ON A WEYL INEQUALITY OF OPERATORS IN BANACH SPACES

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Abstract. Let \( s = (s_n) \) be an injective and surjective \( s \)-number sequence in the sense of Pietsch. We show for a Riesz-operator \( T : X \rightarrow X \) acting on a (complex) Banach space the following Weyl inequality between geometric means of eigenvalues and \( s \)-numbers: For any \( 0 < \delta < 1 \) and all \( n \in \mathbb{N} \),

\[
\left( \prod_{i=1}^{n} |\lambda_i(T)| \right)^{\frac{1}{n}} \leq c_0 \left( 1 + \frac{\delta}{\ln \left(\frac{1}{\delta}\right)} \right) \left( \prod_{i=1}^{\left\lfloor \frac{n}{1+\delta}\right\rfloor} s_i(T) \right)^{\frac{1}{1+\delta}},
\]

where \( c_0 \geq 1 \) is an absolute constant. The proof rests on an elementary mixing multiplicativity of an arbitrary \( s \)-number sequence and a striking result of G. Pisier. The inequality is a contribution to the problem of estimating eigenvalues by \( s \)-numbers first started in a strong sense by H. König (1986, 2001).

1. \( s \)-Numbers

We recall some basic definitions and notions from Banach space theory and from \( s \)-numbers of operators. If \( X \) is a Banach space, we denote by \( X' \) its dual Banach space and by \( B_X \) and \( \overline{B}_X \) the closed and open unit balls of \( X \), respectively. In what follows, \( X, Y, Z \), etc., always denote Banach spaces, \( \mathcal{L}(X,Y) \) is the set of (bounded linear) operators from \( X \) into \( Y \) equipped with the operator norm and \( \mathcal{L} \) stands for the class of all operators between arbitrary Banach spaces.

We use the notion of an \( s \)-number sequence given in [P80b] or [P87]. A rule \( s = (s_n) : \mathcal{L} \rightarrow [0, \infty] \) assigning to every operator \( T \in \mathcal{L} \) a non-negative scalar sequence \( (s_n(T))_{n \in \mathbb{N}} \) is called an \( s \)-number sequence if the following conditions are satisfied:

(S1) Monotonicity:
\[
\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0 \text{ for } T \in \mathcal{L}(X,Y).
\]

(S2) Additivity:
\[
s_{n+m-1}(S + T) \leq s_m(S) + s_n(T) \text{ for } S, T \in \mathcal{L}(X,Y).
\]
Kolmogorov number and the arbitrary $s$ numbers, and with values in the space $1$

Furthermore, we additionally need the following notions:

- (JS) An $s$-number sequence $s = (s_n)$ be an arbitrary $s$-number sequence.

We note that on the class of Hilbert spaces there is only one $s$-number sequence $s$ satisfying the properties (S1) - (S5) that coincides with the singular numbers $[P80b]$. Basic examples for our purposes are the approximation numbers given by

$$a_n(T) := \inf \{ ||T - L|| : L \in \mathcal{L}(X,Y), \text{rank}(L) < n \},$$

the Gelfand numbers given by

$$c_n(T) := \inf \{ ||TJ_M|| : M \subset X, \text{codim}(M) < n \},$$

where $J_M : M \rightarrow X$ is the natural embedding from a subspace $M$ of $X$ into $X$, and the Kolmogorov number given by

$$d_n(T) := \inf \{ ||Q_NT|| : N \subset Y, \dim N < n \},$$

where $Q_N : Y \rightarrow Y/N$ defines the canonical quotient map from $Y$ onto the quotient space $Y/N$.

Moreover, we need also the following characterization of Gelfand and Kolmogorov numbers,

$$c_n(T) = a_n(J_\infty T) \text{ and } d_n(T) = a_n(TQ_1),$$

where $J_\infty : Y \rightarrow l_\infty(B_Y)$ is the metric injection defined by $J_\infty y := (\langle y, a \rangle)_{a \in B_Y}$, and with values in the space $l_\infty(B_Y)$ of bounded sequences and where $Q_1 : l_1(B_X) \rightarrow X$ is the metric surjection from the space of summable sequences $l_1(B_X)$ onto $X$, defined by $Q_1(\langle \xi_x \rangle) := \sum_{x \in B_X} \xi_x x$.

Now we show an elementary but very useful multiplicativity property of an arbitrary $s$-number sequence that we call mixing multiplicativity.

- (MI) Mixing multiplicativity: Let $s = (s_n)$ be an arbitrary $s$-number sequence. Then for $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$,

  (i) $s_{n+m-1}(TS) \leq s_n(T)a_m(S)$ and $s_{n+m-1}(TS) \leq a_n(T)s_m(S)$

  Moreover, if $s = (s_n)$ is injective, then

  (ii) $s_{n+m-1}(TS) \leq c_n(T)s_m(S)$

  and if $s = (s_n)$ is surjective, then

  (iii) $s_{n+m-1}(TS) \leq s_n(T)d_m(S)$. 

