A LAW OF LARGE NUMBERS FOR ARITHMETIC FUNCTIONS

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Abstract. We prove the weighted strong law of large numbers for every integrable i.i.d. sequence where the weights are given by a positive strongly additive function satisfying the Lindeberg condition. This result solves one of the open problems raised in the paper by Berkes and Weber (2007).

1. Main result

Let $f$ be a strongly additive arithmetic function, i.e.,

$$f(mn) = f(m) + f(n) \text{ if } \gcd(m,n) = 1,$$

and

$$f(p^n) = f(p) \quad \text{for all primes } p \text{ and positive integers } n.$$

Erdős-Kac [2] proved that if $f(p) = O(1)$ and $B_p \to \infty$ where $p$ varies along primes, then the sequence $\{f(n)\}$ obeys the central limit theorem, i.e.,

$$\lim_{N \to \infty} \frac{1}{N} \# \{ n \leq N \mid f(n) \leq A_N + xB_N^{1/2} \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du,$$

where

$$A_n = \sum_{p < n} f(p) \quad \text{and} \quad B_n = \sum_{p < n} f^2(p) \quad \text{for all } \varepsilon > 0.$$

Here and in the sequel, we follow the usual convention and denote the summation along the primes by $\sum_p$. Kubilius [4] and Shapiro [5] relaxed the condition $f(p) = O(1)$ to the Lindeberg condition below:

$$\lim_{N \to \infty} \frac{1}{B_N} \sum_{\{ p < N : f(p) \geq \varepsilon B_N^{1/2} \}} \frac{f^2(p)}{p} = 0 \quad \text{for all } \varepsilon > 0.$$

The purpose of this paper is to prove the following theorem and show that the irregularity of a positive strongly additive function $f$ does not have an effect on the weighted law of large numbers.

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Theorem 1.1. Suppose that a positive strongly additive function $f$ satisfies the Lindeberg condition (1.1). Then for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) X_n / \sum_{n=1}^{N} f(n) = E X_1 \text{ a.s.}$$

Berkes-Weber [1] proved the same conclusion by assuming

$$f(p) = o(B_p^{1/2}) \quad \text{and} \quad B_p \to \infty \quad \text{as} \quad p \to \infty,$$

which is stronger than the Lindeberg condition (1.1). They also proved it by assuming the Lindeberg condition and the following smoothness condition:

$$\sup_{n \leq p, p' \leq n^2} \frac{f(p)}{f(p')} = O(1),$$

and posed the question whether the Lindeberg condition alone is sufficient to have the same conclusion. Our result, which is proved by simple calculations without using a randomization technique as is used in [1], gives an affirmative answer to this question.

2. Proof

We use the following asymptotics, which are proved in [1] under the Lindeberg condition (1.1):

$$\sum_{n=1}^{N} f(n) \sim N A_N,$$

(2.1)

$$\sum_{n=1}^{N} f^2(n) \sim N A_N^2.$$

(2.2)

By (2.2), we can take a constant $C > 0$ such that

$$\sum_{n=1}^{N} f^2(n) \leq \frac{C N A_N^2}{2} \quad (N \geq 1).$$

(2.3)

To prove our theorem, we appeal to the characterization by Jamison-Orey-Pruitt [3]:

Lemma 2.1. Let $\{w_k\}$ be a sequence of positive numbers and put $W_N = \sum_{n=1}^{N} w_n$. Then

$$\lim_{N \to \infty} \frac{1}{W_N} \sum_{n=1}^{N} w_n X_n = E X_1 \text{ a.s.}$$

holds for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables if and only if

$$\limsup_{t \to \infty} \frac{1}{t} \# \{n : W_n \leq tw_n\} < \infty.$$

We apply this characterization by putting $w_n = f(n)$. Because of (2.1), it is sufficient to prove

$$\# \{n : n A_n \leq m f(n)\} \leq (1 + C)^2 m.$$

(2.4)
To begin with, we have
\[(2.5) \quad \# \{n : nA \leq mf(n)\} \leq m + m^{2} \sum_{n \geq m} \frac{f^{2}(n)}{n^{2}A_{n}^{2}}.\]

