Abstract. It is proved that all vertex cover algebras of a hypergraph are standard graded if and only if the hypergraph is unimodular. This has interesting consequences on the symbolic powers of monomial ideals.

1. Introduction

A hypergraph $\Delta$ is a collection of subsets of a finite set of vertices $V$. These subsets are called the edges of $\Delta$. For convenience, we assume throughout this paper that there is no inclusion between the edges of $\Delta$. Such a hypergraph is also called a clutter.

A vertex cover of $\Delta$ is a subset of $V$ which meets every edge of $\Delta$. Suppose that $V = \{1, \ldots, n\}$. We may think of a vertex cover of $V$ as a $(0,1)$ vector $c = (c_1, \ldots, c_n)$ that satisfies the condition $\sum_{i \in F} c_i \geq 1$ for all $F \in \Delta$.

Let $w: F \mapsto w_F$ be a weight function from $\Delta$ to the set of positive integers. We call $(\Delta, w)$ a weighted hypergraph. For $k \in \mathbb{N}$ we define a $k$-cover of $(\Delta, w)$ as a vector $c \in \mathbb{N}^n$ that satisfies the condition $\sum_{i \in F} c_i \geq kw_F$ for all $F \in \Delta$.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$. For every vector $c \in \mathbb{N}^n$ we set $x^c = x_1^{c_1} \cdots x_n^{c_n}$. The vertex cover algebra $A(\Delta, w)$ is defined as the subalgebra of the one-variable polynomial ring $R[t]$ generated by all monomials $x^ct^k$, where $c$ is a $k$-cover of $(\Delta, w)$, $k \geq 0$. This algebra was introduced in [4]. It was proved there that $A(\Delta, w)$ is a finitely generated normal Cohen-Macaulay ring.

For every edge $F \in \Delta$ let $P_F$ denote the ideal of $R$ generated by the variables $x_i, i \in F$. Let $I(\Delta, w) := \bigcap_{F \in \Delta} P_F^{w_F}$. It is not hard to see that $A(\Delta, w)$ is the symbolic Rees algebra of the ideal $I(\Delta, w)$; that is,

$$A(\Delta, w) = \bigoplus_{k \geq 0} I(\Delta, w)^{(k)}t^k,$$

where $I(\Delta, w)^{(k)}$ denotes the $k$-th symbolic power $\bigcap_{F \in \Delta} P_F^{w_Fk}$ of $I(\Delta, w)$.

We may view $A(\Delta, w)$ as a graded algebra over $R$ with $A(\Delta, w)_k = I(\Delta, w)^{(k)}t^k$ for $k \geq 0$. It is obvious that $A(\Delta, w)$ is standard graded ($A(\Delta, w)$ is generated by elements of degree one) if and only if $I(\Delta, w)^{(k)} = I(\Delta, w)^k$ for all $k \geq 0$. Therefore, it is of great interest to know when $A(\Delta, w)$ is a standard graded algebra over $R$. 
This problem has a satisfactory answer in the case \( w \) is the canonical weight; i.e. \( w_F = 1 \) for all \( F \in \Delta \). In this case, we will use the notation \( A(\Delta) \) and \( I^\ast(\Delta) \) instead of \( A(\Delta, w) \) and \( I(\Delta, w) \). Notice that ideals of the form \( I^\ast(\Delta) \) are exactly squarefree monomial ideals of \( R \). It was proved in [5] Theorem 1.4 (and implicitly in [2] Proposition 3.4 and [3] Theorem 3.5) that \( A(\Delta) \) is standard graded over \( R \) if and only if the blocker of \( \Delta \) is a Mengerian hypergraph (see the last section for more details).

The aim of this paper is to characterize hypergraphs \( \Delta \) for which \( A(\Delta, w) \) is a standard graded algebra over \( R \) for all weight functions \( w \). If \( \Delta \) is a graph, such a characterization was already obtained in [4] Theorem 5.1 and Theorem 5.4, where it is proved that \( A(\Delta, w) \) is a standard graded algebra over \( R \) for all weight functions \( w \) if and only if \( \Delta \) is bipartite. Hence for all \( k \geq 1 \) or, equivalently, if and only if every \( k \)-cover of \( \Delta \) is unimodular (see e.g. [1, Theorem 5, p. 164]). In particular, a graph is unimodular if and only if it is bipartite. Hence the aforementioned result of [4] is a special case of Theorem 1.1

An interesting case of Theorem 1.1 concerns the hypergraph of all \( n + 1 \) points in general position. Since every set of \( n + 1 \) points in general position in \( \mathbb{P}^n \) can be transformed into this case, we obtain the following consequence.

