ON THE FARRELL COHOMOLOGY OF THE
MAPPING CLASS GROUP OF NON-ORIENTABLE SURFACES

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Abstract. We study the unstable cohomology of the mapping class groups
$N_g$ of non-orientable surfaces of genus $g$. In particular, we determine for all
genus $g$ and all primes $p$ when the group $N_g$ is $p$-periodic.

To this purpose we show that $N_g$ is a subgroup of the mapping class group
$\Gamma_{g-1}$ of an orientable surface of genus $g - 1$ and deduce that $N_g$ has finite
virtual cohomological dimension. Furthermore, we describe precisely which
finite groups of odd order are subgroups of $N_g$.

1. Introduction

Because of their close relation to moduli spaces of Riemann surfaces, the map-
ing class groups of orientable surfaces have been the focus of much mathematical
research for a long time. Less well studied is the mapping class group of non-
orientable surfaces, although recently the study of mapping class groups has also
been extended to the non-orientable case. This paper contributes to this pro-
gramme. While Wahl [W] proved the analogue of Harer’s (co)homology stability in
the non-oriented case, we concentrate here on the unstable part of the cohomology.
In particular, we study the question of $p$-periodicity.

Recall that a group $G$ of finite virtual cohomological dimension ($vcd$) is said to be
$p$-periodic if the $p$-primary component of its Farrell cohomology ring, $\hat{H}^*(G, \mathbb{Z})_p$,
contains an invertible element of positive degree. Farrell cohomology extends Tate
cohomology of finite groups to groups of finite $vcd$. In degrees above the $vcd$ it
agrees with the ordinary cohomology of the group. For the mapping class group in
the oriented case, the question of $p$-periodicity has been examined by Xia [X] and
by Glover, Mislin and Xia [GMX]. Here we determine exactly for which genus and
prime $p$ the non-orientable mapping class groups are $p$-periodic. In the process we
also establish that these groups are of finite cohomological dimension and present
a classification theorem for finite group actions on non-orientable surfaces.

Let $N_g$ be a non-orientable surface of genus $g$, i.e. the connected sum of $g$
projective planes. The associated mapping class group $N_g$ is defined to be the group
of connected components of the group of homeomorphisms of $N_g$. The mapping
class groups of the projective plane and the Klein bottle are well known to be the
trivial group and the Klein 4-group respectively, namely

$N_1 = \{e\}$ and $N_2 = C_2 \times C_2$. 

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Throughout this paper we may therefore assume that \( g \geq 3 \). Our main result can now be stated as follows.

**Theorem 1.1.** \( \mathcal{N}_g \) is not \( p \)-periodic in the following two cases.

1. Assume \( p = 2 \). Then \( \mathcal{N}_g \) is not \( p \)-periodic.
2. Assume \( p \) is odd and \( g \equiv 2 \pmod{p} \). Write \( g = lp + 2 \) with \( l = kp - t \) for some \( k > 0 \) and \( 0 \leq t < p \). If \( k > t - 2 \), then \( \mathcal{N}_g \) is not \( p \)-periodic.

In all other cases \( \mathcal{N}_g \) is \( p \)-periodic.

In particular, \( \mathcal{N}_g \) is \( p \)-periodic whenever \( p \) is odd and \( g \) is not equal to 2 mod \( p \).

On the other hand, for odd \( p \), \( \mathcal{N}_g \) is not \( p \)-periodic for all \( g > p^3 \) with \( g \) equal to 2 mod \( p \).

In outline, we will first show that the mapping class group \( \mathcal{N}_g \) of a non-orientable surface of genus \( g \) is a subgroup of the mapping class group \( \Gamma_{g-1} \) of an orientable surface of genus \( g - 1 \). Many properties of \( \Gamma_{g-1} \) are thus inherited by \( \mathcal{N}_g \). In particular it follows that \( \mathcal{N}_g \) is of finite virtual cohomological dimension and its Farrell cohomology is well-defined. We then recall that a group \( G \) is not \( p \)-periodic precisely when \( G \) has a subgroup isomorphic to \( \mathbb{C}_p \times \mathbb{C}_p \), the product of two cyclic groups of order \( p \). Motivated by this we prove a classification theorem for actions of finite groups on non-orientable surfaces. From this it is straightforward to deduce necessary and sufficient conditions for \( \mathbb{C}_p \times \mathbb{C}_p \) to act on \( \mathcal{N}_g \). Finally, we discuss some open questions.

