ON THE LIMIT POINTS OF \((a_n \xi)_{n=1}^\infty \mod 1\)
FOR SLOWLY INCREASING INTEGER SEQUENCES \((a_n)_{n=1}^\infty\)

ARTŪRAS DUBICKAS

(Communicated by Ken Ono)

ABSTRACT. In this paper, we are interested in sequences of positive integers \((a_n)_{n=1}^\infty\) such that the sequence of fractional parts \((a_n \xi)_{n=1}^\infty\) has only finitely many limit points for at least one real irrational number \(\xi\). We prove that, for any sequence of positive numbers \((g_n)_{n=1}^\infty\) satisfying \(g_n \geq 1\) and \(\lim_{n \to \infty} g_n = \infty\) and any real quadratic algebraic number \(\alpha\), there is an increasing sequence of positive integers \((a_n)_{n=1}^\infty\) such that \(a_n \leq n g_n\) for every \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} \{a_n \alpha\} = 0\). The above bound on \(a_n\) is best possible in the sense that the condition \(\lim_{n \to \infty} g_n = \infty\) cannot be replaced by a weaker condition. More precisely, we show that if \((a_n)_{n=1}^\infty\) is an increasing sequence of positive integers satisfying \(\lim \inf_{n \to \infty} a_n/n < \infty\) and \(\xi\) is a real irrational number, then the sequence of fractional parts \((a_n \xi)_{n=1}^\infty\) has infinitely many limit points.

1. INTRODUCTION

By an old result of Weyl \[16\], for every increasing sequence of positive integers \((a_n)_{n=1}^\infty\), the set of real numbers \(\xi\) for which the sequence of fractional parts \((a_n \xi)_{n=1}^\infty\) is not uniformly distributed in \([0, 1]\) is of Lebesgue measure zero. In particular, for almost all real \(\xi\), the set \((a_n \xi)_{n=1}^\infty\) is everywhere dense in \([0, 1]\). Of course, all rational numbers \(\xi\) are trivial exceptions, because the set of limit points of \((a_n \xi)_{n=1}^\infty\) is finite if \(\xi \in \mathbb{Q}\). Another exception is related to the so-called PV-numbers, named after Pisot and Vijayaraghavan (see \[11\] and \[15\]). For instance, taking the PV-number \(\sqrt{2}+1\) and setting \(S_n = (\sqrt{2}+1)^n - (\sqrt{2}-1)^n \in \mathbb{N}\), we have \(\lim_{n \to \infty} (S_n \sqrt{2}) = 0\). More precisely, \(S_n \sqrt{2} \to 1\) as \(n \to \infty\). So there is a geometrically growing sequence \((a_n)_{n=1}^\infty\) and a quadratic number \(\xi\) such that \((a_n \xi)_{n=1}^\infty\) has a unique limit point. Erdös asked whether, for every sufficiently fast growing sequence of integers \((a_n)_{n=1}^\infty\), there are some non-trivial exceptional \(\xi \notin \mathbb{Q}\) for which \((a_n \xi)_{n=1}^\infty\) is not dense in \([0, 1]\). For every lacunary sequence \((a_n)_{n=1}^\infty\), namely, the sequence satisfying \(a_{n+1} \geq \tau a_n\) for some \(\tau > 1\) and each \(n \in \mathbb{N}\), the question of Erdös was answered in the affirmative by de Mathan \[5\] and Pollington \[12\], independently. See also Hilfssatz III in Khintchine’s paper \[8\].

However, if \((a_n)_{n=1}^\infty\) is a slowly increasing sequence of positive integers, then it can be no exceptional \(\xi\) in the sense that the sequence \((a_n \xi)_{n=1}^\infty\) is everywhere
dense in $[0, 1)$ for every real irrational number $\xi$. In this direction, Furstenberg \[4\] proved a remarkable result which implies that if an increasing sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ is a multiplicative semigroup which is not generated by powers of a single integer, then the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$ for each irrational real number $\xi$. The set $A$ is said to be a \textit{multiplicative semigroup} if it is closed under multiplication, namely, if $aa' \in A$ for any $a, a' \in A$. For example, the set of integers of the form $p^kq^m$, where $p < q$ are two fixed primes and $k, m$ run over all non-negative integers, is a multiplicative semigroup. It is easy to see that a semigroup with at least two generators must satisfy the condition $\lim_{n \to \infty} a_{n+1}/a_n = 1$.

