to the maximal ideal \mathfrak{m} . Then the following statements are equivalent:*

- (i) $M/H_{\mathfrak{m}}^0(M)$ is a Cohen-Macaulay module and $\mathfrak{m}H_{\mathfrak{m}}^0(M) = 0$.
- (ii) Every system of parameters of M has the property of parametric decomposition.

Before we give proofs for Theorem 1.1 and its corollary in Section 3, we need some basic facts on good systems of parameters and sequentially Cohen-Macaulay modules, which will be summarized in Section 2. In Section 4 we shall show that the Hilbert-Samuel polynomial of a sequentially Cohen-Macaulay module M with respect to a good parameter ideal can be computed effectively by using the dimension filtration \mathcal{D} of M (Theorem 4.1).

2. PRELIMINARIES

Throughout this paper, R is a Noetherian local commutative ring with maximal ideal \mathfrak{m} and M is a finitely generated R -module with $\dim M = d$. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of the module M , and we denote by \mathfrak{q} the ideal generated by x_1, \dots, x_d . For positive integers n , we set

$$\Lambda_{d,n} = \{(\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq d \text{ and } \sum_{i=1}^d \alpha_i = d + n - 1\}.$$

Let $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \dots, \alpha_d) \in \Lambda_{d,n}$. Then $\mathfrak{q}^n M \subseteq \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$, and if the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$ holds true for a system

of parameters \underline{x} of M , we say that \underline{x} has the *property of parametric decomposition*. Recall that a filtration $\mathcal{D} : H_{\mathfrak{m}}^0(M) = D_0 \subset D_1 \subset \dots \subset D_t = M$ of submodules of M is said to be a *dimension filtration* if D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \dots, 1$, and a system of parameters $\underline{x} = x_1, \dots, x_d$ of M is called a *good system of parameters* of M if $D_i \cap (x_{d_i+1}, \dots, x_d)M = 0$ for all $0 \leq i \leq t-1$, where $d_i = \dim D_i$.

Now, let us briefly give some facts on the dimension filtration and good systems of parameters (see [CC], [CN]). Because of the Noetherian property of M , the dimension filtration of M exists uniquely. Therefore, in the sequel we always denote by

$$\mathcal{D} : H_{\mathfrak{m}}^0(M) = D_0 \subset D_1 \subset \dots \subset D_t = M$$

with $\dim D_i = d_i$ the dimension filtration of M . In this case, we also say that the dimension filtration \mathcal{D} of M has length t . Moreover, let $\bigcap_{\mathfrak{p} \in \text{Ass} M} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of 0 of M ; then $D_i = \bigcap_{\dim(R/\mathfrak{p}) \geq d_{i+1}} N(\mathfrak{p})$. Put $N_i = \bigcap_{\dim(R/\mathfrak{p}) \leq d_i} N(\mathfrak{p})$. Therefore $D_i \cap N_i = 0$ and $\dim(M/N_i) = d_i$. By the Prime Avoidance there exists a system of parameters $\underline{x} = (x_1, \dots, x_d)$ such that $x_{d_i+1}, \dots, x_d \in \text{Ann}(M/N_i)$. It follows that $D_i \cap (x_{d_i+1}, \dots, x_d)M \subseteq N_i \cap D_i = 0$ for all $0 \leq i \leq t-1$. Thus $\underline{x} = x_1, \dots, x_d$ is a good system of parameters of M , and therefore the set of good systems of parameters of M is non-empty. Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of M . It is easy to see that x_1, \dots, x_{d_i} is a good system of parameters of D_i and $x_1^{n_1}, \dots, x_{d_i}^{n_{d_i}}$ is a good system of parameters of M for any d -tuple of positive integers n_1, \dots, n_d .

Lemma 2.1. *Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of M . Then $D_i = 0 :_M x_j$ for all $d_i < j \leq d_{i+1}$ and $0 \leq i \leq t-1$, and therefore $0 :_M x_i^l = 0 :_M x_i$ for all $l \geq 1$.*

Proof. Since $D_i \cap (x_{d_i+1}, \dots, x_d)M = 0$, we have $D_i \subseteq 0 :_M x_j$ for all $j \geq d_i$. Thus it suffices to prove that $0 :_M x_j \subseteq D_j$ for every $d_i < j \leq d_{i+1}$. Assume that $0 :_M x_j \not\subseteq D_i$. Let s be the largest integer such that $0 :_M x_j \not\subseteq D_{s-1}$. Then $t \geq s > i$ and $0 :_M x_j = 0 :_{D_s} x_j$. Since $d_s \geq d_{i+1} \geq j$, x_j is a parameter element of D_s and $\dim(0 :_M x_j) < d_s$. Hence $0 :_M x_j \subseteq D_{s-1}$ by the maximality of D_{s-1} . This contradicts the choice of s . Therefore $0 :_M x_j = D_i$. \square

