ON THE FARRELL COHOMOLOGY OF THE MAPPING CLASS GROUP OF NON-ORIENTABLE SURFACES

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Abstract. We study the unstable cohomology of the mapping class groups $\mathcal{N}_g$ of non-orientable surfaces of genus $g$. In particular, we determine for all genus $g$ and all primes $p$ when the group $\mathcal{N}_g$ is $p$-periodic.

To this purpose we show that $\mathcal{N}_g$ is a subgroup of the mapping class group $\Gamma_{g-1}$ of an orientable surface of genus $g-1$ and deduce that $\mathcal{N}_g$ has finite virtual cohomological dimension. Furthermore, we describe precisely which finite groups of odd order are subgroups of $\mathcal{N}_g$.

1. Introduction

Because of their close relation to moduli spaces of Riemann surfaces, the mapping class groups of orientable surfaces have been the focus of much mathematical research for a long time. Less well studied is the mapping class group of non-orientable surfaces, although recently the study of mapping class groups has also been extended to the non-orientable case. This paper contributes to this programme. While Wahl [W] proved the analogue of Harer’s (co)homology stability in the non-oriented case, we concentrate here on the unstable part of the cohomology. In particular, we study the question of $p$-periodicity.

Recall that a group $G$ of finite virtual cohomological dimension ($vcd$) is said to be $p$-periodic if the $p$-primary component of its Farrell cohomology ring, $\hat{H}^*(G, \mathbb{Z})_{(p)}$, contains an invertible element of positive degree. Farrell cohomology extends Tate cohomology of finite groups to groups of finite $vcd$. In degrees above the $vcd$ it agrees with the ordinary cohomology of the group. For the mapping class group in the oriented case, the question of $p$-periodicity has been examined by Xia [X] and by Glover, Mislin and Xia [GMX]. Here we determine exactly for which genus and prime $p$ the non-orientable mapping class groups are $p$-periodic. In the process we also establish that these groups are of finite cohomological dimension and present a classification theorem for finite group actions on non-orientable surfaces.

Let $\mathcal{N}_g$ be a non-orientable surface of genus $g$, i.e., the connected sum of $g$ projective planes. The associated mapping class group $\mathcal{N}_g$ is defined to be the group of connected components of the group of homeomorphisms of $N_g$. The mapping class groups of the projective plane and the Klein bottle are well known to be the trivial group and the Klein 4-group respectively, namely

\[ \mathcal{N}_1 = \{e\} \quad \text{and} \quad \mathcal{N}_2 = C_2 \times C_2. \]
Throughout this paper we may therefore assume that $g \geq 3$. Our main result can now be stated as follows.

**Theorem 1.1.** $N_g$ is not $p$-periodic in the following two cases.

1. Assume $p = 2$. Then $N_g$ is not $p$-periodic.
2. Assume $p$ is odd and $g \equiv 2 \pmod{p}$. Write $g = lp + 2$ with $t = kp - t$ for some $k > 0$ and $0 \leq t < p$. If $k > t - 2$, then $N_g$ is not $p$-periodic.

In all other cases $N_g$ is $p$-periodic.$

In particular, $N_g$ is $p$-periodic whenever $p$ is odd and $g$ is not equal to $2$ mod $p$.

On the other hand, for odd $p$, $N_g$ is not $p$-periodic for all $g > p^2$ with $g$ equal to $2$ mod $p$.

In outline, we will first show that the mapping class group $N_g$ of a non-orientable surface of genus $g$ is a subgroup of the mapping class group $\Gamma_{g-1}$ of an orientable surface of genus $g-1$. Many properties of $\Gamma_{g-1}$ are thus inherited by $N_g$. In particular it follows that $N_g$ is of finite virtual cohomological dimension and its Farrell cohomology is well-defined. We then recall that a group $G$ is not $p$-periodic precisely when $G$ has a subgroup isomorphic to $C_p \times C_p$, the product of two cyclic groups of order $p$. Motivated by this we prove a classification theorem for actions of finite groups on non-orientable surfaces. From this it is straightforward to deduce necessary and sufficient conditions for $C_p \times C_p$ to act on $N_g$. Finally, we discuss some open questions.