### 2. Preliminaries

Let \( \Sigma_{g-1} \) be a closed orientable surface of genus \( g - 1 \), embedded in \( \mathbb{R}^3 \) such that \( \Sigma_{g-1} \) is invariant under reflections in the \( xy \)-, \( yz \)-, and \( xz \)-planes. Define a (orientation-reversing) homeomorphism \( J : \Sigma_{g-1} \to \Sigma_{g-1} \) by

\[
J(x, y, z) = (-x, -y, -z).
\]

\( J \) is reflection in the origin. Under the action of \( J \) on \( \Sigma_{g-1} \), the orbit space is homeomorphic to a non-orientable surface \( \mathcal{N}_g \) of genus \( g \) with associated orientation double cover

\[
p : \Sigma_{g-1} \longrightarrow \mathcal{N}_g.
\]

Let \( \Gamma_{g-1}^\pm \) denote the extended mapping class group, i.e. the group of connected components of the homeomorphisms of \( \Sigma_{g-1} \), not necessarily orientation-preserving. \( \Gamma_{g-1} \) as usual will denote its index 2 subgroup corresponding to the orientation preserving homeomorphisms.

Birman and Chillingworth [BC] give the following description of the mapping class group \( \mathcal{N}_g \). Let \( C(J) \subset \Gamma_{g-1}^\pm \) be the group of connected components of

\[
S(J) := \{ \varphi \in \text{Homeo}(\Sigma_{g-1}) | \exists \tilde{\varphi} \text{ isotopic to } \varphi \text{ such that } \tilde{\varphi}J = J \tilde{\varphi} \},
\]

the subgroup of homeomorphisms that commute with \( J \) up to isotopy. By definition, \( J \) generates a normal subgroup of \( C(J) \). Birman and Chillingworth identify the quotient group with the mapping class group of the orbit space \( \mathcal{N}_g = \Sigma_{g-1}/\langle J \rangle \),

\[
\mathcal{N}_g \cong \frac{C(J)}{\langle J \rangle}.
\]

The following result has proved very useful, as many properties of \( \Gamma_{g-1} \) are inherited by \( \mathcal{N}_g \).
Key-Lemma 2.1. $N_g$ is isomorphic to a subgroup of $\Gamma_{g-1}$.

Proof. Consider the projection

$$\pi : C(J) \rightarrow \frac{C(J)}{(J)} \cong N_g.$$ 

For a subgroup $G$ of $N_g$ write

$$\pi^{-1}(G) = G^+ \cup G^- \subset C(J),$$

where

$$G^+ := \pi^{-1}(G) \cap \Gamma_{g-1} \quad \text{and} \quad G^- := \pi^{-1}(G) \cap (\Gamma_{g-1}^+ \setminus \Gamma_{g-1}).$$

Note that $G^- = JG^+$. We claim that $\pi|_{G^+} : G^+ \rightarrow G$ is an isomorphism. Indeed, injectivity holds, as the only non-zero element $J$ in the kernel of $\pi$ is not an element of $G^+$. Surjectivity is also immediate as every element in $G$ has exactly two pre-images under $\pi$ which differ by $J$. Thus exactly one of them is an element in the orientable mapping class group $\Gamma_{g-1}$, that is, an element of $G^+$. \[ \square \]

Recall that Farrell cohomology is defined only for groups of finite virtual cohomological dimension.

Corollary 2.2. The non-orientable mapping class group $N_g$ has finite virtual cohomological dimension with

$$vcd N_g \leq 4g - 9.$$ 

Proof. The mapping class group $\Gamma_{g-1}$ is virtually torsion free. Furthermore, from Harer [H], we know that $\Gamma_{g-1}$ is of finite virtual cohomological dimension $4(g-1) - 5$. Hence every subgroup of $\Gamma_{g-1}$ will also have finite virtual cohomological dimension with $vcd$ less than or equal to $4g - 9$, cf. [Br, Exercise 1, p. 229]. The corollary now follows from the Key-Lemma. \[ \square \]

3. Classifying finite group actions on $N_g$

The purpose of this section is to give necessary and sufficient criteria for when a finite group is isomorphic to a subgroup of $N_g$. For the purpose of this paper we are only interested in groups of odd order.