Later, a simpler proof of Furstenberg’s theorem was given by Boshernitzan \[4\], whereas the papers of Berend \[11\], \[12\], \[2\], \[3\], Kra \[9\] and Urban \[14\] contain various generalizations of Furstenberg’s result. See also \[13\] for a collection of many slowly increasing sequences $(a_n)_{n=1}^{\infty}$ such that, for each $\xi \notin \mathbb{Q}$, the sequence $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$. Such are, for instance, the sequences $a_n = n$, $a_n = P(n)$, where $P(x) \in \mathbb{Z}[x]$ has degree $\geq 1$, $a_n = P(p_n)$, where $p_n$ is the $n$th prime. Nevertheless, a similar question on whether, for the sequence of positive integers $(a_n)_{n=1}^{\infty}$ of the form $p^k + q^m$, where $p < q$ are two fixed primes and $k, m$ run over all non-negative integers, the sequence $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$ remains open \[10\].

In this paper, we investigate whether, for a given increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$, there is an \textit{exceptional} real irrational number $\xi$ in the sense that the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has only finitely many limit points. Then no Furstenberg type theorem holds. \textit{How slowly can such a sequence $(a_n)_{n=1}^{\infty}$ for which at least one exceptional $\xi \notin \mathbb{Q}$ exists increase?} The above examples show that for each rapidly increasing sequence, e.g., a lacunary sequence $(a_n)_{n=1}^{\infty}$, such exceptional $\xi$ exist, but for most ‘natural’ slowly increasing sequences such exceptional $\xi$ do not exist.

We shall prove that there is a sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ satisfying $a_n \leq ng_n$ for each $n \in \mathbb{N}$ such that $\{a_n\xi\}_{n=1}^{\infty}$ has only finitely many limit points for some $\xi \notin \mathbb{Q}$, if and only if $\lim_{n \to \infty} g_n = \infty$, no matter how slowly $g_n$ tends to infinity. Moreover, it turns out that it is possible to construct an ‘extreme’ sequence $(a_n)_{n=1}^{\infty}$ for which the sequence $\{a_n\xi\}_{n=1}^{\infty}$, where $\xi \notin \mathbb{Q}$, has not just finitely many, but only one limit point, say, 0. In fact, our construction of an ‘extreme’ sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ of slowest possible growth involves the properties of this exceptional $\xi$ (which will be taken as an arbitrary real quadratic algebraic number $\alpha$) and the properties of some recurrence sequences related to some algebraic integer in the field $\mathbb{Q}(\alpha)$.

**Theorem 1.** Let $\alpha$ be a real quadratic algebraic number, and let $g_1, g_2, g_3, \ldots$ be a sequence of real numbers such that $g_n \geq 1$ for each $n \geq 1$ and $\lim_{n \to \infty} g_n = \infty$. Then there exists an increasing sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ satisfying $a_n \leq ng_n$ for each $n \in \mathbb{N}$ such that $\lim_{n \to \infty} (a_n\alpha) = 0$.

The bound $a_n \leq ng_n$ for $n \in \mathbb{N}$ on the growth of $(a_n)_{n=1}^{\infty}$ in Theorem 1 is the best possible in the sense that the condition $\lim_{n \to \infty} g_n = \infty$ cannot be weakened. Indeed, suppose that there is a constant $g \geq 1$ and an increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$ satisfying $a_n \leq gn$ for infinitely many $n \in \mathbb{N}$. Then $\liminf_{n \to \infty} a_n/n \leq g < \infty$, so the sequence $A = (a_n)_{n=1}^{\infty}$ has a positive upper
Theorem 2. Let $(a_n)_{n=1}^\infty$ be an increasing sequence of positive integers with positive upper density, i.e., $\liminf_{n\to\infty} a_n/n < \infty$, and let $\xi$ be an irrational real number. Then the sequence of fractional parts $\{(p/q)^n\xi\}_{n=1}^\infty$ has infinitely many limit points.