Theorem. Banach spaces and $n$-dimensional Hilbert spaces and the mixing multiplicativity of Pisier’s isomorphism (S2), the rank property (S4), and the ideal property (S3) of an $s$-number sequence immediately yield

$$s_{n+m-1}(TS) = s_{n+m-1}(T(S-L)+TL)$$

$$\leq s_n(T(S-L)) + s_m(TL) = s_n(T(S-L)) \leq s_n(T)||S-L||,$$

implying by definition of the approximation numbers the desired inequality $s_{n+m-1}(TS) \leq s_n(T)a_m(S)$.

The inequalities (ii) and (iii) follow from (i) and the characterization of Gelfand and Kolmogorov numbers, respectively. For example, (ii) follows from $s_{n+m-1}(TS) = s_{n+m-1}(J\lambda T) \leq a_n(J\lambda T)s_m(S) = c_n(T)s_m(S)$.

\[\Box\]

2. A Weyl Inequality

For a Riesz operator $T \in \mathcal{L}(X)$ acting on a complex Banach space (cf. [K86], [P87], [CS90]) we assign an eigenvalue sequence $(\lambda_n(T))$ as follows: The eigenvalues are arranged in an order of non-increasing absolute values and each eigenvalue is counted according to its algebraic multiplicity,

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0.$$ 

If $T$ possesses less than $n$ eigenvalues $\lambda$ with $\lambda \neq 0$, we put $\lambda_n(T) = \lambda_{n+1}(T) = \cdots = 0$. In this section we establish for an arbitrary injective and surjective $s$-number sequence a Weyl inequality for Riesz operators. The following Weyl inequality complements Weyl inequalities for $s$-numbers given in [K86], [K01], [KRT80], [P80a], [P87], [H05] and [CHi07]. In the sequel, $[x]$ denotes the integer part of $x$ for $1 \leq x < \infty$ and if $0 < x \leq 1$ we put $[x] := 1$.

Theorem. Let $s = (s_n)$ be an injective and surjective $s$-number sequence. Then for any $0 < \delta \leq 1$ there exists a constant $c(\delta) \geq 1$ such that for all (complex) Banach spaces $X$, all Riesz operators $T \in \mathcal{L}(X)$ and all $n \in \mathbb{N}$ the inequality

$$\left( \prod_{i=1}^{n} |\lambda_i(T)| \right)^{\frac{1}{n}} \leq c(\delta) \left( \prod_{i=1}^{[\frac{n}{\delta}] + 1} s_i(T) \right)^{\frac{1}{[\frac{n}{\delta}]}}$$

holds. For the constant $c(\delta)$ we obtain the estimate $c(\delta) \leq c_0(1 + \frac{1}{\delta} \ln \frac{n}{\delta})$, where $c_0 \geq 1$ is an absolute constant.

Proof. In order to prove the inequality we need a striking result of Pisier [P89a], [P89b] concerning the existence of isomorphisms between arbitrary $n$-dimensional Banach spaces and $n$-dimensional Hilbert spaces and the mixing multiplicativity of an injective and surjective $s$-number sequence.

Pisier’s isomorphism states the following:

For each $\alpha > \frac{1}{2}$, there is a constant $b(\alpha)$ such that for any $n$ and any $n$-dimensional (real or complex) Banach space $E$, there is an isomorphism $u : l^2_n \rightarrow E$ such that

$$d_k(u) \leq b(\alpha) \left( \frac{n}{k} \right)^{\alpha} \text{ and } c_k(u^{-1}) \leq b(\alpha) \left( \frac{n}{k} \right)^{\alpha}$$