Theorem 3.1. Let $N_g$ denote a non-orientable surface of genus $g$, and let $A$ be a finite group of odd order. Then $A$ is isomorphic to a subgroup of $\text{Homeo}(N_g)$ if and only if $A$ has partial presentation

$$(c_1, \ldots, c_h, y_1, \ldots, y_t | \ldots)$$

such that

1. $h \geq 1$;
2. $\prod_{i=1}^{h} c_i^2 \prod_{i=1}^{t} y_i = 1$;
3. the order of $y_i$ in $A$ is $m_i$;
4. the Riemann-Hurwitz equation holds:

$$g - 2 = |A|(h - 2) + |A| \sum_{i=1}^{t} (1 - \frac{1}{m_i}).$$

The proof of the theorem is an application of the theory of covering spaces. Different versions of the theorem can be found in the literature; see for example [I]. For completeness and convenience for the reader we include a proof.
Proof: Assume \( A \) has a partial presentation of the form described in the theorem, and let \( N_h \) be a non-orientable surface of genus \( h \geq 1 \). Represent \( N_h \) as a \( 2h \)-sided polygon with sides to be identified in pairs, where the polygon is bounded by the cycle \( c_1c_2c_3\ldots c_hc_1 \). At a vertex add \( t \) (non-intersecting) loops \( y_1, \ldots, y_t \) so that the resulting 2-cells bounded by \( y_1, \ldots, y_t \) are mutually disjoint and are contained in the polygon; see Figure 1. Choose a direction for each of the loops \( y_1, \ldots, y_t \) and call the resulting one-vertex graph \( G \). Note that we can give \( N_h \) the structure of a CW-complex so that \( G \) is cellularly embedded in \( N_h \). A covering graph \( \tilde{G} \) is obtained from \( G \) as follows. Its vertex set and edge set are \( A \) and \( E \times A \) respectively, where \( E \) is the edge set of the graph \( G \). If \( e \) is an edge of \( G \), then the edge \((e, a)\) of \( \tilde{G} \) runs from the vertex \( a \) to the vertex \( ae \). The forgetful map of graphs \( p : \tilde{G} \to G \) is a covering map which we now extend to a branched covering map \( p : S \to N_h \) of surfaces as follows.

Label the regions of \( N_h \) as \( D_1, D_2, \ldots, D_t \) and \( D_{t+1} \), where \( D_1, D_2, \ldots, D_t \) are bounded by the loops \( y_1, y_2, \ldots, y_t \) and \( D_{t+1} \) is the remaining region. For each cycle \( C \) in \( G \), \( p^{-1}(C) \) is a collection of cycles in \( \tilde{G} \). The cycles \( y_i \) have \( \frac{|A|}{m_i} \) corresponding cycles in \( \tilde{G} \), for each \( i \in \{1, \ldots, t\} \). Finally, the cycle \( c_1c_2c_3\ldots c_hc_1 \) has \( |A| \) cycles above it in \( \tilde{G} \), because the order of \( \prod_{j=1}^{h-1} c_{j+1}^2 \prod_{i=1}^{t} y_i \) in \( A \) is 1.

Each of these cycles in \( \tilde{G} \) attach a 2-cell. Then extend \( p \) by mapping the interior of each 2-cell onto the interior of the 2-cell \( D_n \) by using the maps \( z \mapsto z^d \), where \( d = m_i \) for \( n \in \{1, \ldots, t\} \) and \( d = 1 \) for \( n = t + 1 \). We obtain a surface \( S \) which admits a CW-structure with \( \tilde{G} \) cellularly embedded in \( S \).

We now argue by contradiction that \( S \) is non-orientable. Suppose that \( S \) is an orientable surface, and let \( A^0 \subset A \) be the subgroup of homeomorphisms which preserve the orientation. Now \( A^0 \neq A \) since \( N_g \) is non-orientable. So, \( A^0 \) is a subgroup of index 2 in \( A \), which contradicts our assumption that \( A \) is of odd order. So \( S \) is non-orientable. Finally, its genus \( g \) is determined by the Riemann-Hurwitz formula, condition (4).