In this respect we recall the paper of Vijayaraghavan [15] once again. He proved that, for any rational non-integer number $p/q > 1$ and any real number $\xi \neq 0$, the sequence of fractional parts $\{(p/q)^n\xi\}_{n=1}^\infty$ has infinitely many limit points.

In the next section, we shall prove two auxiliary results necessary for the proof of Theorem I. Section 3 contains the proof of Theorem I. We do not know whether a similar construction of the slowly increasing sequence $(a_n)_{n=1}^\infty$ is possible for other real numbers $\alpha$ (see the end of Section 3). In Section 4, we prove Theorem 2. The proofs of both theorems are completely self contained.

2. Auxiliary results

Lemma 3. Let $\alpha$ be a real quadratic algebraic number. Then there exist $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ such that the number $\beta = p\alpha + q$ is a positive quadratic reciprocal unit with minimal polynomial $x^2 - tx + 1$, where $t \geq 4$ is an even integer.

Proof. Suppose that the minimal polynomial of $\alpha$ is

$$ax^2 + bx + c = a(x - \alpha)(x - \alpha'),$$

where $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $c \neq 0$. Since $\alpha$ is a real quadratic number, the discriminant $\Delta = b^2 - 4ac$ is a positive integer which is not a perfect square. Hence the Pell equation $X^2 - \Delta Y^2 = 1$ has a solution $X, Y \in \mathbb{N}$ with $X \geq 2$. Set $p = 2aY$ and $q = bY + X$, so that

$$\beta = 2aY\alpha + bY + X.$$

Then $\beta' = 2b\alpha' + aY + \alpha$. From $\alpha + \alpha' = -b/a$ it follows that

$$\beta + \beta' = 2aY(\alpha + \alpha') + 2bY + 2X = 2aY(-b/a) + 2bY + 2X = 2X.$$

Similarly, using $a\alpha' = c/a$, $\alpha + \alpha' = -b/a$ and $X^2 - (b^2 - 4ac)Y^2 = 1$, we obtain

$$\beta\beta' = 4a^2Y^2\alpha\alpha' + 2aY(bY + X)(\alpha + \alpha') + (bY + X)^2$$

$$= 4acY^2 - 2bY(bY + X) + b^2Y^2 + 2bXY + X^2 = (4ac - b^2)Y^2 + X^2 = 1.$$

This proves that $\beta$ is a reciprocal real quadratic unit with minimal polynomial $x^2 - 2Xx + 1$. From $\beta = (\beta^2 + 1)/(2X)$, we conclude that $\beta$ is positive. \hfill $\square$

Lemma 4. Let $\beta > 1$ be a reciprocal quadratic unit with minimal polynomial $x^2 - tx + 1$, where $t \geq 4$ is an even integer. Set $T_m = \beta^m + \beta^{-m}$ and $U_m = (\beta^m - \beta^{-m})/\sqrt{(t/2)^2 - 1}$. Then $T_m, U_m \in \mathbb{N}$,