### 2. Preliminaries

Let $\Sigma_{g-1}$ be a closed orientable surface of genus $g-1$, embedded in $\mathbb{R}^3$ such that $\Sigma_{g-1}$ is invariant under reflections in the $xy$-, $yz$-, and $xz$-planes. Define a (orientation-reversing) homeomorphism $J : \Sigma_{g-1} \to \Sigma_{g-1}$ by

$$J(x, y, z) = (-x, -y, -z).$$

$J$ is reflection in the origin. Under the action of $J$ on $\Sigma_{g-1}$, the orbit space is homeomorphic to a non-orientable surface $N_g$ of genus $g$ with associated orientation double cover

$$p : \Sigma_{g-1} \longrightarrow N_g.$$

Let $\Gamma_{g-1}^\pm$ denote the extended mapping class group, i.e. the group of connected components of the homeomorphisms of $\Sigma_{g-1}$, not necessarily orientation-preserving. $\Gamma_{g-1}$ as usual will denote its index 2 subgroup corresponding to the orientation preserving homeomorphisms.

Birman and Chillingworth [BC] give the following description of the mapping class group $N_g$. Let $C(J) \subset \Gamma_{g-1}^\pm$ be the group of connected components of

$$S(J) := \{ \varphi \in \text{Homeo}(\Sigma_{g-1}) | \exists \tilde{\varphi} \text{ isotopic to } \varphi \text{ such that } \tilde{\varphi}J = J\tilde{\varphi} \},$$

the subgroup of homeomorphisms that commute with $J$ up to isotopy. By definition, $J$ generates a normal subgroup of $C(J)$. Birman and Chillingworth identify the quotient group with the mapping class group of the orbit space $N_g = \Sigma_{g-1}/\langle J \rangle$,

$$N_g \cong \frac{C(J)}{\langle J \rangle}.$$

The following result has proved very useful, as many properties of $\Gamma_{g-1}$ are inherited by $N_g$. 

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**Key-Lemma 2.1.** $\mathcal{N}_g$ is isomorphic to a subgroup of $\Gamma_{g-1}$.

*Proof.* Consider the projection

$$\pi : C(J) \longrightarrow \frac{C(J)}{\langle J \rangle} \cong \mathcal{N}_g.$$  

For a subgroup $G$ of $\mathcal{N}_g$ write

$$\pi^{-1}(G) = G^+ \cup G^- \subset C(J),$$

where

$$G^+ := \pi^{-1}(G) \cap \Gamma_{g-1} \quad \text{and} \quad G^- := \pi^{-1}(G) \cap (\Gamma_{g-1} \setminus \Gamma_{g-1}).$$

Note that $G^- = J G^+$. We claim that $\pi|_{G^+} : G^+ \to G$ is an isomorphism. Indeed, injectivity holds, as the only non-zero element $J$ in the kernel of $\pi$ is not an element of $G^+$. Surjectivity is also immediate as every element in $G$ has exactly two pre-images under $\pi$ which differ by $J$. Thus exactly one of them is an element in the orientable mapping class group $\Gamma_{g-1}$, that is, an element of $G^+$. $\square$

Recall that Farrell cohomology is defined only for groups of finite virtual cohomological dimension.

**Corollary 2.2.** The non-orientable mapping class group $\mathcal{N}_g$ has finite virtual cohomological dimension with

$$vcd \mathcal{N}_g \leq 4g - 9.$$  

*Proof.* The mapping class group $\Gamma_{g-1}$ is virtually torsion free. Furthermore, from Harer [H], we know that $\Gamma_{g-1}$ is of finite virtual cohomological dimension $4(g-1)-5$. Hence every subgroup of $\Gamma_{g-1}$ will also have finite virtual cohomological dimension with $vcd$ less than or equal to $4g - 9$, cf. [Br] Exercise 1, p. 229. The corollary now follows from the Key-Lemma. $\square$

## 3. Classifying finite group actions on $\mathcal{N}_g$

The purpose of this section is to give necessary and sufficient criteria for when a finite group is isomorphic to a subgroup of $\mathcal{N}_g$. For the purpose of this paper we are only interested in groups of odd order.

