ON CLOSED $\delta$-PINCHED MANIFOLDS WITH DISCRETE ABELIAN GROUP ACTIONS

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Abstract. Let $M^n$ be a closed odd $n$-manifold with sectional curvature $\delta < \sec_M \leq 1$, and let $M$ admit an effective isometric $\mathbb{Z}_p^k$-action with $p$ prime. The main results in the paper are: (1) if $\delta > 0$ and $n \geq 5$, then there exists a constant $p(n, \delta)$, depending only on $n$ and $\delta$, such that $p \geq p(n, \delta)$ implies that (i) $k \leq \frac{n+1}{2}$, (ii) the universal covering space of $M$ is homeomorphic to $S^n$ if $k > \frac{3}{8}n + 1$, (iii) the fundamental group $\pi_1(M)$ is cyclic if $k > \frac{n+1}{4} + 1$; (2) if $\delta = 0$ and $n = 3$, then $k \leq 4$ for $p = 2$ and $k \leq 2$ for $p \geq 3$, and $\pi_1(M)$ is cyclic if $p \geq 5$ and $k = 2$.

0. Introduction

In the past decade, the closed positively curved $n$-manifold, $M^n$, with symmetry has been researched. Much investigation focuses on the case that $M^n$ admits an effective isometric torus $T^k$-action with $k$ large ([8], [3], [5], [13]–[16], [18]). In this field, the original work is by Grove and Searle [8], and a dramatic improvement after it is the work by Wilking [18]. Roughly, their work shows that the manifold $M$ is constrained to have the cohomology of a symmetry space if $k$ is large enough. (The present paper was inspired by and coincides with this line.) A natural step is to further replace the $T^k$-action by a disconnected group action. Fang and Rong [4] studied $M^n$ which admits an effective isometric $\mathbb{Z}_p^k$-action or a $T^1 \oplus \mathbb{Z}_p^k$-action with $p$ prime for $n$ even or odd respectively, where $p \geq p(n)$, a constant depending only on $n$. A natural problem [4] is: to research $M^n$ with $n$ odd which admits an effective isometric $\mathbb{Z}_p^k$-action.

Due to the problem above, the present paper obtains the following result.

Theorem A. Let $M^n$ be a closed odd $n$-manifold with sectional curvature $0 < \delta < \sec_M \leq 1$ and $n \geq 5$. Suppose $M$ admits an effective isometric $\mathbb{Z}_p^k$-action. Then there exists a constant $p(n, \delta)$, depending only on $n$ and $\delta$, such that $p \geq p(n, \delta)$ implies:

1. $k \leq \frac{n+1}{2}$.
2. The universal covering space of $M$ is homeomorphic to $S^n$ if $k > \frac{3}{8}n + 1$.
3. The fundamental group $\pi_1(M)$ is cyclic if $k > \frac{n+1}{4} + 1$.

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When \( n \geq 5 \), it is easy to check that ‘\( k = \frac{n+1}{2} \)’ implies ‘\( k > \frac{3}{8}n + 1 \)’, and that ‘\( k > \frac{3}{8}n + 1 \)’ implies ‘\( k > \frac{n+1}{4} + 1 \)’. Thus if \( k = \frac{n+1}{2} \), or if \( k > \frac{3}{8}n + 1 \) in Theorem A, then \( M \) is homeomorphic to \( S^n/\mathbb{Z}_h \) (but we cannot make sure whether \( Z_h \) is conjugate to a linear action).

Remark 0.2. Theorem A was originally inspired by the results in [8] and [4]. Let \( \omega \) be a closed \( n \)-manifold of positive sectional curvature. The main result in [8] asserts that if \( M^n \) admits an effective isometric torus \( T^k \)-action, then \( k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \) and ‘=’ implies that \( M \) is diffeomorphic to a sphere, a lens space or a complex projective space. The main results in [4] are: there exists a constant \( p(n) \), depending only on \( n \), such that if a simply connected \( M^n \) admits an effective isometric \( \mathbb{Z}_p^k \)-action or \( T^1 \oplus \mathbb{Z}_p^k \)-action with prime \( p \geq p(n) \) for \( n = 2m \) or \( 2m + 1 \) respectively, then \( k \leq m \); and if \( k = m \) in addition or if \( m \geq 7 \) and \( k \geq \left\lceil \frac{3m}{2} \right\rceil + 2 \), then \( M \) is homeomorphic to a sphere or a complex projective space.