Recall that M is said to be a *sequentially Cohen-Macaulay module* if each quotient D_i/D_{i-1} in the dimension filtration of M is Cohen-Macaulay. Notice that the notion of sequentially Cohen-Macaulay modules was introduced by Stanley [St] for the graded case, and it was studied for the local case in [Sch], [CN]. Also notice that a special type of sequentially Cohen-Macaulay rings called approximately Cohen-Macaulay rings was studied very early by Goto [G].

3. PROOF OF THEOREM 1.1

The following result is an immediate consequence of the definition of a good system of parameters in a sequentially Cohen-Macaulay module.

Lemma 3.1. *Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of a sequentially Cohen-Macaulay module M and $\mathfrak{q} = (x_1, \dots, x_d)$. Then*

$$\mathfrak{q}^n M \cap D_i = \mathfrak{q}^n D_i$$

for all $n \geq 1$ and $0 \leq i \leq t-1$.

Proof. Since D_{i+1}/D_i is a Cohen-Macaulay module with $\dim D_{i+1}/D_i = d_{i+1} > d_i = \dim D_i$ and $\mathfrak{q}D_{i+1} = (x_1, \dots, x_{d_{i+1}})D_{i+1}$ for $0 \leq i \leq t-1$, it follows from well-known facts in commutative algebra that

$$\mathfrak{q}^n D_{i+1} \cap D_i = \mathfrak{q}^n D_i.$$

Therefore

$$\begin{aligned} \mathfrak{q}^n M \cap D_i &= (\mathfrak{q}^n D_t \cap D_{t-1}) \cap D_i = \mathfrak{q}^n D_{t-1} \cap D_i \\ &\dots = \mathfrak{q}^n D_{i+1} \cap D_i = \mathfrak{q}^n D_i \end{aligned}$$

for all $n \geq 1$ and $0 \leq i \leq t-1$ as required. \square

Let s be a positive integer and $y_1, \dots, y_s \in \mathfrak{m}$. For each $1 \leq i \leq s$ and $\alpha = (\alpha_1, \dots, \alpha_i) \in \Lambda_{i,n}$, we set $Q_i = (y_1, \dots, y_i)R$, $Q = (y_1, \dots, y_s)R$ and $Q_i(\alpha) = (y_1^{\alpha_1}, \dots, y_i^{\alpha_i})$, $Q(\alpha) = (y_1^{\alpha_1}, \dots, y_s^{\alpha_s})$. The following result is due to Heinzer-Ratliff-Shah [HRS, Theorem 2.4]. But we give here the module version of this result proved by Goto-Shimoda [GS2, Lemma 2.1].

Lemma 3.2. *Let s be a positive integer and y_1, \dots, y_s an M -regular sequence. Then*

$$Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$$

for all $n \geq 1$.

Because the techniques and methods of proof of Theorem 1.1 heavily depend on the works of Goto-Shimoda [GS1] and [GS2], let us summarize the auxiliary results in [GS1] into the following.

Lemma 3.3. *With the notation as above the following assertions hold true:*

(i) *Let $y \in R$ and assume that $0 :_M y^\ell \subseteq yM$ for all $\ell \geq 1$. Then y is a non-zerodivisor on M .*

(ii) *Suppose that $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \geq 1$. Then*

$$Q_{s-1}M : y_s^\ell \subseteq QM + (0 :_M y_s^\ell)$$

for all $\ell \geq 1$.

(iii) *Suppose that $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \geq 1$. Then*

$$Q_i^n M = \bigcap_{\alpha \in \Lambda_{i,n}} Q_i(\alpha)M$$

for all $1 \leq i \leq s$ and $\ell \geq 1$.

In the above lemma, the key is assertion (ii), which is given in the proof of Lemma 3.2 in [GS1]. By this lemma one gets the following.