Conversely, assume \( A \) acts on the non-orientable surface \( S = N_g \). As \( A \) is of odd order, \( A \) acts without reflections and its singular set is discrete. Thus the quotient map \( p : S \to S/A \) is a branched covering, and \( S/A \) is a non-orientable surface of genus \( h \geq 1 \). Represent \( S/A \) as a \( 2h \)-sided polygon with sides \( c_1, c_1, c_2, c_2, \ldots, c_h, c_h \) to be identified in pairs, and in which the branch points of \( p \) are in the interior of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A non-orientable surface of genus \( h \).}
\end{figure}
the polygon. Now add mutually disjoint loops \( y_1, \ldots, y_t \) around each branch point, all starting at the same vertex as indicated in Figure 1. Let us call the resulting one-vertex graph \( G \). Its inverse image \( p^{-1}(G) \) is a Cayley graph for the group \( A \): vertices correspond to the elements of \( A \) and at each vertex there are \( 2(h + t) \) directions corresponding to generators \( c_i \) and \( y_i \). The three conditions for the partial presentations are easily verified. First note that \( h \) is positive as \( S/A \) is non-orientable. As \( \prod_{j=1}^{h} c_j \prod_{i=1}^{t} y_i = 1 \) is a closed curve in \( S/A \), so it is in \( S \) and hence must represent the identity in \( A \). The order \( m_i \) of \( y_i \) is precisely the branch number of the singular point that \( y_i \) encircles. Thus the formula in condition (4) follows from the Riemann-Hurwitz equation. \( \square \)

As we are interested in subgroups of the mapping class group, we state the following result, which is well-known at least for orientable surfaces.

**Theorem 3.2.** A finite group \( G \) is a subgroup of the mapping class group \( N_g \) if and only if it is a subgroup of \( \text{Homeo}(N_g) \).

**Proof.** If \( G \) is a finite subgroup of \( N_g \), then it follows by the Nielsen realisation problem for non-orientable surfaces \([K]\) that \( G \) lifts to a subgroup of \( \text{Homeo}(N_g) \). Conversely, let \( G \) be a finite subgroup of \( \text{Homeo}(N_g) \). An application of the Lefschetz Fixed Point Formula shows that for all \( g \geq 3 \), any element of finite order in \( \text{Homeo}(N_g) \) cannot be homotopic to the identity. Hence the kernel of the canonical projection \( \text{Homeo}(N_g) \to N_g \) when restricted to a finite subgroup \( G \in \text{Homeo}(N_g) \) must be trivial. \( \square \)

Theorem 3.1 and Theorem 3.2 together imply that a finite group \( A \) of odd order is a subgroup of the mapping class group \( N_g \) if and only if it has partial presentation such that conditions (1) to (4) in Theorem 3.1 hold.

### 4. The \( p \)-periodicity of \( N_g \)

Using the result of the previous section we can now prove our main result. Theorem 1.1 is equivalent to the following three lemmata. Recall (see e.g. \([Br\], Theorem 6.7\)) that a group of finite vcd is \( p \)-periodic if and only if it does not contain an elementary abelian subgroup of rank two.

**Lemma 4.1.** \( N_g \) is not 2-periodic.

**Proof.** It will suffice to exhibit a subgroup of \( N_g \) isomorphic to \( C_2 \times C_2 \). Let \( R_1 \) and \( R_2 \) be homeomorphisms of \( \Sigma_{g-1} \) (embedded in \( \mathbb{R}^3 \) as before) which are rotations by \( \pi \), given by the formulæ

\[
R_1(x, y, z) = (-x, -y, z),
R_2(x, y, z) = (x, -y, -z).
\]

Clearly, \( J, R_1 \) and \( R_2 \) are all involutions. For \( g \geq 3 \) the induced actions on the first homology groups \( H_1(\Sigma_{g-1}) \) are non-trivial and all different; they define non-trivial, distinct elements of order two in \( \Gamma_{g-1}^\pm \). From their defining formulæ it is clear that they commute with each other. Hence, they generate a subgroup

\[
H = C_2 \times C_2 \times C_2 \subset C(J) \subset \Gamma_{g-1}^\pm,
\]

and thus

\[
\pi(H) \cong C_2 \times C_2 \subset \frac{C(J)}{(J)} \cong N_g.
\]