When \( n = 3 \), we get the following result (mainly due to the Hamilton’s work. See Theorem 5.1 below).

**Theorem B.** Let \( M^3 \) be a closed \( 3 \)-manifold of positive sectional curvature. If \( M \) admits an effective isometric \( \mathbb{Z}_p^k \)-action with \( p \) prime, then \( k \leq 4 \) for \( q = 2 \) and \( k \leq 2 \) for \( q \geq 3 \). In addition, \( \pi_1(M) \) is cyclic if \( q \geq 5 \) and \( k = 2 \).

Remark 0.3. In Theorem B, ‘\( q \geq 5 \) and \( k = 2 \)’ is optimal for ‘\( \pi_1(M) \) is cyclic’. One can check that space forms \( S^3/D_q^k \) and \( S^3/T^k \) admit effective isometric \( T^1 \oplus \mathbb{Z}_3 \)- and \( T^1 \oplus \mathbb{Z}_2 \)-actions respectively (ref. [12]), where \( D_q^k \) and \( T^k \) are groups in the proof of the lemma in the Appendix.

The rest of the paper is organized as follows:

In Section 1, we show the choice of \( p(n, \delta) \) in Theorem A.

In Sections 2-4, we will prove parts 1-3 of Theorem A respectively.

In Section 5, we will give the proof of Theorem B.

1. **The choice of \( p(n, \delta) \) in Theorem A**

The choice of \( p(n, \delta) \) in Theorem A is due to the following two results.

**Theorem 1.1** [14]. Let \( M^n \) be a closed \( n \)-manifold with sectional curvature \( 0 < \delta < \sec_M \leq 1 \). Then \( \pi_1(M) \) has a finite normal cyclic subgroup with index less than \( \omega(n, \delta) \), a constant depending only on \( n \) and \( \delta \).

Remark 1.2. X. Rong supplied a conjecture [14]: Let \( M^n \) be a closed \( n \)-manifold of positive sectional curvature. Then \( \pi_1(M) \) has a finite normal cyclic subgroup with index less than \( \omega(n) \), a constant depending only on \( n \). It should be pointed out that, according to the proofs of the present paper, the conclusions in Theorem A will hold for manifolds of positive sectional curvature (i.e., \( \delta = 0 \)) once the conjecture is verified.

**Theorem 1.3** [7]. Let \( M^n \) be a closed \( n \)-manifold with non-negative sectional curvature. Then the total Betti number of \( M \), with respect to any coefficient field, is less than \( c(n) \), a constant depending only on \( n \).

**Assertion.** We choose the constant \( p(n, \delta) \) in Theorem A satisfying that \( p(n, \delta) \geq \max\{\omega(n, \delta), c(n)\} \) and \( p(n + 1, \delta) \geq p(n, \delta) > 2 \).
Remark 1.4. Weinstein’s theorem [2] asserts that a closed positively curved manifold \( M \) of odd dimension is orientable. Thus if \( \mathbb{Z}_p^l \) acts isometrically on \( M \) with prime \( p > 2 \), then \( \mathbb{Z}_p^l \) preserves the orientation of \( M \). In addition, if \( \mathbb{Z}_p^l \) has a non-empty fixed point set, which is totally geodesic, then it is of even codimension.

2. The proof of Part 1 of Theorem A

Lemma 2.1. Let \( M^n \) be a closed \( n \)-manifold with sectional curvature \( 0 < \delta \leq \sec M \leq 1 \). Then the \( \mathbb{Z}_p^2 \) group with prime \( p \geq p(n, \delta) \) cannot act on \( M \) freely and isometrically.

Proof. We argue by contradiction. Assume that \( \mathbb{Z}_p^2 \) acts on \( M \) freely and isometrically. Note that \( M = M/\mathbb{Z}_p^2 \) is also a Riemannian manifold with \( \delta < \sec M \leq 1 \). Let \( \pi : \tilde{M} \to M \) be the universal covering map. Then \( \tilde{M} = \tilde{M}/\pi_1(\tilde{M}) \), and there is the following exact sequence [1, p. 66]:

\[
0 \to \pi_1(M) \to \pi_1(\tilde{M}) \xrightarrow{f} \mathbb{Z}_p \oplus \mathbb{Z}_p \to 0.
\]