Lemma 3.4. *Suppose that (1) $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \geq 1$ and that*

(2) $0 :_M y_i^\ell = 0 :_M y_i$ for all $1 \leq i \leq s$ and $\ell \geq 1$. Then for all integers $1 \leq i \leq j \leq s$, the element y_j is a non-zerodivisor on $M/[Q_{i-1}M + (0 :_M y_j)]$, so that one has the equality

$$Q_{i-1}M : y_j^2 = Q_{i-1}M + (0 :_M y_j).$$

Proof. Since conditions (1) and (2) are independent of the order of y_1, \dots, y_s , we may assume that $i = j$. Then we have

$$Q_{i-1}M : y_i^\ell \subseteq Q_iM + (0 :_M y_i)$$

for all $\ell \geq 1$, thanks to (ii) and (iii) of Lemma 3.3. Let $L = M/[Q_{i-1}M + (0 :_M y_i)]$ and let $\alpha \in M$ be such that $y_i^\ell \bar{\alpha} = 0$ in L with $\ell \geq 1$, where $\bar{\alpha}$ denotes the image of α in L . Then $y_i^{\ell+1} \alpha \in Q_{i-1}M$, so that $\alpha \in Q_{i-1}M : y_i^{\ell+1} \subseteq Q_iM + (0 :_M y_i)$. Hence $\bar{\alpha} \in y_i L$, which shows, by (i) of Lemma 3.3, that y_i is a non-zero-divisor on L . Then the second conclusion is now clear. \square

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. (i) \Rightarrow (ii). Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of M and $\mathfrak{q} = (x_1, \dots, x_d)$. We prove by induction on the length t of the dimension filtration \mathcal{D} of M that \underline{x} has the property of parametric decomposition. The case $t = 0$ is trivial. Let $t \geq 1$. Set $\bar{M} = M/D_{t-1}$. Since \bar{M} is a Cohen-Macaulay module, the sequence x_1, \dots, x_d is \bar{M} -regular. Then $\mathfrak{q}^n \bar{M} = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha) \bar{M}$ by Lemma 3.2.

Therefore $\bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M \subseteq \mathfrak{q}^n M + D_{t-1}$. Since $x_1^{\alpha_1}, \dots, x_d^{\alpha_d}$ is a good system of parameters of M for each $\alpha \in \Lambda_{d,n}$, it follows from Lemma 3.1 that $\mathfrak{q}(\alpha)M \cap D_{t-1} = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_{d_{t-1}}^{\alpha_{d_{t-1}}})D_{t-1}$. Now, applying the inductive hypothesis on D_{t-1} we can show with the same method that was used in the proof of Proposition 2.2 of [GS2] that $\bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M = \mathfrak{q}^n M$.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i). Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of M having the property of parametric decomposition and $\mathfrak{q} = (x_1, \dots, x_d)$. Since $0 :_M x_i^l = 0 :_M x_i$ for all $l \geq 1$ and $1 \leq i \leq d$ by Lemma 2.1, we get by Lemma 3.4 that $\mathfrak{q}_i M : x_j^2 = \mathfrak{q}_i M + (0 :_M x_j)$ for all $1 \leq i \leq j \leq d$. Therefore the implication follows by Theorem 3.9 of [CC], and the proof of Theorem 1.1 is complete. \square

Proof of Corollary 1.2. (i) \Rightarrow (ii). It is easy to see from the hypothesis that M is a sequentially Cohen-Macaulay module with the dimension filtration $\mathcal{D} : H_{\mathfrak{m}}^0(M) \subset M$. Moreover, by Lemma 3.1 we have

$$(x_1, \dots, x_d)M \cap H_{\mathfrak{m}}^0(M) = (x_1, \dots, x_d)H_{\mathfrak{m}}^0(M) \subseteq \mathfrak{m}H_{\mathfrak{m}}^0(M) = 0$$

for any system of parameters x_1, \dots, x_d of M . This means that every system of parameters of M is good; therefore it has the property of parametric decomposition by Theorem 1.1.

(ii) \Rightarrow (i). First, it follows by Theorem 1.1 that M is sequentially Cohen-Macaulay. Remember by the definition of the dimension filtration of M that $D_0 = H_{\mathfrak{m}}^0(M)$ and $\dim D_i > 0$ for all $i > 0$. Therefore the implication is proved if we can show that $\mathfrak{m}D_{t-1} = 0$. Suppose the contrary. Then there is an element $x_1 \in \mathfrak{m}$ so that $x_1 D_{t-1} \neq 0$ and $\dim M/x_1 M = d - 1$. Since $d \geq 2$, we can choose $x_2 \in \mathfrak{m}$ such that $x_2 D_{t-1} = 0$ and $\dim M/(x_1, x_2)M = d - 2$. We observe that the sequence x_1, x_2 and $x_1, x_1 + x_2$ are part of systems of parameters of M . Therefore

$$(x_1^2, x_1 + x_2)M \cap (x_1, (x_1 + x_2)^2)M = (x_1, x_2)^2 M = (x_1^2, x_2)M \cap (x_1, x_2^2)M.$$