$$T_m\beta - T_{m+1} = \beta^{-m+1}(1 - \beta^{-2})$$

and

$$U_m\beta - U_{m+1} = \beta^{-m+1}(1 - \beta^{-2})/\sqrt{(t/2)^2 - 1} - 1$$

for each $m \in \mathbb{N}$. Furthermore, $\gcd(T_m, T_{m+1}) = \gcd(U_m, U_{m+1}) = 2$ for each $m \geq 1$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Clearly, $T_0 = 2$, $T_1 = t$ and $T_{m+1} = tT_m - T_{m-1}$ for each $m \geq 1$. Similarly, $U_0 = 0$, $U_1 = 2$ and $U_{m+1} = tU_m - U_{m-1}$ for each $m \geq 1$. This proves that $T_m, U_m \in \mathbb{N}$ for each $m \in \mathbb{N}$. The numbers $T_1, T_2, \ldots$ are all even, hence $\gcd(T_m, T_{m+1}) \geq 2$. If, however, some $d > 2$ divides $T_m$ and $T_{m+1}$, then from the recurrence relation on $T_{m+1}, T_m, T_{m-1}$ we see that $d$ also divides $T_{m-1}$, and so on up to $d|T_0$, i.e., $d|2$, which is impossible. This proves that $\gcd(T_m, T_{m+1}) = 2$. The proof of $\gcd(U_m, U_{m+1}) = 2$ is the same.

From the representation $T_m = \beta^m + \beta^{-m}$, we have

$$T_m^\beta - T_{m+1} = \beta(\beta^m + \beta^{-m}) - (\beta^{m+1} + \beta^{-m-1}) = \beta^{-m+1}(1 - \beta^2).$$

Likewise,

$$\sqrt{(t/2)^2 - 1}(U_m\beta^{-1} - U_{m-1}) = \beta^{-1}(\beta^m - \beta^{-m}) - (\beta^{m-1} - \beta^{-m-1}) = \beta^{-m+1}(1 - \beta^2).$$

This finishes the proof. □

3. Proof of Theorem

Suppose that $\alpha$ is a real quadratic algebraic number and $\alpha'$ is its reciprocal over $\mathbb{Q}$. There are two cases, $\alpha > \alpha'$ and $\alpha < \alpha'$. In the first case, take $\beta = p\alpha + q$ with $p, q$ as in Lemma 3. Then $\beta > 1 > \beta' = \beta^{-1}$. In the second case, the role of $\alpha$ belongs to $\alpha'$. So we take $\beta = p\alpha' + q$ with $p, q$ as in Lemma 3. Then $\beta > 1 > \beta' = p\alpha + q = \beta^{-1}$. Note that, in both cases, we have $\beta > 1$, so Lemma 4 can be applied. Below, we shall construct the sequence $a_1 < a_2 < a_3 < \ldots$ using $T_m$, $m = 1, 2, \ldots$ (in the first case) and $U_m$, $m = 1, 2, \ldots$ (in the second case).

Note that by replacing each $g_n$ with $g_n = \inf_{j \geq n} g_j$, we can assume that the sequence $g_1, g_2, g_3, \ldots$ is non-decreasing. By replacing each $g_n$ with its integer part $[g_n]$, we can assume that each $g_n$ is a positive integer. Finally, by reducing each positive gap $k = g_{m+1} - g_m$, where $k \geq 2$, to the gap with $k = 1$, we can assume without loss of generality that $g_{n+1} - g_n \leq 1$.

Take $\beta > 1$ as above (namely, $\beta = p\alpha + q$ or $\beta = p\alpha' + q$),

$$c = 8p\beta^3$$

and

$$k_m = \lfloor c(\beta^m/g_m) \rfloor = \lfloor 8c\beta^{m+5}/g_m \rfloor.$$

Let

$$A_m = \{pkT_{m+1} + p\ell T_m \mid k = 1, \ldots, k_{m+1}, \ \ell = 1, \ldots, k_m\},$$

$$A'_m = \{pkU_{m+1} + p\ell U_m \mid k = 1, \ldots, k_{m+1}, \ \ell = 1, \ldots, k_m\}.$$ 

Consider the sets $B = \bigcup_{m=1}^\infty A_m$ and $B' = \bigcup_{m=1}^\infty A'_m$. Denote their distinct elements by $b_1 < b_2 < b_3 < \ldots$ and $b'_1 < b'_2 < b'_3 < \ldots$, respectively. The required sequence $A = \{a_1 < a_2 < a_3 < \ldots\}$ will be obtained from $B$ in the first case and from $B'$ in the second case. In both cases, we just replace several first elements of $B$ (resp. $B'$) by smaller positive integers.

