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A LAW OF LARGE NUMBERS FOR ARITHMETIC FUNCTIONS

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ABSTRACT. We prove the weighted strong law of large numbers for every integrable i.i.d. sequence where the weights are given by a positive strongly additive function satisfying the Lindeberg condition. This result solves one of the open problems raised in the paper by Berkes and Weber (2007).

1. MAIN RESULT

Let f be a strongly additive arithmetic function, i.e.,

$$\begin{aligned} f(mn) &= f(m) + f(n) \text{ if } \gcd(m,n) = 1, \quad \text{and} \\ f(p^n) &= f(p) \quad \text{for all primes } p \text{ and positive integers } n. \end{aligned}$$

Erdős-Kac [2] proved that if $f(p) = O(1)$ and $B_p \to \infty$ where p varies along primes, then the sequence $\{f(n)\}$ obeys the central limit theorem, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N \mid f(n) \le A_N + xB_N^{1/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

where

$$A_n = \sum_{p<n} \frac{f(p)}{p}, \qquad B_n = \sum_{p<n} \frac{f^2(p)}{p}.$$

Here and in the sequel, we follow the usual convention and denote the summation along the primes by $\sum_p$. Kubilius [4] and Shapiro [5] relaxed the condition $f(p) = O(1)$ to the Lindeberg condition below:

$$\lim_{N\to\infty} \frac{1}{B_N} \sum_{\{p<N: |f(p)| \ge \varepsilon B_N^{1/2}\}} \frac{f^2(p)}{p} = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{1.1}$$

The purpose of this paper is to prove the following theorem and show that the irregularity of a positive strongly additive function f does not have an effect on the weighted law of large numbers.

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Theorem 1.1. *Suppose that a positive strongly additive function f satisfies the Lindeberg condition* (1.1). *Then for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables, we have*

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n)X_n \Big/ \sum_{n=1}^{N} f(n) = EX_1 \quad a.s.$$

Berkes-Weber [1] proved the same conclusion by assuming

$$f(p) = o(B_p^{1/2}) \quad \text{and} \quad B_p \to \infty \quad \text{as} \quad p \to \infty,$$

which is stronger than the Lindeberg condition (1.1). They also proved it by assuming the Lindeberg condition and the following smoothness condition:

$$\sup_{n \le p, p' \le n^2} \frac{f(p)}{f(p')} = O(1),$$

and posed the question whether the Lindeberg condition alone is sufficient to have the same conclusion. Our result, which is proved by simple calculations without using a randomization technique as is used in [1], gives an affirmative answer to this question.

2. Proof

We use the following asymptotics, which are proved in [1] under the Lindeberg condition (1.1):

$$\sum_{n=1}^{N} f(n) \sim NA_N, \tag{2.1}$$

$$\sum_{n=1}^{N} f^2(n) \sim NA_N^2. \tag{2.2}$$

By (2.2), we can take a constant $C > 0$ such that

$$\sum_{n=1}^{N} f^2(n) \le \frac{CNA_N^2}{2} \qquad (N \ge 1). \tag{2.3}$$

To prove our theorem, we appeal to the characterization by Jamison-Orey-Pruitt [3]:

Lemma 2.1. *Let $\{w_k\}$ be a sequence of positive numbers and put $W_N = \sum_{n=1}^{N} w_n$. Then*

$$\lim_{N\to\infty} \frac{1}{W_N} \sum_{n=1}^{N} w_n X_n = EX_1 \quad a.s.$$

holds for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables if and only if

$$\limsup_{t\to\infty} \frac{1}{t} \#\{n : W_n \le tw_n\} < \infty.$$

We apply this characterization by putting $w_n = f(n)$. Because of (2.1), it is sufficient to prove

$$\#\{n : nA_n \le mf(n)\} \le (1 + C)^2 m. \tag{2.4}$$

To begin with, we have

$$\#\{n : nA_n \le mf(n)\} \le m + m^2 \sum_{n>m} \frac{f^2(n)}{n^2 A_n^2}. \tag{2.5}$$

To bound the second term, we first prove

$$\sum_{m<n\le M} \frac{f^2(n)}{n^2A_n^2} \le \frac{C}{m} + C \sum_{m<p<M} \frac{f(p)}{p^2A_p} \qquad (m < M). \tag{2.6}$$

By using the partial summation method, we have

$$\begin{aligned}
&\sum_{m<n\le M} \frac{f^2(n)}{n^2A_n^2}\\
&= \sum_{m<n<M} \left(\sum_{k=m+1}^{n} f^2(k)\right)\left(\frac{1}{n^2A_n^2} - \frac{1}{(n+1)^2A_{n+1}^2}\right) + \left(\sum_{k=m+1}^{M} f^2(k)\right)\frac{1}{M^2A_M^2}\\
&= \sum_{m<n<M} \left(\sum_{k=m+1}^{n} f^2(k)\right)\left(\frac{1}{A_n^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) + \frac{1}{(n+1)^2}\left(\frac{1}{A_n^2} - \frac{1}{A_{n+1}^2}\right)\right)\\
&\quad + \left(\sum_{k=m+1}^{M} f^2(k)\right)\frac{1}{M^2A_M^2}.
\end{aligned}$$

Thanks to (2.3), we have $\sum_{k=m+1}^{n} f^2(k) \le CnA_n^2/2$ and hence

$$\begin{aligned}
\sum_{m<n\le M} \frac{f^2(n)}{n^2A_n^2} &\le C \sum_{m<n<M} \left(\frac{1}{n(n+1)} + \frac{A_{n+1}^2 - A_n^2}{2nA_{n+1}^2}\right) + \frac{C}{2M}\\
&= \frac{C}{m+1} - \frac{C}{M} + C \sum_{m<n<M} \frac{A_{n+1}^2 - A_n^2}{2nA_{n+1}^2} + \frac{C}{2M}.
\end{aligned}$$

Since $A_{n+1}^2 - A_n^2$ vanishes if n is not prime and

$$\frac{A_{p+1}^2 - A_p^2}{2pA_{p+1}^2} = \frac{(A_{p+1} + A_p)(A_{p+1} - A_p)}{2pA_{p+1}^2} \le \frac{2A_{p+1}f(p)}{2p^2A_{p+1}^2} \le \frac{f(p)}{p^2A_p}$$

for prime p, we have (2.6).

By applying (2.6), we have

$$\begin{aligned}
\sum_{m<p<M} \frac{f(p)}{p^2A_p} \le \sum_{m<n\le M} \frac{f(n)}{n^2A_n} &\le \left(\sum_{m<n\le M} \frac{1}{n^2}\right)^{1/2} \left(\sum_{m<n\le M} \frac{f^2(n)}{n^2A_n^2}\right)^{1/2}\\
&\le \frac{1}{\sqrt{m}}\left(\frac{C}{m} + C \sum_{m<p<M} \frac{f(p)}{p^2A_p}\right)^{1/2}.
\end{aligned}$$

Therefore

$$m^2\left(\sum_{m<p<M} \frac{f(p)}{p^2A_p}\right)^2 \le C + Cm \sum_{m<p<M} \frac{f(p)}{p^2A_p},$$

and thereby

$$\sum_{m<p<M} \frac{f(p)}{p^2A_p} \le \frac{Cm + \sqrt{C^2m^2 + 4Cm^2}}{2m^2} \le \frac{C+1}{m}. \tag{2.7}$$

By letting $M \to \infty$, we see that (2.6) and (2.7) are valid even in the case $M = \infty$. Combining these with (2.5), we have (2.4). $\square$

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