**Theorem 3.1.** Let $\mathcal{N}_g$ denote a non-orientable surface of genus $g$, and let $A$ be a finite group of odd order. Then $A$ is isomorphic to a subgroup of $\text{Homeo}(\mathcal{N}_g)$ if and only if $A$ has partial presentation

$$\langle c_1, \ldots, c_h, y_1, \ldots, y_t | \ldots \rangle$$

such that

1. $h \geq 1$;
2. $\prod_{j=1}^h c_j^2 \prod_{i=1}^t y_i = 1$;
3. the order of $y_i$ in $A$ is $m_i$;
4. the Riemann-Hurwitz equation holds:

$$g - 2 = |A|(h - 2) + |A| \sum_{i=1}^t (1 - \frac{1}{m_i}).$$

The proof of the theorem is an application of the theory of covering spaces. Different versions of the theorem can be found in the literature; see for example [T]. For completeness and convenience for the reader we include a proof.
Proof: Assume $A$ has a partial presentation of the form described in the theorem, and let $N_h$ be a non-orientable surface of genus $h \geq 1$. Represent $N_h$ as a $2h$-sided polygon with sides to be identified in pairs, where the polygon is bounded by the cycle $c_1 c_2 c_3 \ldots c_h$. At a vertex add $t$ (non-intersecting) loops $y_1, \ldots, y_t$ so that the resulting 2-cells bounded by $y_1, \ldots, y_t$ are mutually disjoint and are contained in the polygon; see Figure 1. Choose a direction for each of the loops $y_1, \ldots, y_t$ and call the resulting one-vertex graph $G$. Note that we can give $N_h$ the structure of a CW-complex so that $G$ is cellularly embedded in $N_h$. A covering graph $\tilde{G}$ is obtained from $G$ as follows. Its vertex set and edge set are $A$ and $E \times A$ respectively, where $E$ is the edge set of the graph $G$. If $e$ is an edge of $G$, then the edge $(e, a)$ of $\tilde{G}$ runs from the vertex $a$ to the vertex $ae$. The forgetful map of graphs $p : \tilde{G} \to G$ is a covering map which we now extend to a branched covering map $p : S \to N_h$ of surfaces as follows.

Label the regions of $N_h$ as $D_1, D_2, \ldots, D_t$ and $D_{t+1}$, where $D_1, D_2, \ldots, D_t$ are bounded by the loops $y_1, y_2, \ldots, y_t$ and $D_{t+1}$ is the remaining region. For each cycle $C$ in $G$, $p^{-1}(C)$ is a collection of cycles in $\tilde{G}$. The cycles $y_i$ have $\frac{|A|}{m_i}$ corresponding cycles in $\tilde{G}$, for each $i \in \{1, \ldots, t\}$. Finally, the cycle $c_1 c_2 c_3 \ldots c_h y_1 \ldots y_t$, bounding $D_{t+1}$, has $|A|$ cycles above it in $\tilde{G}$, because the order of $\prod_{j=1}^{h} c_j \prod_{i=1}^{t} y_i$ in $A$ is 1.

To each of these cycles in $\tilde{G}$ attach a 2-cell. Then extend $p$ by mapping the interior of each 2-cell onto the interior of the 2-cell $D_n$ by using the maps $z \to z^d$, where $d = m_i$ for $n \in \{1, \ldots, t\}$ and $d = 1$ for $n = t + 1$. We obtain a surface $S$ which admits a CW-structure with $\tilde{G}$ cellularly embedded in $S$.