**Corollary 1.2.** Let \( P_0, ..., P_n \) be the defining ideals of \( n + 1 \) points in general position in \( \mathbb{P}^n \). Then

\[
(P_0^{w_0} \cap \cdots \cap P_n^{w_n})^k = P_0^{w_0k} \cap \cdots \cap P_n^{w_nk},
\]

for all integers \( w_0, ..., w_n \geq 1 \) and \( k \geq 0 \).

2. **Decomposition property versus unimodularity**

We adhere to the notions of the preceding section.

Let \( \Delta = \{F_1, ..., F_m\} \) be a hypergraph on the set of vertices \( \{1, ..., n\} \). Let \( w \) be a weight function on \( \Delta \) and \( w = (w_{F_1}, ..., w_{F_m}) \). Note that \( w \in \mathbb{Z}^n \). Let \( M \) be the edge-vertex incidence matrix of \( \Delta \). By definition, a vector \( c \in \mathbb{N}^n \) is a \( k \)-cover of the weighted hypergraph \( (\Delta, w) \) if and only if \( M \cdot c \leq k w \).

It is obvious that the vertex cover algebra \( A(\Delta, w) \) is standard graded over \( R \) if and only if every monomial of \( A(\Delta, w)_1 \) for all \( k \geq 1 \) or, equivalently, if and only if every \( k \)-cover of \( (\Delta, w) \) can be written as a sum of \( k \) \( 1 \)-covers of \( (\Delta, w) \) for all \( k \geq 1 \). This observation leads us to the following notion in hypergraph theory.

Let \( Q \) be an arbitrary polyhedron in \( \mathbb{R}^n \) and \( k Q = \{kc | c \in Q \}, k \geq 1 \). We say that \( Q \) has the integer decomposition property if for each \( k \geq 1 \) and each integral vector \( c \in k Q \) there exist integral vectors \( c_1, ..., c_k \in Q \) such that \( c = c_1 + \cdots + c_k \).
If \( Q = \{ c \geq 0 \mid M \cdot c \geq w \} \), then \( kQ = \{ c \geq 0 \mid M \cdot c \geq kw \} \). Therefore, we have the following lemma.

**Lemma 2.1.** \( A(\Delta, w) \) is standard graded over \( R \) if and only if the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \) has the integer decomposition property.

If \( Q = \{ c \geq 0 \mid M \cdot c \leq w \} \), there is the following characterization of the integer decomposition property in terms of \( M \).

**Theorem 2.2.** (Baum and Trotter; see e.g. [6] Theorem 19.4). Let \( M \) be an \( m \times n \) integral matrix. Then the polyhedron \( \{ c \geq 0 \mid M \cdot c \leq w \} \) has the integer decomposition property for all integral vectors \( w \in \mathbb{Z}^m \) if and only if \( M \) is totally unimodular.

Since \( M \) is totally unimodular if and only if \( -M \) is totally unimodular and since \( w \) may have negative components, we can replace \( M \) by \( -M \) and \( w \) by \( -w \) in Theorem 2.2. Therefore, Theorem 2.2 remains true if we replace the polyhedron \( \{ c \geq 0 \mid M \cdot c \leq w \} \) by the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \), which appears in Lemma 2.1.

As a consequence, if the incidence matrix of a hypergraph \( \Delta \) is totally unimodular, then \( A(\Delta, w) \) is standard graded for all weight functions \( w \). This proves the sufficient part of Theorem 1.1. The necessary part of Theorem 1.1 does not follow from Theorem 2.2 because of the condition \( w \in \mathbb{Z}^m \).

By Lemma 2.1, Theorem 1.1 follows from the following modification of Theorem 2.2 for integral matrices with non-negative entries and for positive integral vectors \( w \in \mathbb{Z}_+^m \).

**Theorem 2.3.** Let \( M \) be an \( m \times n \) integral matrix with non-negative entries. Then the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \) has the integer decomposition property for all integral vectors \( w \in \mathbb{Z}_+^m \) if and only if \( M \) is totally unimodular.

The proof of this theorem is given at the end of the next section. Following the proof of [6] Theorem 19.4], we need to prepare some results on the relationship between the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \) and the total unimodularity of \( M \).