Note that we can assume \( \pi_1(\tilde{M}) = \langle \pi_1(M), \alpha, \beta \rangle \) with \( f(\alpha) \) and \( f(\beta) \) being generators of \( \mathbb{Z}_p \oplus \mathbb{Z}_p \). According to the choice of \( p(n, \delta) \) and Theorem 1.1, \( \pi_1(M) \) contains a normal cyclic subgroup, say \( \langle \gamma \rangle \), such that \( [\pi_1(M) : \langle \gamma \rangle] < p \). Then \( \langle \alpha^h, \beta^j \rangle \in \langle \gamma \rangle \) for some \( h, j \geq 0 \) such that \( 0 < h < p \) and \( 0 < j < p \), so \( \langle \alpha^h, \beta^j \rangle \) is a cyclic subgroup. Hence

\[
\langle f(\alpha^h), f(\beta^j) \rangle = \langle (f(\alpha))^h, (f(\beta))^j \rangle = \langle f(\alpha), f(\beta) \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p
\]
is a cyclic group, a contradiction. \( \square \)

Remark 2.2. Using Lemma 2.1, we can get that \( \mathbb{Z}_p^k \) with \( p \geq p(n, \delta) \) has an isotropy subgroup of rank \( k-1 \) if \( M \) admits an effective isometric \( \mathbb{Z}_p^k \) action (see the following proof). According to [1], parts 1 and 2 of Theorem A can be derived from the proofs in [1] if \( M \) is simply connected.

Proof of Part 1 of Theorem A. By Lemma 2.1, \( \mathbb{Z}_p^k \) cannot act freely on \( M \) for \( k \geq 2 \), i.e., we can find an isotropy subgroup \( \mathbb{Z}_p^l \) with \( l \geq 1 \). Take a component \( N \) of \( F(\mathbb{Z}_p^l, M) \), the fixed point set of \( \mathbb{Z}_p^l \).

Claim. \( N = F(\mathbb{Z}_p^l, M) \). Note that \( \alpha(N) \) is also a component of \( F(\mathbb{Z}_p^l, M) \) for any \( \alpha \in \mathbb{Z}_p^l \). If the claim is not true, then \( F(\mathbb{Z}_p^l, M) \) contains at least \( p \) components and thus \( \sum_{i=0}^n \text{rank}(H_i(F(\mathbb{Z}_p^l, M); \mathbb{Z}_p)) \geq 2p \). On the other hand, by Theorem 2.3 below and Theorem 1.3

\[
\sum_{i=0}^n \text{rank}(H_i(F(\mathbb{Z}_p^l, M); \mathbb{Z}_p)) \leq \sum_{i=0}^n \text{rank}(H_i(M; \mathbb{Z}_p)) \leq p.
\]

Thus we get a contradiction.

By Remark 1.4, \( N \) is a totally geodesic submanifold of even codimension. Consider the induced action \( \mathbb{Z}_p^k|_N = \mathbb{Z}_p^k/\mathbb{Z}_p^l \cong \mathbb{Z}_p^{k-1} \) on \( N \). Repeating the process above, we can find \( \mathbb{Z}_p^{k-1} \)-fixed point component \( N_0 \) of codimension \( 2m \).

Note that \( \mathbb{Z}_p^{k-1} \) can act on the normal space of \( N_0 \) as a subgroup of \( SO(2m) \). Then \( 2m \geq 2(k-1) \), i.e., \( k \leq m + 1 \leq \frac{n-1}{2} + 1 = \frac{n+1}{2} \).

\( \square \)

Theorem 2.3 [1 p. 163]. Let the group \( G \cong \mathbb{Z}_q \) with \( q \) prime act on a closed \( n \)-manifold \( M^n \). Then \( \sum_{i=0}^n \text{rank}(H_i(F(G, M); \mathbb{Z}_q)) \leq \sum_{i=0}^n \text{rank}(H_i(M; \mathbb{Z}_q)) \).
We will use the following connectedness theorem by B. Wilking.

**Theorem 3.1** [8]. Let $N^n$ be a closed $n$-manifold of positive sectional curvature, and let $L^j \subset N^n$ be a closed totally geodesic embedded submanifold. Then the inclusion map $L^j \hookrightarrow N^n$ is $(2l - n + 1)$-connected.

**Remark 3.2.** We say that the inclusion map $L \hookrightarrow N$ is $i$-connected if the homotopy groups $\pi_j(N, L) = 0$ for $0 \leq j \leq i$. Then $\pi_i(N) \cong \pi_i(L)$ if $i \geq 2$, and $H_j(N, L; \mathbb{Z}) = 0$ for $0 \leq j \leq i$ (the Hurewicz theorem).