We now argue by contradiction that $S$ is non-orientable. Suppose that $S$ is an orientable surface, and let $A^0 \subset A$ be the subgroup of homeomorphisms which preserve the orientation. Now $A^0 \neq A$ since $N_g$ is non-orientable. So, $A^0$ is a subgroup of index 2 in $A$, which contradicts our assumption that $A$ is of odd order. So $S$ is non-orientable. Finally, its genus $g$ is determined by the Riemann-Hurwitz formula, condition (4).

Conversely, assume $A$ acts on the non-orientable surface $S = N_g$. As $A$ is of odd order, $A$ acts without reflections and its singular set is discrete. Thus the quotient map $p : S \to S/A$ is a branched covering, and $S/A$ is a non-orientable surface of genus $h \geq 1$. Represent $S/A$ as a $2h$-sided polygon with sides $c_1, c_2, c_3, \ldots, c_h$ to be identified in pairs, and in which the branch points of $p$ are in the interior of

![Figure 1. A non-orientable surface of genus $h$.](image)
the polygon. Now add mutually disjoint loops \(y_1, \ldots, y_t\) around each branch point, all starting at the same vertex as indicated in Figure 1. Let us call the resulting one-vertex graph \(G\). Its inverse image \(p^{-1}(G)\) is a Cayley graph for the group \(A\): vertices correspond to the elements of \(A\) and at each vertex there are \(2(h + t)\) directions corresponding to generators \(c_i\) and \(y_i\). The three conditions for the partial presentations are easily verified. First note that \(h\) is positive as \(S/A\) is non-orientable. As \(\prod_{j=1}^{h} c_j \prod_{i=1}^{t} y_i = 1\) is a closed curve in \(S/A\), so it is in \(S\) and hence must represent the identity in \(A\). The order \(m_i\) of \(y_i\) is precisely the branch number of the singular point that \(y_i\) encircles. Thus the formula in condition (4) follows from the Riemann-Hurwitz equation.

As we are interested in subgroups of the mapping class group, we state the following result, which is well-known at least for orientable surfaces.

**Theorem 3.2.** A finite group \(G\) is a subgroup of the mapping class group \(N_g\) if and only if it is a subgroup of \(\text{Homeo}(N_g)\).

**Proof.** If \(G\) is a finite subgroup of \(N_g\), then it follows by the Nielsen realisation problem for non-orientable surfaces \([K]\) that \(G\) lifts to a subgroup of \(\text{Homeo}(N_g)\). Conversely, let \(G\) be a finite subgroup of \(\text{Homeo}(N_g)\). An application of the Lefschetz Fixed Point Formula shows that for all \(g \geq 3\), any element of finite order in \(\text{Homeo}(N_g)\) cannot be homotopic to the identity. Hence the kernel of the canonical projection \(\text{Homeo}(N_g) \to N_g\) when restricted to a finite subgroup \(G \in \text{Homeo}(N_g)\) must be trivial.

Theorem 3.1 and Theorem 3.2 together imply that a finite group \(A\) of odd order is a subgroup of the mapping class group \(N_g\) if and only if it has partial presentation such that conditions (1) to (4) in Theorem 3.1 hold.

### 4. The \(p\)-periodicity of \(N_g\)

Using the result of the previous section we can now prove our main result. Theorem 1.1 is equivalent to the following three lemmata. Recall (see e.g. \([B]\) Theorem 6.7) that a group of finite \(vcd\) is \(p\)-periodic if and only if it does not contain an elementary abelian subgroup of rank two.