Let us first show that, in the first case,

$$\lim_{n \to \infty} \{b_n \alpha\} = 0.$$

Suppose that $b_n \in A_m$. Such $m \in \mathbb{N}$ is not necessarily unique, but $m \to \infty$ provided that $n \to \infty$, and, vice versa, $n \to \infty$ as $m \to \infty$. By the above, $b_n = pkT_{m+1} + p\ell T_m$ with some $k, \ell \in \mathbb{N}$ satisfying $1 \leq k, \ell \leq \max\{k_m, k_{m+1}\} \leq c\beta^{m+1}/g_m$. From $\beta = p\alpha + q$ it follows that

$$\{b_n \alpha\} = \{(kT_{m+1} + \ell T_m)p\alpha\} = \{(kT_{m+1} + \ell T_m)\beta\}.$$
Using the upper bound for \( k \) and \( \ell \), the formulae \( c = 8p\beta^5 \) and Lemma 3, we deduce that

\[
\{b_n\}_{\alpha} = \{(kT_{m+1} + \ell T_m)\beta\} = k(T_{m+1}\beta - T_m) + \ell(T_m\beta - T_{m+1})
\]

\[
\leq \beta^{-m}(1 - \beta^{-2})(k + \ell\beta) \leq \beta^{-m}(1 - \beta^{-2})(1 + \beta)c\beta^m + 1/gm
\]

\[
= (\beta + \beta^2)(1 - \beta^{-2})c/gm < 16p\beta^7/gm
\]

for each sufficiently large \( m \). (Certainly, this holds for those \( m \) for which \( gm > 16p\beta^7 \).) If \( n \to \infty \), then \( m \to \infty \) and \( y_n \to \infty \). Hence \( \lim_{n \to \infty} \{b_n\}_{\alpha} = 0 \), as claimed.

Similarly, in the second case, the equality \( p\alpha + q = \beta' = \beta^{-1} \) combined with the representation \( b_n = pkU_{m+1} + p\ell U_m \) yields \( \{b_n\}_{\alpha} = \{(kU_{m+1} + \ell U_m)\beta^{-1}\} \). Using the fact that \( U_m\beta^{-1} - U_{m-1} \) is ‘small’ (see Lemma 4), in exactly the same manner as above we can prove that, in the second case, \( \lim_{n \to \infty} \{b_n\}_{\alpha} = 0 \).

Our next goal is to show that the elements of the set \( A_m = \{pkT_{m+1} + p\ell T_m \mid k = 1, \ldots, k_{m+1}, \ell = 1, \ldots, k_m\} \) are distinct for \( m \geq m_1 \). Assume that \( pkT_{m+1} + p\ell T_m = pk'T_{m+1} + p'\ell T_m \), where \( \ell \neq \ell' \). Then \( (k - k')T_{m+1}/2 = (\ell' - \ell)T_m/2 \). By Lemma 1, the integers \( T_{m+1}/2 \) and \( T_m/2 \) are coprime. It follows that \( T_{m+1}/2 \) divides \( |\ell - \ell'| \). Therefore, \( \beta^{-m+1} < T_{m+1} \leq 2|\ell - \ell'| \leq 2k_m \leq 2c\beta^m/gm \). Setting \( m_1 \) to be the least integer for which \( g_m \geq 2c \), we derive that \( \beta^{-m+1} < \beta^m \) for \( m \geq m_1 \), a contradiction.

Likewise, the elements of the set \( A'_m = \{pkU_{m+1} + p\ell U_m \mid k = 1, \ldots, k_{m+1}, \ell = 1, \ldots, k_m\} \) are distinct for \( m \geq m_2 \).

Let us take an integer \( M \geq \max\{m_1, m_2\} \), where \( M \) is so large that

\[
m \leq k_m < \beta^2k_{m-1} \quad \text{for} \quad m \geq M.
\]

Such an \( M \) exists, because the quotient \( k_m/k_{m-1} \) is ‘approximately’ \( \beta g_m/g_{m-1} \), which is less than or equal to \( \beta(1 + g_{m-1})/g_m - 1 < \beta(1 + \varepsilon) \) for \( m \) large enough.