3. Unimodularity versus integrality

Recall that a matrix is called unimodular if each maximal minor equals 0, \( \pm 1 \) and that a rational polyhedron is called integral if all of its vertices are integral.

By a result of Veinott and Dantzig (see e.g. [6] Theorem 19.2]) unimodular integral matrices can be characterized by the integrality of associated polyhedra. This result can be modified for matrices with non-negative entries as follows.

**Theorem 3.1.** Let \( M \) be an integral \( m \times n \) matrix with non-negative entries of full row rank. Then \( M \) is unimodular if and only if the polyhedron \( \{ c \geq 0 \mid M \cdot c = w \} \) is integral for all integral vectors \( w \in \mathbb{Z}_+^m \).

**Proof.** If \( M \) is unimodular, then \( \{ c \geq 0 \mid M \cdot c = w \} \) is integral by the aforementioned result of Veinott and Dantzig. For the converse assume that \( \{ c \geq 0 \mid M \cdot c = w \} \) is integral for each integral vector \( w \in \mathbb{Z}_+^m \). Let \( E \) be an arbitrary non-singular maximal square submatrix of \( M \). We have to prove that \( \det E = \pm 1 \). Since \( \det E \) is an integer, \( E^{-1} \) is not an integral matrix if \( \det E \neq \pm 1 \). Therefore, it suffices to show that \( E^{-1} \cdot a \) is an integral vector for all \( a \in \mathbb{Z}_+^m \).
First, we can find an integral vector \( a' \in \mathbb{Z}_+^m \) such that \( E^{-1} \cdot a + a' \in \mathbb{Q}_+^m \). Let \( b := E^{-1} \cdot a + a' \) and \( w := E \cdot b \). Then \( w = a + E \cdot a' \in \mathbb{Z}_+^m \) because \( E \) has non-negative entries and no row of \( E \) is zero. Let \( b^* \) be the vector obtained from \( b \) by adding \( n - m \) zero-components so that \( M \cdot b^* = E \cdot b = w \). Then \( b^* \) lies on at least \( n - m \) facets of the polyhedron \( \{ c \geq 0 \mid M \cdot c = w \} \). Hence \( b^* \) is a vertex of the polyhedron \( \{ c \geq 0 \mid M \cdot c = w \} \). By the assumption, \( b^* \) is integral. Therefore, \( b \) and hence \( E^{-1} \cdot a = b - a' \) are integral vectors.

The following corollary is again a modification of a result of Hoffman and Kruskal for integral matrices with non-negative entries (see e.g. [6, Corollary 19.2a]).

**Corollary 3.2.** An integral \( m \times n \) matrix \( M \) with non-negative entries is totally unimodular if and only if the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \) is integral for all integral vectors \( w \in \mathbb{Z}_+^m \).

**Proof.** Let \( I \) denote the unit matrix of rank \( m \). It is well known that \( M \) is totally unimodular if and only if the composed matrix \( (I, M) \) is unimodular. On the other hand, the vertices of the polyhedron \( \{ c \geq 0 \mid M \cdot c \geq w \} \) are integral if and only if the vertices of the polyhedron \( \{ b \geq 0 \mid (I, M) \cdot b = w \} \) in \( \mathbb{R}^{m+n} \) are integral. Therefore, the assertion follows from Theorem 3.1.

We shall use Corollary 3.2 to prove Theorem 2.3. As observed in the preceding section, Theorem 1.1 follows from Theorem 2.3.

**Proof of Theorem 2.3.** The necessity already follows from Theorem 2.2. To prove the sufficiency we assume that the polyhedron \( Q = \{ c \geq 0 \mid M \cdot c \geq w \} \) has the integer decomposition property for all integral vectors \( w \in \mathbb{Z}_+^m \). By Corollary 3.2, we only need to show that \( Q \) is integral. Let \( a \) be an arbitrary vertex of \( Q \). Suppose that \( a \) is not integral. Let \( k \) be the least common multiple of the denominators occurring in \( a \). Put \( c := ka \). Then \( c \) is an integral vector in \( kQ \). Therefore, there are integral vectors \( c_1, \ldots, c_k \in Q \) such that \( c = c_1 + \cdots + c_k \). From this it follows that \( a = (c_1 + \cdots + c_k)/k \). Since \( a \) is a vertex of \( Q \), we must have \( c_1 = \cdots = c_k \) and hence \( a = c_1 \in \mathbb{N}^n \).