**Lemma 4.1.** \(N_g\) is not \(2\)-periodic.

**Proof.** It will suffice to exhibit a subgroup of \(N_g\) isomorphic to \(C_2 \times C_2\). Let \(R_1\) and \(R_2\) be homeomorphisms of \(\Sigma_{g-1}\) (embedded in \(\mathbb{R}^3\) as before) which are rotations by \(\pi\), given by the formulæ

\[
R_1(x, y, z) = (-x, -y, z),
\]

\[
R_2(x, y, z) = (x, -y, -z).
\]

Clearly, \(J\), \(R_1\) and \(R_2\) are all involutions. For \(g \geq 3\) the induced actions on the first homology groups \(H_1(\Sigma_{g-1})\) are non-trivial and all different; they define non-trivial, distinct elements of order two in \(\Gamma_{g-1}^\pm\). From their defining formulæ it is clear that they commute with each other. Hence, they generate a subgroup

\[H = C_2 \times C_2 \times C_2 \subset C(J) \subset \Gamma_{g-1}^\pm,\]

and thus

\[\pi(H) \cong C_2 \times C_2 \subset \frac{C(J)}{J} \cong N_g.\]
Thus $\mathcal{N}_g$ is never $2$-periodic, \hfill \Box

**Lemma 4.2.** Let $p$ be odd. $\mathcal{N}_g$ is not $p$-periodic in the following three cases.

1. If $g = lp + 2$ and $l = kp$ where $k > 0$, then $\mathcal{N}_g$ is not $p$-periodic.
2. If $g = lp + 2$ and $l = kp - 1$ where $k > 0$, then $\mathcal{N}_g$ is not $p$-periodic.
3. If $g = 2p + 2$, $l = kp - t$ and $k > t - 2$ where $2 \leq t < p$, then $\mathcal{N}_g$ is not $p$-periodic.

**Proof.** In all three cases we will use Theorem 3.1 (and Theorem 3.2) to exhibit subgroups $C_p \times C_p \subset \mathcal{N}_g$. Hence, in these cases $\mathcal{N}_g$ is not $p$-periodic.

**Case (1):** Let $h := k + 2 \geq 3$. A presentation of $A = C_p \times C_p = \langle c_1 \rangle \times \langle c_2 \rangle$ can now be given as follows:

$$A = \langle c_1, \ldots, c_h \rangle c_3 = c_1^{p-1}c_2^{p-1}, c_4 = \ldots = c_h = 1, c_1c_2 = c_2c_1, c_1^p = c_2^p = 1 \rangle.$$ 

One checks that the four conditions of Theorem 3.1 are satisfied; here

$$g - 2 = p^2(h - 2).$$

**Case (2):** Let $h := k + 1 \geq 2$. A presentation of $A = C_p \times C_p = \langle c_1 \rangle \times \langle c_2 \rangle$ is given by

$$A = \langle c_1, \ldots, c_h, y_1 | y_1 = c_1^{p-2}c_2^{p-2}, c_3 = \ldots = c_h = 1, c_1c_2 = c_2c_1, c_1^p = c_2^p = 1 \rangle.$$

Again one easily checks that the four conditions of Theorem 3.1 are satisfied; in this case

$$g - 2 = p^2(h - 2) + p^2(1 - \frac{1}{p}).$$

**Case (3):** Let $h := k + 2 - t \geq 1$. As also $t \geq 2$, a presentation of $A = C_p \times C_p = \langle y_1 \rangle \times \langle y_2 \rangle$ is now given by

$$A = \langle c_1, \ldots, c_h, y_1, y_2, \ldots, y_t | c_1 = y_1^\frac{1}{p} y_2^\frac{1}{p} \ldots y_t^\frac{1}{p}, y_2 = y_3 = \ldots = y_t, c_2 = c_3 = \ldots = c_h = 1, y_1y_2 = y_2y_1, y_1^p = y_2^p = 1 \rangle.$$

This presentation satisfies the conditions of Theorem 3.1 with

$$g - 2 = p^2(h - 2) + p^2t(1 - \frac{1}{p}).$$

Hence in all these three cases, i.e. whenever condition (2) of Theorem 1.1 holds, the mapping class group $\mathcal{N}_g$ is not $p$-periodic. \hfill \Box

**Lemma 4.3.** Let $p$ be odd and assume that $g$ does not satisfy any of the three conditions of Lemma 4.2; then $\mathcal{N}_g$ is $p$-periodic.