**Corollary 3.3** [8]. Let $N^n$ and $L^j$ be the manifolds in Theorem 3.1. If $n$ is odd and $l = n - 2$, then the universal covering space of $N$ is an integer homology sphere.

**Remark 3.4.** In Theorem A, to prove that the universal covering space of $M$, $\tilde{M}$ is homeomorphic to $S^n$, one only needs to verify that $\tilde{M}$ is an integer homology sphere ([5], [17]).

**Proof of Part 2 of Theorem A.** By Remark 2.2, there is a $\mathbb{Z}_p^{k-1}$-fixed point component $N_0$. Analyzing the representation of $\mathbb{Z}_p^{k-1}$ on the normal space of $N_0$, we can take a $\mathbb{Z}_p$-fixed point component $N$ such that the effective part of $\mathbb{Z}_p^{k-1}|N$ is isomorphic to a $\mathbb{Z}_p^{k-1}$-group.

If $5 \leq n \leq 11$, then $k > \frac{3}{2}n + 1$ implies $k = \frac{n+1}{2}$ (see part 1 of Theorem A), and thus $\dim(N) = n - 2$. Hence the universal covering space of $M$ is homeomorphic to $S^n$ by Corollary 3.3 (see Remark 3.4).

Assume that $n \geq 13$. It is not hard to check that $\dim(N) \geq \frac{3}{2}(n - 1)$ by part 1 of Theorem A (note that $n$ is odd). Then the inclusion map $i : N \hookrightarrow M$ is at least $\frac{n-1}{2}$-connected by Theorem 3.1, so $i_*(\pi_1(N)) = \pi_1(M)$ (see Remark 3.2). Let $\pi : \tilde{M} \to M$ be the universal covering map. Then $\pi^{-1}(N)$ is simply connected, and so $i : \pi^{-1}(N) \hookrightarrow \tilde{M}$ is also $\frac{n-1}{2}$-connected.

On the other hand, we can assume $\dim(N) \leq n - 4$ by Corollary 3.3; then the effective $\mathbb{Z}_p^{k-1}$-action on $N$ satisfies $k - 1 > \frac{3}{2}\dim(N) + 1$. By induction we can assume that $\pi^{-1}(N)$, the universal covering space of $N$, is homeomorphic to a sphere. Then $\tilde{M}$ is an integer homology sphere because $i : \pi^{-1}(N) \hookrightarrow \tilde{M}$ is $\frac{n-1}{2}$-connected (see Remark 3.2). Hence $\tilde{M}$ is homeomorphic to $S^n$ indeed by Remark 3.4.

**4. The proof of Part 3 of Theorem A**

In the proof of Part 3 of Theorem A, we will use:

**Lemma 4.1** [5]. Let $N^n$ ($n \geq 5$) be a closed positively curved $n$-manifold, and let $L$ be a closed totally geodesic embedded submanifold of codimension 2. Then $\pi_1(N)$ is cyclic.

**Proof of Part 3 of Theorem A.** As the proof of Part 2 of Theorem A, take a $\mathbb{Z}_p$-fixed point component $N$ which admits an effective $\mathbb{Z}_p^{k-1}$-action.

When $n = 5$ and 7, $k > \frac{4n+1}{4}$ implies $k = \frac{4n+1}{4}$. Then as in the proof of Part 2 of Theorem A, $\dim(N) = n - 2$, so $\pi_1(M)$ is cyclic by Lemma 4.1.

Assume that $n \geq 9$. One can check that $\dim(N) \geq \frac{4n+1}{4}$ by Part 1 of Theorem A. Then the inclusion map $i : N \hookrightarrow M$ is at least 2-connected by Theorem 3.1, so $i_*(\pi_1(N)) = \pi_1(M)$ (see Remark 3.2). On the other hand, we can assume
dim(N) ≤ n − 4 by Lemma 4.1; then the effective \( \mathbb{Z}_p^{k-1} \)-action on \( N \) satisfies \( k - 1 > \frac{\dim(N)+1}{2} + 1 \). By induction we can get that \( \pi_1(N) \) is cyclic, so \( \pi_1(M) \) is cyclic because \( \pi_1(N) \cong \pi_1(M) \).

\[ \Box \]

5. The Proof of Theorem B

In this section, the following remarkable result by R. Hamilton will be a basis.