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Thus $\mathcal{N}_g$ is never 2-periodic, □

**Lemma 4.2.** Let $p$ be odd. $\mathcal{N}_g$ is not $p$-periodic in the following three cases.

1. If $g = lp + 2$ and $l = kp$ where $k > 0$, then $\mathcal{N}_g$ is not $p$-periodic.
2. If $g = lp + 2$ and $l = kp - 1$ where $k > 0$, then $\mathcal{N}_g$ is not $p$-periodic.
3. If $g = lp + 2$, $l = kp - t$ and $k > t - 2$ where $2 \leq t < p$, then $\mathcal{N}_g$ is not $p$-periodic.

**Proof.** In all three cases we will use Theorem 3.1 (and Theorem 3.2) to exhibit subgroups $C_p \times C_p \subset \mathcal{N}_g$. Hence, in these cases $\mathcal{N}_g$ is not $p$-periodic.

**Case (1):** Let $h := k + 2 \geq 3$. A presentation of $A = C_p \times C_p = \langle c_1 \rangle \times \langle c_2 \rangle$ can now be given as follows:

$$A = \langle c_1, \ldots, c_h | c_3 = c_1^{p-1}c_2^{-1}, c_4 = \ldots = c_h = 1, c_1c_2 = c_2c_1, c_1^p = c_2^p = 1 \rangle.$$ 

One checks that the four conditions of Theorem 3.1 are satisfied; here

$$g - 2 = p^2(h - 2).$$

**Case (2):** Let $h := k + 1 \geq 2$. A presentation of $A = C_p \times C_p = \langle c_1 \rangle \times \langle c_2 \rangle$ is given by

$$A = \langle c_1, \ldots, c_h, y_1 | y_1 = c_1^{p-2}c_2^{p-2}, c_3 = \ldots = c_h = 1, c_1c_2 = c_2c_1, c_1^p = c_2^p = 1 \rangle.$$ 

Again one easily checks that the four conditions of Theorem 3.1 are satisfied; in this case

$$g - 2 = p^2(h - 2) + p^2(1 - \frac{1}{p}).$$

**Case (3):** Let $h := k + 2 - t \geq 1$. As also $t \geq 2$, a presentation of $A = C_p \times C_p = \langle y_1 \rangle \times \langle y_2 \rangle$ is now given by

$$A = \langle c_1, \ldots, c_h, y_1, y_2, \ldots, y_t | c_1 = \frac{y_1}{y_2} \frac{y_3}{y_4} \ldots \frac{y_{t-1}}{y_t}, y_2y_3 = \ldots = y_t, c_2 = c_3 = \ldots = c_h = 1, y_1y_2 = y_2y_1, y_1^p = y_2^p = 1 \rangle.$$ 

This presentation satisfies the conditions of Theorem 3.1 with

$$g - 2 = p^2(h - 2) + p^2t(1 - \frac{1}{p}).$$

Hence in all these three cases, i.e. whenever condition (2) of Theorem 1.1 holds, the mapping class group $\mathcal{N}_g$ is not $p$-periodic. □

**Lemma 4.3.** Let $p$ be odd and assume that $g$ does not satisfy any of the three conditions of Lemma 4.2; then $\mathcal{N}_g$ is $p$-periodic.