**Proof.** Let $p$ be odd and suppose that there exists a subgroup $A = C_p \times C_p$ contained in $\mathcal{N}_g$. Then by Theorem 3.1 (and Theorem 3.2), $A$ acts on $N_g$ and the Riemann-Hurwitz Formula must be satisfied for some $h \geq 1$ where $h$ is the genus of the quotient surface $N_g/A$. ($h$ cannot be zero as the sphere cannot arise as the quotient of a non-oriented surface.) Let $s$ be the number of singular points of the action of $A$ on $N_g$, and let $a$ be an element in the stabiliser of some singular point $x$. By Key-Lemma 2.1, $a$ lifts to an element of $\Gamma_{g-1}$ and by the Nielsen realization problem to a homeomorphism, also denoted by $a$, of $\Sigma_{g-1}$. The singular point $x$ lifts to two points in $\Sigma_{g-1}$, and under the action of $a$ these form two separate orbits as the group $A$ and hence the element $a$ are of odd order. So $a$ is in the stabiliser of these two points and therefore must act freely on the tangent planes at these points.
Remark 4.4. A group is $p$-periodic if and only if it does not contain a subgroup isomorphic to $C_p \times C_p$. Therefore, any subgroup of a $p$-periodic group is $p$-periodic. Hence by the Key-Lemma 2.1, the $p$-periodicity of any $\Gamma_{g-1}$ implies the $p$-periodicity of $\mathcal{N}_g$. (In particular, as $\Gamma_{g-1}$ is always $p$ periodic for odd $p$ and $g$ not equal to 2 mod $p$, so is $\mathcal{N}_g$.) However, comparing our results with those of Xia [X], we note here that the converse is false. For example, when $p = 5$ and $g = 7$, $\Gamma_6$ is not $p$-periodic but $\mathcal{N}_7$ is. However, for a fixed $p$ there are at most finitely many such $g$ where $\Gamma_{g-1}$ is not $p$-periodic but $\mathcal{N}_g$ is.

5. The $p$-period and other open questions

We will briefly discuss three questions that arise from our study.

5.1. The $p$-period. Recall that the $p$-period $d$ of a $p$-periodic group $G$ is the least positive degree of an invertible element in its Farrell cohomology group $\tilde{H}^*(G, \mathbb{Z})_{(p)}$. The question thus arises as to what the $p$-period of $\mathcal{N}_g$ is when $\mathcal{N}_g$ is $p$-periodic.

For any group $G$ of finite $vcd$, an invertible element in $\tilde{H}^*(G, \mathbb{Z})_{(p)}$ restricts to an invertible element in the Farrell cohomology of any subgroup of $G$. Thus the $p$-period of a subgroup divides the $p$-period of $G$.

The main result of [GMX] is that for all $g$ such that $\Gamma_{g-1}$ is $p$-periodic, the $p$-period divides $2(p-1)$. Hence for all such $g$, the $p$-period of $\mathcal{N}_g$ also divides $2(p-1)$. However, as we noted above, there are pairs $p$ and $g$ for which $\mathcal{N}_g$ is $p$-periodic but $\Gamma_{g-1}$ is not. We expect that the methods of [GMX] can be pushed to cover also these cases. It remains also to find lower bounds for the $p$-period.

5.2. Punctured mapping class groups. In the oriented case Lu [L1], [L2] has studied the $p$-periodicity of the mapping class groups with marked points, and proved that they are all $p$-periodic of period 2. One might expect a similar result to hold for the mapping class group of non-orientable surfaces with marked points.

5.3. The virtual cohomological dimension. We have established in Corollary 2.2 that $\mathcal{N}_g$ has finite virtual cohomological dimension and that this dimension is less than or equal to $4g - 9$. It seems an interesting project to determine the $vcd$ of $\mathcal{N}_g$.
References


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