For any integer \( n > k_{M-1}k_M \), there is a unique integer \( m \geq M \) such that

\[
k_{m-1}k_m < n \leq k_mk_{m-1}.
\]

Since all \( k_{m+1}k_m \) elements of \( A_m \) (resp. \( A'_m \)) are distinct, the \( n \)th element of \( B \) (resp. \( B' \)) does not exceed the \( n \)th element of \( A_m \) (resp. \( A'_m \)). The largest element of \( A_m \) is \( pk_{m+1}T_{m+1} + pk_mT_m \). Hence, using the bounds \( k_{m+1} < \beta^4k_{m-1}, T_{m+1} < 2\beta^m + 1 \) and \( \beta^m < 2g_mk_m/c \), we obtain

\[
b_n \leq pk_{m+1}T_{m+1} + pk_mT_m < 2pk_{m+1}T_{m+1} \leq 4p\beta^4k_{m-1}k_m + 1 \beta^m + 1 \leq 4p\beta^5k_{m-1}k_m + 1 \beta^m + 1 < 8p\beta^5k_{m-1}k_m + 1 \beta^m + 1 < k_{m-1}k_m c = k_{m-1}k_m g_m.
\]

This is less than \( ng_n \), because \( m \leq k_{m-1}k_m \), the sequence \( g_1, g_2, \ldots \) is non-decreasing, and \( k_{m-1}k_m < n \). Consequently, \( b_n < ng_n \) for each \( n > k_{M-1}k_M \). Similarly, using \( U_{m+1} < \beta^{m+1} \), we obtain

\[
b'_n \leq pk_{m+1}U_{m+1} + pk_mU_m < 2pk_{m+1}U_{m+1} \leq 2p\beta^4k_{m-1}k_m + 1 \beta^{m+1} \leq 4p\beta^5k_{m-1}k_m + 1 \beta^{m+1} \leq k_{m-1}k_m g_m < ng_n
\]

for each \( n > k_{M-1}k_M \). This proves the required upper bound for \( b_n \) and \( b'_n \) provided that \( n \) is large enough.

Trivially, \( b_n \geq n \) and \( b'_n \geq n \) for each positive integer \( n \). Thus, by the above, there exists a positive integer \( n_0 \), say \( n_0 = k_{M-1}k_M \), such that \( n \leq b_n < ng_n \) and \( n \leq b'_n < ng_n \) for each \( n \geq n_0 + 1 \). In the first case, \( \alpha > \alpha' \), the required
increasing sequence of positive integers \( A = \{a_1 < a_2 < a_3 < \ldots \} \) can be obtained from \( B = \bigcup_{m=1}^{\infty} A_m = \{b_1 < b_2 < b_3 < \ldots \} \) by setting \( a_n = n \) for \( n \leq n_0 \) and \( a_n = b_n \) for \( n \geq n_0 + 1 \). In the second case, if \( \alpha' > \alpha \), the required increasing sequence of positive integers \( A = \{a_1 < a_2 < a_3 < \ldots \} \) can be obtained from \( B' = \bigcup_{m=1}^{\infty} A'_m = \{b'_1 < b'_2 < b'_3 < \ldots \} \) by setting \( a_n = n \) for \( n \leq n_0 \) and \( a_n = b'_n \) for \( n \geq n_0 + 1 \). In both cases, we have \( a_n \leq n_g n \) for each \( n \geq 1 \). This completes the proof of the theorem. \( \square \)

Suppose that \( \xi \) is either a real algebraic number of degree \( \geq 3 \) or a real transcendental number. Is there a slowly increasing sequence of positive integers \( a_1 < a_2 < a_3 < \ldots \) satisfying, for instance, \( a_n \leq n[(\log n)^\alpha] \) for each \( n \geq 3 \), such that \( \lim_{n \to \infty} a_n \xi \) is either a real algebraic number of degree \( \geq 3 \) or a real transcendental number. This result is, of course, weaker than the same inequality for every \( \xi \) satisfying, for instance, \( \lim \sup_{n \to \infty} a_n \xi \) is either a real algebraic number of degree \( \geq 3 \) or a real transcendental number. Therefore, \( 0 < d_m^2 a_n \gamma - 2^m a_n \ell \) and define \( A = \bigcup_{m=1}^{\infty} A_m = \{a_1 < a_2 < a_3 < \ldots \} \).