**Theorem 5.1** \( \square \). Let \( M^3 \) be a closed 3-manifold of positive sectional curvature. If a group \( G \) acts isometrically on \( M \), then \( M \) admits a metric of positive constant sectional curvature for which \( G \) acts isometrically.

Using Theorem 5.1, we give the following lemma.

**Lemma 5.2.** Let \( M^3 \) be a closed 3-manifold of positive sectional curvature. If the isometry group of \( M \), \( \text{Iso}(M) \), contains a \( \mathbb{Z}_q^k \) subgroup with \( q \) prime, then \( k \leq 4 \) for \( q = 2 \) and \( k \leq 2 \) for \( q \geq 3 \).

**Proof.** We first prove that \( k \leq 2 \) for \( q \geq 3 \). We claim that \( \mathbb{Z}_q^2 \) cannot act freely on \( M \). Assuming the claim, we will prove that \( k \leq 2 \). Note that we can assume \( k \geq 2 \). By the claim, we can find \( e \neq \alpha \in \mathbb{Z}_q^k \) such that \( \alpha \) has a non-empty fixed point set and that \( \dim(F(\alpha, M)) = 1 \) (see Remark 1.4). Note that \( M \) is a positive 3-space form by Theorem 5.1; then by Theorem 2.3 and the lemma in the Appendix,

\[
\sum_{i=0}^{1} \text{rank}(H_i(F(\alpha, M); \mathbb{Z}_q)) \leq \sum_{i=0}^{3} \text{rank}(H_i(M; \mathbb{Z}_q)) \leq 4.
\]

Thus \( F(\alpha, M) \) contains at most 2 components; then \( \mathbb{Z}_q^k \) preserves each component of \( F(\alpha, M) \). That is, the isotropy group of \( O \), a component of \( F(\alpha, M) \), contains a \( \mathbb{Z}_q^{k-1} \) subgroup. Therefore \( k \leq 2 \) because \( \mathbb{Z}_q^{k-1} \) can act faithfully on the normal space of \( O \) (note that \( O \) is of codimension 2).

Now we prove the above claim. Assume that \( \mathbb{Z}_q^2 \) acts freely on \( M \). Note that \( M/\mathbb{Z}_q^2 \) is a 3-manifold of positive sectional curvature, and thus it is also a positive 3-space form by Theorem 5.1. Because \( \pi_1(M/\mathbb{Z}_q^2)/\pi_1(M) \cong \mathbb{Z}_q^2 \), \( H_1(M/\mathbb{Z}_q^2; \mathbb{Z}) \) contains a \( \mathbb{Z}_q^3 \) subgroup (recall that \( H_1(M/\mathbb{Z}_q^2; \mathbb{Z}) \cong \pi_1(M/\mathbb{Z}_q^2)/C \), where \( C \) is the commutator subgroup of \( \pi_1(M/\mathbb{Z}_q^2) \)). This is a contradiction to the lemma in the Appendix.

The proof for \( q = 2 \) is similar to the above (the difference is that there exists a \( \mathbb{Z}_3^k \) subgroup (note that we can assume that \( k \geq 4 \)) which preserves the orientation of \( M \) such that it cannot act freely on \( M \). \[ \Box \]

Next we will give Lemma 5.3. Note that Lemmas 5.2 and 5.3 together imply Theorem B.

**Lemma 5.3.** Let \( M^3 \) be a closed 3-manifold of positive sectional curvature. If \( \text{Iso}(M) \) contains a \( \mathbb{Z}_q^2 \) subgroup with prime \( q \geq 5 \), then \( \pi_1(M) \) is cyclic.

Before proving Lemma 5.3, we first observe the following lemma.

**Lemma 5.4.** Let \( M^3 \) be a closed 3-manifold of positive sectional curvature. If \( \text{Iso}(M) \) contains a \( \mathbb{Z}_q^2 \) subgroup with prime \( q \geq 5 \), then \( \text{Iso}(M) \) contains a \( T^1 \oplus \mathbb{Z}_q \) subgroup, and \( \mathbb{Z}_q \) preserves every exceptional \( T^1 \)-orbit.
Proof. We first prove that \( \text{Iso}(M) \) contains a \( T^1 \oplus \mathbb{Z}_q \) subgroup. According to p. 108 in [12], \( k = \text{rank}(\text{Iso}(M)) \geq 1 \); i.e., \( \text{Iso}(M) \) contains a torus \( T^k \) subgroup with \( k \geq 1 \), because \( M \) is a space form of positive constant sectional curvature by Theorem 5.1. If \( \text{rank}(\text{Iso}(M)) \geq 2 \), the conclusion is obvious. If \( \text{rank}(\text{Iso}(M)) = 1 \), recall that the identity component \( \text{Iso}_0(M) \) of \( \text{Iso}(M) \) is \( T^1 \), or \( \text{SO}(3) \), or \( \text{SU}(2) \).