**Proof.** Let $p$ be odd and suppose that there exists a subgroup $A = C_p \times C_p$ contained in $\mathcal{N}_g$. Then by Theorem 3.1 (and Theorem 3.2), $A$ acts on $\mathcal{N}_g$ and the Riemann-Hurwitz Formula must be satisfied for some $h \geq 1$ where $h$ is the genus of the quotient surface $\mathcal{N}_g/A$. ($h$ cannot be zero as the sphere cannot arise as the quotient of a non-oriented surface.) Let $s$ be the number of singular points of the action of $A$ on $\mathcal{N}_g$, and let $a$ be an element in the stabiliser of some singular point $x$. By Key-Lemma 2.1, $a$ lifts to an element of $\Gamma_{g-1}$ and by the Nielsen realization problem to a homeomorphism, also denoted by $a$, of $\Sigma_{g-1}$. The singular point $x$ lifts to two points in $\Sigma_{g-1}$, and under the action of $a$ these form two separate orbits as the group $A$ and hence the element $a$ are of odd order. So $a$ is in the stabiliser of these two points and therefore must act freely on the tangent planes at these points.
Remark 4.4. A group is $p$-periodic if and only if it does not contain a subgroup isomorphic to $C_p \times C_p$. Therefore, any subgroup of a $p$-periodic group is $p$-periodic. Hence by the Key-Lemma 2.1, the $p$-periodicity of any $\Gamma_{g-1}$ implies the $p$-periodicity of $N_g$. (In particular, as $\Gamma_{g-1}$ is always $p$ periodic for odd $p$ and $g$ not equal to 2 mod $p$, so is $N_g$.) However, comparing our results with those of Xia [Xi], we note here that the converse is false. For example, when $p = 5$ and $g = 7$, $\Gamma_6$ is not $p$-periodic but $N_7$ is. However, for a fixed $p$ there are at most finitely many such $g$ where $\Gamma_{g-1}$ is not $p$-periodic but $N_g$ is.

5. The $p$-period and other open questions

We will briefly discuss three questions that arise from our study.

5.1. The $p$-period. Recall that the $p$-period $d$ of a $p$-periodic group $G$ is the least positive degree of an invertible element in its Farrell cohomology group $\tilde{H}^*(G, \mathbb{Z})_{(p)}$. The question thus arises as to what the $p$-period of $N_g$ is when $N_g$ is $p$-periodic.

For any group $G$ of finite $vcd$, an invertible element in $\tilde{H}^*(G, \mathbb{Z})_{(p)}$ restricts to an invertible element in the Farrell cohomology of any subgroup of $G$. Thus the $p$-period of a subgroup divides the $p$-period of $G$.

The main result of [GMX] is that for all $g$ such that $\Gamma_{g-1}$ is $p$-periodic, the $p$-period divides $2(p-1)$. Hence for all such $g$, the $p$-period of $N_g$ also divides $2(p-1)$. However, as we noted above, there are pairs $p$ and $g$ for which $N_g$ is $p$-periodic but $\Gamma_{g-1}$ is not. We expect that the methods of [GMX] can be pushed to cover also these cases. It remains also to find lower bounds for the $p$-period.

5.2. Punctured mapping class groups. In the oriented case Lu [L1], [L2] has studied the $p$-periodicity of the mapping class groups with marked points, and proved that they are all $p$-periodic of period 2. One might expect a similar result to hold for the mapping class group of non-orientable surfaces with marked points.

5.3. The virtual cohomological dimension. We have established in Corollary 2.2 that $N_g$ has finite virtual cohomological dimension and that this dimension is less than or equal to $4g - 9$. It seems an interesting project to determine the $vcd$ of $N_g$. 

(for otherwise $a$ would be homotopic to a homeomorphism that fixes a whole disk, but all such homeomorphisms are well-known to give rise to elements of infinite order in the mapping class group). This also implies that the action of $a$ on the tangent plane at $x$ in $N_g$ is free. It follows that the stabiliser of each singular point is isomorphic to $C_p$, as these are the only non-trivial subgroups of $A$ that are also subgroups of $GL_2(\mathbb{R})$. So by the Riemann-Hurwitz equation, for some $h \geq 1$,

$$g - 2 = p^2(h - 2) + ps(p - 1).$$

From this it follows that $g = lp + 2$ for some $l \geq 1$, and furthermore that $l = p(h - 2 + s) - s$. Note that $l = -s$ (mod $p$). Now write $s = qp + t$ for some $q \geq 0$ and $0 \leq t < p$. Then $l = p(h + q(p - 2) - 2 + t) - t$. Thus we are in the situation of Lemma 4.2. Indeed, as $h \geq 1$, we can write $l = kp - t$ with $k = h + q(p - 2) + t = h + 2 - t$. Lemma 4.3 follows from this.

□
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