To bound the second term, we first prove
\[(2.6) \quad \sum_{m < n \leq M} \frac{f^{2}(n)}{n^{2}A_{n}^{2}} \leq \frac{C}{m} + \sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}}, \quad (m < M).\]

By using the partial summation method, we have
\[
\begin{align*}
\sum_{m < n \leq M} \frac{f^{2}(n)}{n^{2}A_{n}^{2}} &= \sum_{m < n < M} \left( \sum_{k = m+1}^{n} f^{2}(k) \right) \left( \frac{1}{n^{2}A_{n}^{2}} - \frac{1}{(n+1)^{2}A_{n+1}^{2}} \right) + \left( \sum_{k = m+1}^{M} f^{2}(k) \right) \frac{1}{M^{2}A_{M}^{2}} \\
&= \sum_{m < n < M} \left( \sum_{k = m+1}^{n} f^{2}(k) \right) \left( \frac{1}{n^{2}} - \frac{1}{(n+1)^{2}} \right) + \frac{1}{(n+1)^{2}} \left( \frac{1}{A_{n}^{2}} - \frac{1}{A_{n+1}^{2}} \right) \\
&\quad + \left( \sum_{k = m+1}^{M} f^{2}(k) \right) \frac{1}{M^{2}A_{M}^{2}}.
\end{align*}
\]

Thanks to (2.3), we have \(\sum_{k = m+1}^{n} f^{2}(k) \leq CnA_{n}^{2}/2\) and hence
\[
\sum_{m < n \leq M} \frac{f^{2}(n)}{n^{2}A_{n}^{2}} \leq C \sum_{m < n < M} \left( \frac{1}{n(n+1)} + \frac{A_{n+1}^{2} - A_{n}^{2}}{2nA_{n+1}^{2}} \right) + \frac{C}{2M} \\
= \frac{C}{m+1} - \frac{C}{M} + C \sum_{m < n < M} \frac{A_{n+1}^{2} - A_{n}^{2}}{2nA_{n+1}^{2}} + \frac{C}{2M}.
\]

Since \(A_{n+1}^{2} - A_{n}^{2}\) vanishes if \(n\) is not prime and
\[
\frac{A_{p+1}^{2} - A_{p}^{2}}{2pA_{p+1}^{2}} = \frac{(A_{p+1} + A_{p})(A_{p+1} - A_{p})}{2pA_{p+1}^{2}} \leq \frac{2A_{p+1}f(p)}{2p^{2}A_{p+1}^{2}} \leq \frac{f(p)}{p^{2}A_{p}}
\]
for prime \(p\), we have (2.6).

By applying (2.6), we have
\[
\begin{align*}
\sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}} &\leq \sum_{m < n \leq M} \frac{f(n)}{n^{2}A_{n}} \leq \left( \sum_{m < n \leq M} \frac{1}{n^{2}} \right)^{1/2} \left( \sum_{m < n \leq M} \frac{f^{2}(n)}{n^{2}A_{n}^{2}} \right)^{1/2} \\
&\leq \frac{1}{\sqrt{m}} \left( \frac{C}{m} + C \sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}} \right)^{1/2}.
\end{align*}
\]

Therefore
\[
m^{2} \left( \sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}} \right)^{2} \leq C + Cm \sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}},
\]

and thereby
\[(2.7) \quad \sum_{m < p < M} \frac{f(p)}{p^{2}A_{p}} \leq C + \frac{\sqrt{C^{2}m^{2} + 4Cm^{2}}}{2m^{2}} \leq \frac{C + 1}{m}.\]
By letting $M \to \infty$, we see that (2.6) and (2.7) are valid even in the case $M = \infty$. Combining these with (2.5), we have (2.4). □

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