Endow \( \text{Iso}(M) \) with a bi-invariant metric. Then the conjugate map

\[
\text{Iso}(M) \times \text{Iso}_0(M) \rightarrow \text{Iso}_0(M), (g, g_0) \mapsto g g_0 g^{-1}
\]

induces an \( \text{Iso}(M) \subset O(n) \) action on \( T_e(\text{Iso}_0(M)) \), the tangent space at \( e \), where \( n = \text{dim}(\text{Iso}(M)) \). Note that \( n = 1 \) or 3; then the \( \mathbb{Z}_q^2 \subset \text{Iso}(M) \subset O(n) \) action on \( \mathbb{R}^n \) has a non-empty fixed point set. Take a line \( tX \) fixed by \( \mathbb{Z}_q^2 \), where \( X \in T_e(\text{Iso}_0(M)) \). The subgroup \( \exp(tX) \), the closure of \( \exp(tX) \), commutes with the subgroup \( \mathbb{Z}_q^2 \subset \text{Iso}(M) \), where \( \exp : T_e(\text{Iso}_0(M)) \rightarrow \text{Iso}_0(M) \) is the exceptional map. Note that \( \exp(tX) \) contains a \( T^1 \)-subgroup; i.e., we find a \( T^1 \oplus \mathbb{Z}_q \) subgroup.

Assume that there is \( Z_r \subset T^1 \) with \( r \) prime such that \( F(Z_r, M) \neq \emptyset \). Note that \( T^1 \) preserves the orientation of \( M \); then \( \text{dim}(F(Z_r, M)) = 1 \) (see Remark 1.4). By Theorem 2.3 and the lemma in the Appendix

\[
\sum_{i=0}^{1} \text{rank}(H_i(F(Z_r, M); \mathbb{Z}_r)) \leq \sum_{i=0}^{3} \text{rank}(H_i(M; \mathbb{Z}_r)) \leq 6.
\]

Thus \( F(Z_r, M) \) contains at most 3 components. Therefore if \( q \geq 5 \), then the \( \mathbb{Z}_q \) in \( T^1 \oplus \mathbb{Z}_q \) preserves every component of \( F(Z_r, M) \) (note that \( \mathbb{Z}_q \) preserves \( F(Z_r, M) \)).

In the rest, we will only need to give the proof of Lemma 5.3, in which the following results will be used.

**Lemma 5.5** [5]. Let \( M \) be a closed Riemannian manifold on which \( T^1 \) acts isometrically. If there is an isometry \( \phi \) on \( M^* \), then \( \chi(F(\phi, M^*)) = \text{Lef}(\phi; M^*) \), where \( M^* = M/T^1 \).

Recall that the Lefschetz number \( \text{Lef}(\phi; M^*) = \sum_i (-1)^i \text{trace}(\phi_*^i) \), where \( \text{trace}(\phi_*^i) \) is the trace of the induced map by \( \phi \) on \( H_i(M^*; \mathbb{Q}) \). Lemma 5.5 generalizes the result ([10], p. 63): any isometry \( \phi \) on a closed Riemannian manifold \( M \) satisfies \( \chi(F(\phi, M)) = \text{Lef}(\phi; M) \).

**Lemma 5.6** [8]. Let \( M \) be a closed manifold of positive sectional curvature on which \( T^1 \) acts isometrically. If \( T^1 \) has fixed point set of codimension 2, then \( \pi_1(M) \) is cyclic.

Note that Lemma 4.1 is an extending version of Lemma 5.6.

**Lemma 5.7** [1] p. 91. Let \( G \) be a connected compact Lie group, and let \( X \) be an arcwise connected \( G \)-space. Then the projectional map \( p : X \rightarrow X/G \) induces an onto map on fundamental groups.

**Proof of Lemma 5.3.** By Lemma 5.4, \( \text{Iso}(M) \) contains a \( T^1 \oplus \mathbb{Z}_q \) subgroup, and \( \mathbb{Z}_q \) preserves every exceptional \( T^1 \)-orbit.