4. Remarks

For a vector \( w \) we will use the notation \( w \gg 0 \) if all components of \( w \) are large enough. In the proof of Theorem 3.1 we may choose \( b \gg 0 \), which implies \( w \gg 0 \). Therefore, we obtain the following stronger statement for the converse of Theorem 3.1.

**Theorem 4.1.** Let \( M \) be an integral \( m \times n \) matrix with non-negative entries of full row rank. Then \( M \) is unimodular if the polyhedron \( \{ c \geq 0 \mid M \cdot c = w \} \) is integral for all integral vectors \( w \gg 0 \) of \( \mathbb{Z}_+^m \).

Similarly as above, this result yields the following improvement of Theorem 1.1 where \( w \gg 0 \) means \( w \gg 0 \).

**Theorem 4.2.** If \( A(\Delta, w) \) is a standard graded algebra for all weight functions \( w \gg 0 \), then \( \Delta \) is a unimodular hypergraph.

The above theorem doesn’t hold for a sequence of weight functions \( w \gg 0 \). In fact, if \( \Delta \) is a unimodular hypergraph, then \( A(\Delta) \) is standard graded over \( R \) by Theorem 1.1. But \( A(\Delta) \) need not be standard graded over \( R \) if \( A(\Delta, w) \) is standard...
graded over $R$ for a sequence of weight functions $w \gg 0$. This follows from the following observation.

**Lemma 4.3.** For every hypergraph $\Delta$ there exists a number $d$ such that $A(\Delta, w)$ is a standard graded algebra over $R$ for all weight functions $w_F = kd$, $F \in \Delta$, $k \geq 1$.

**Proof.** It is known that $A(\Delta)$ is always a finitely generated algebra over $R$ [4 Theorem 4.1]. Therefore, there exists a number $d$ such that the Veronese subalgebra $A(\Delta)^{(kd)}$ is standard graded over $R$ for all $k \geq 1$ [4 Theorem 2.1]. But $A(\Delta)^{(kd)} = A(\Delta, w)$ where $w$ is the weight function $w_F = kd$, $F \in \Delta$. □

On the other hand, one may ask whether $A(\Delta, w)$ is a standard graded algebra over $R$ for all weight functions $w$ if $A(\Delta)$ is standard graded.

This question has a positive answer if $\Delta$ is a graph. In this case, if $A(\Delta)$ is standard graded over $R$, then $\Delta$ is bipartite [4 Theorem 5.1]; whence $A(\Delta, w)$ is standard graded for all weight functions $w$ by [4 Theorem 5.4]. However, we can give an example of a hypergraph $\Delta$ such that $A(\Delta)$ is standard graded over $R$, whereas $A(\Delta, w)$ is not standard graded over $R$ for all weight functions $w$. For that we shall need the following result.

Let $M$ be the edge-vertex incidence matrix of a hypergraph $\Delta$. Let $m \times n$ be the size of $M$. One calls $\Delta$ a **Mengerian hypergraph** if

$$\min\{a \cdot c \mid a \in \mathbb{N}^n, M \cdot a \geq 1\} = \max\{b \cdot 1 \mid b \in \mathbb{N}^m, M^T \cdot b \leq c\},$$

for all $c \in \mathbb{N}^n$, where $1$ denotes the vector $(1, \ldots, 1) \in \mathbb{N}^n$. One calls the hypergraph of the minimal covers of $\Delta$ the **blocker** of $\Delta$, which we denote by $\Delta^*$. It is proved in [5 Theorem 1.4] that $A(\Delta)$ is standard graded over $R$ if and only if $\Delta^*$ is Mengerian.

**Example 4.4.** Let $\Delta$ be the hypergraph on 6 vertices which has the edges

$$\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}.$$

Then $\Delta^*$ is the simplicial complex with the edges

$$\{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}.$$

It is known that $\Delta^*$ is Mengerian, whereas $\Delta$ is not (see e.g. [4 Example 1.8]). Therefore, $A(\Delta)$ is standard graded over $R$. On the other hand, since unimodular hypergraphs are Mengerian (see e.g. [1 Corollary 1, p. 170]), $\Delta$ is not unimodular. Therefore, $A(\Delta, w)$ is not standard graded over $R$ for all weight functions $w$.

Finally, we would like to point out that one can test totally unimodular matrices (and hence unimodular hypergraphs) in polynomial time (see e.g. [6 Chapter 19]).

**References**


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