for $1 \leq k \leq n$, where $b(\alpha) \leq b_0(\alpha - \frac{1}{2})^{-\frac{1}{2}}$ and $b_0 > 0$ is an absolute constant.
Hence, moreover, the proof uses ideas presented in [CHe91] and [DJ93]. If \( \lambda_n(T) = 0 \), then there is nothing to prove. So we assume \( \lambda_n(T) \neq 0 \). By [K86], [P87] or [CS90] we can find an \( n \)-dimensional Banach space \( X_n \) of \( X \) invariant under \( T \) such that the restriction \( T_n \) of \( T \) to \( X_n \) has precisely \( \lambda_1(T), \ldots, \lambda_n(T) \) as its eigenvalues. By Pisier we get for \( \frac{1}{2} < \alpha \leq 1 \) an isomorphism \( u : l^2_n \to X_n \) such that
\[
d_k(u) \leq b(\alpha) \left( \frac{n}{k} \right)^{\alpha} \quad \text{and} \quad c_k(u^{-1}) \leq b(\alpha) \left( \frac{n}{k} \right)^{\alpha}
\]
for \( 1 \leq k \leq n \). Applying Weyl’s inequality to the Hilbert space operator \( u^{-1}T_nu \) and inserting the principle of related operators [P87] we arrive at
\[
\left( \prod_{k=1}^{n} \lambda_k(T) \right)^{\frac{1}{n}} \leq \left( \prod_{k=1}^{n} s_k(u^{-1}T_nu) \right)^{\frac{1}{n}}.
\]
For estimating the right-hand side of the inequality we put \( m := \left[ \frac{n}{\alpha} \right] \) for \( 0 < \delta \leq 1 \). Then we have
\[
\left( \prod_{k=1}^{m} s_k(u^{-1}T_nu) \right)^{\frac{1}{m}} \leq \left( \prod_{k=1}^{m} s_{[\delta k]+k-1}(u^{-1}T_nu) \right)^{\frac{1}{m}}.
\]
The mixing multiplicativity (MI) (ii) and (iii) of an injective and surjective \( s \)-number sequence yields for the single \( s \)-numbers the estimate
\[
s_{[\delta k]+k-1}(u^{-1}T_nu) \leq c\left[ \frac{\delta}{2} \right](u^{-1})s_k(T_n)d\left[ \frac{\delta}{2} \right](u)
\]
\[
\leq b^2(\alpha) \left( \frac{n}{[\delta k]} \right)^{2\alpha} s_k(T_n)
\]
\[
\leq b^2(\alpha) \left( \frac{n}{[\delta k]} \right)^{2\alpha} s_k(T) \quad \text{for} \quad 1 \leq k \leq m.
\]
Hence,
\[
\left( \prod_{k=1}^{m} s_{[\delta k]+k-1}(u^{-1}T_nu) \right)^{\frac{1}{m}} \leq b^2(\alpha) \left( \prod_{k=1}^{m} \frac{n}{[\delta k]} \right)^{\frac{2\alpha}{m}} \left( \prod_{k=1}^{m} s_k(T) \right)^{\frac{1}{m}}.
\]
From \( \left[ \frac{\delta k}{2} \right] \geq \frac{\delta k}{4} \), \( m = \left[ \frac{n}{1+\delta} \right] \geq \frac{n}{2(1+\delta)} \) and \( e^m \geq \frac{m^m}{m!} \) we obtain for the first term on the right-hand side of the inequality the estimate
\[
\left( \prod_{k=1}^{m} \frac{n}{[\delta k]} \right)^{\frac{1}{m}} \leq \left( \frac{4}{\delta} \right)^{2\alpha} e^{2\alpha} \left( \frac{m}{n} \right)^{2\alpha}
\]
\[
\leq \left( \frac{4}{\delta} \right)^{2\alpha} e^{2\alpha} (2(1+\delta))^{2\alpha} \leq 16^2 e^2 \left( \frac{1}{3} \right)^{2\alpha}.
\]
Combining the previous inequalities we arrive at
\[
\left( \prod_{k=1}^{m} |\lambda_k(T)| \right)^\frac{1}{n} \leq 16^2 e^2 b^2 (\alpha) \left( \frac{1}{\delta} \right)^{2\alpha} \left( \prod_{k=1}^{m} s_k(T) \right)^\frac{1}{m}.
\]

Finally, it remains to estimate the constant appearing on the right-hand side of the inequality by choosing an appropriate \( \alpha \), \( \frac{1}{2} < \alpha \leq 1 \). To this end, first let \( \frac{1}{2} < \delta < 1 \). If we choose \( \alpha = 1 \), then for the constant \( c(\delta) \) we obtain \( c(\delta) \leq 2 \cdot 16^2 e^2 b_0^2 \). If \( 0 < \delta \leq \frac{1}{e} \), then we choose \( \alpha = \frac{1}{2} + \frac{1}{2 \ln(\frac{1}{\delta})} \), which guarantees \( \frac{1}{2} < \alpha \leq 1 \) and for \( c(\delta) \) we check that
\[
c(\delta) \leq 2 \cdot 16^2 e^2 b_0^2 \ln \left( \frac{1}{\delta} \right) \).
\]

Summarizing the previous estimates we obtain
\[
c(\delta) \leq 2 \cdot 16^2 e^2 b_0^2 \max\left\{ e, \frac{1}{\delta} \ln \left( \frac{1}{\delta} \right) \right\} \leq 2 \cdot 16^2 e^4 b_0^2 \left( 1 + \frac{1}{\delta} \ln \left( \frac{1}{\delta} \right) \right) \text{ for } 0 < \delta \leq 1.
\]

\( \square \)

References


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