**Claim.** we can assume that \( M^* \) is homeomorphic to \( S^2 \), where \( M^* = M/T^1 \). Let \( \alpha \) be the generator of \( \mathbb{Z}_q \), and let \( \hat{\alpha} \) denote the induced action by \( \alpha \) on \( M^* \). Assuming the claim, by Lemma 5.5 we can get

\[
\chi(F(\hat{\alpha}, M^*)) = \text{Lef}(\hat{\alpha}; M^*) = 2
\]
(note that \(\alpha\) preserves the orientation of \(M\). See Remark 1.4). Then \(F(\hat{\alpha}, M^*)\) is \(M^*\) or two points. If \(F(\hat{\alpha}, M^*) = M^*\), then \(\alpha\) preserves every \(T^1\)-orbit on \(M\), and thus \(T^1 \oplus \mathbb{Z}_2\)-action is not effective, a contradiction. Then \(F(\hat{\alpha}, M^*)\) contains only two points. In other words, \(\alpha\) preserves only two \(T^1\)-orbits on \(M\). Since \(\alpha\) preserves every exceptional \(T^1\)-orbit on \(M\), there are at most two exceptional orbits on \(M\); i.e., \(M^*\) contains at most two singular points. Hence \(M\) is a gluing of two solid tori, so \(M\) is homeomorphic to a lens space, and thus \(\pi_1(M)\) is cyclic.

Next we will prove the claim above. By Lemma 5.6, we can assume that the \(T^1\)-action on \(M\) has an empty fixed point set. Then \(M^*\) is an orientable manifold of dimension 2. On the other hand, \(\pi_1(M^*)\) is finite by Lemma 5.7. According to the classification of closed orientable surfaces, \(M^*\) is homeomorphic to \(S^2\).

\[\square\]

**Appendix**

**Lemma.** Let \(M^3\) be a positive 3-space form. Then its first homology group \(H_1(M; \mathbb{Z})\) \(\cong 0\), or \(\mathbb{Z}_h\) with \(h \geq 2\), or \(\mathbb{Z}_2 \oplus \mathbb{Z}_l\) with \(l\) odd.

**Proof.** According to p. 111 in [12] (cf. [11]), \(\pi_1(M)\) is one group of the following list: \(C_m, D^*_4, D^*_{2\times(2n+1)}, T^*, T^*_{S, 3k}, O, I^*\) and the direct product of any of these groups with a cyclic group of relatively prime order. In this list,

- \(C_m\) is a cyclic group of order \(m\),
- \(D^*_{4m} = \{x, y | x^2 = (xy)^2 = y^m\}\),
- \(T^*, O, I^* = \{x, y | x^2 = (xy)^2 = y^n, x = 1\}\) for \(n = 3, 4, 5\) respectively,
- \(D^*_{2\times(2n+1)} = \{x, y | x^2 = 1, y^{2n+1} = 1, y = y^{-1}x\}\) with \(k \geq 2\) and \(m \geq 1, k \geq 1\),
- \(T^*_{3k} = \{x, y | x^2 = (xy)^3 = y^2, yz^{-1} = y, zy^{-1} = xz, z^{3k} = 1\}\) with \(k \geq 1\).

Recall that \(H_1(M; \mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]\), where \([\pi_1(M), \pi_1(M)]\) denotes the commutator subgroup of \(\pi_1(M)\). One can check that \(H_1(M; \mathbb{Z}) \cong \mathbb{Z}_m, \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}_{2k}\) and \(\mathbb{Z}_{2k}\) when \(\pi_1(M) \cong C_m, D^*_{4m}\) with \(m\) odd or even, \(T^*, O, I^*, D^*_{2\times(2n+1)}\) and \(T^*_{S, 3k}\) respectively (for example, \([D^*_{4m}, D^*_{4m}] = \{x y x^{-1} y^{-1} = x y x^{-1} y^{-1} = (x y)^2 y^{-m-2} = y^{-2}\}\), so \(D^*_{4m}/[D^*_{4m}, D^*_{4m}] \cong \mathbb{Z}_4\) or \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) for \(m\) odd or even respectively).

In the proof above, \(T^*, O^*\) and \(I^*\) are a binary tetrahedral group of order 24, a binary octahedral group of order 48, and a binary icosahedral group of order 120 respectively.

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**References**


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