Since M/D_{t-1} is Cohen-Macaulay, it follows from Lemma 3.1 that

$$\begin{aligned} x_1 D_{t-1} &= (x_1^2, x_1 + x_2) D_{t-1} \cap (x_1, (x_1 + x_2)^2) D_{t-1} \\ &= (x_1^2, x_2) D_{t-1} \cap (x_1, x_2^2) D_{t-1} = x_1^2 D_{t-1}. \end{aligned}$$

Thus $x_1 D_{t-1} = 0$ by Nakayama's lemma, which is impossible. Hence $\mathfrak{m} D_{t-1} = 0$. \square

4. HILBERT-SAMUEL POLYNOMIALS

A parameter ideal \mathfrak{q} is called a *good parameter ideal* if it is generated by a good system of parameters. Then, in this section we shall show that for a sequentially Cohen-Macaulay M the Hilbert-Samuel function $H_{\mathfrak{q},M}(n) = \ell(M/\mathfrak{q}^{n+1}M)$ has a special expression with non-negative coefficients, which can be computed by the dimension filtration, and this function coincides with the Hilbert-Samuel polynomial $P_{\mathfrak{q},M}(n)$ for any good parameter ideal \mathfrak{q} of M and all $n \geq 1$. Moreover, the sequentially Cohen-Macaulayness of M can be characterized by this expression of the Hilbert-Samuel function.

Theorem 4.1. *Let $\mathcal{D} : D_0 \subset D_1 \subset \dots \subset D_t = M$ be the dimension filtration of M and set $\mathcal{D}_i = D_i/D_{i-1}$ for all $1 \leq i \leq t$, $\mathcal{D}_0 = D_0$. Then the following statements are equivalent:*

- (i) M is a sequentially Cohen-Macaulay module.
- (ii) For any good parameter ideal \mathfrak{q} of M , it holds that

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^t \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $n \geq 0$.

- (iii) There exists a good parameter ideal \mathfrak{q} of M such that

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^t \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $n \geq 0$.

Proof. (i) \Rightarrow (ii). We argue by the induction on the length t of the dimension filtration \mathcal{D} of M . The case $t = 0$ is obvious. Assume that $t > 0$. By virtue of Lemma 3.1, we have a short exact sequence

$$0 \rightarrow D_{t-1}/\mathfrak{q}^{n+1}D_{t-1} \rightarrow M/\mathfrak{q}^{n+1}M \rightarrow M/\mathfrak{q}^{n+1}M + D_{t-1} \rightarrow 0.$$

Therefore, we have $\ell(M/\mathfrak{q}^{n+1}M) = \ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) + \ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t)$. Since D_{t-1} is a sequentially Cohen-Macaulay module and its dimension filtration is of length $t-1$, it follows from the inductive hypothesis that

$$\ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) = \sum_{i=0}^{t-1} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i).$$

Notice that \mathcal{D}_t is Cohen-Macaulay of dimension $d = d_t$, so we have

$$\ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t) = \binom{n+d}{d} \ell(\mathcal{D}_t/\mathfrak{q}\mathcal{D}_t).$$

Hence

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^t \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i),$$

for all $n \geq 0$.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). Since the following sequence is exact:

$$D_{t-1}/\mathfrak{q}^{n+1}D_{t-1} \rightarrow M/\mathfrak{q}^{n+1}M \rightarrow M/\mathfrak{q}^{n+1}M + D_{t-1} \rightarrow 0,$$

we get $\ell(M/\mathfrak{q}^{n+1}M) \leq \ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) + \ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t)$. Therefore, by induction on the length of the dimension filtration we can show that

$$\ell(M/\mathfrak{q}^{n+1}M) \leq \sum_{i=0}^t \ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i).$$

On the other hand, since

$$\ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) \leq \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $0 \leq i \leq t$, it follows from the hypothesis that

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^t \ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) = \sum_{i=0}^t \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i).$$

Therefore $\ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) = \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$ for all $n \geq 0$ and $0 \leq i \leq t$. Thus \mathcal{D}_i is Cohen-Macaulay for all $0 \leq i \leq t$, and this completes the proof. \square

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