By the choice of \( \ell_m \), it is easy to see that \( \{a_n \gamma \} \to 0 \) as \( n \to \infty \). Furthermore, for each \( n = \ell_m \), we have \( a_n = a_m \leq \ell_m 2^{d_m} < \ell_m g_{\ell_m} \leq n_g n \), because the elements of \( A_m \) are distinct. So the inequality \( a_n \leq n_g n \) holds for infinitely many \( n \in \mathbb{N} \). The number \( \gamma \) is a transcendental Liouville number if \( \lim_{n \to \infty} a_n \gamma = 0 \) as \( n \to \infty \). This completes the proof of the theorem.

**Theorem 5.** For any sequence \( 1 \leq g_1 \leq g_2 \leq \ldots \) satisfying \( \lim_{n \to \infty} g_n = \infty \), there is a transcendental Liouville number \( \gamma \) for which there is a sequence of positive integers \( \{a_n\}_{n=1}^{\infty} \) satisfying \( a_n \leq n_g n \) for infinitely many \( n \in \mathbb{N} \) such that \( \{a_n \gamma \} \to 0 \) as \( n \to \infty \).

**Proof.** Take \( \gamma = \sum_{k=1}^{\infty} 2^{-d_k} \), where \( (d_k)_{k=1}^{\infty} \) is a sequence of positive integers increasing so fast that \( d_{k+1} > 3d_k \) and \( g_3 > 3^k \), where \( \ell_k = [2^{d_{k+1}/2}] \). Then \( 0 < 2^{d_m} a_n - u_m < 2^{-d_{m+1} d_m+1} + 1 \) with some \( u_m \in \mathbb{N} \). Therefore, \( 0 < \ell^{2^{d_m} \gamma} < \ell^{2^{d_m} \gamma} = 0 \) and \( \gamma \) is either a real algebraic number of degree \( \geq 3 \) or a real transcendental number.

This result is, of course, weaker than the same inequality \( a_n \leq n_g n \) of Theorem 1 which holds for all \( n \in \mathbb{N} \).

### 4. Proof of Theorem 2

Set \( g = \liminf_{n \to \infty} a_n / n < \infty \). Suppose that the sequence \( \{a_n \xi\}_{n=1}^{\infty} \) has only \( t \) limit points for some \( \xi \notin \mathbb{Q} \). Let us denote the number of elements of \( A \) lying in \([1, x] \) by \( A(x) \). The condition \( g = \liminf_{n \to \infty} a_n / n < \infty \) implies that \( A(n) > n/(2g) \) for infinitely many \( n \in \mathbb{N} \).

Put \( L = [3gt] \). We claim that the sequence \( A = \{a_n\}_{n=1}^{\infty} \) contains at least \( t + 1 \) elements in infinitely many intervals \([N + 1, N + L] \), where \( N \in \mathbb{N} \). Indeed, if at most \( t \) elements of \( A \) lie in each of the intervals \([kL + 1, kL + L] \), \( k = 0, 1, 2, \ldots \), except for, say, \( l \) intervals, then the number of elements of \( A \) up to \( kL \) is \( \leq (L + (k - l)t \), i.e., \( A(kL) \leq lL + (k - l)t \). For a given \( n \in \mathbb{N} \), take \( k \in \mathbb{N} \) such that \( (k - 1)L < n \leq kL \). Then, using \( L \geq 3gt \), we find that

\[
A(n) \leq A(kL) \leq kt + L < (n/L + 1)t + L \leq n/(3g) + t + L.
\]
So the inequality \( A(n) > n/(2q) \) cannot hold for infinitely many \( n \in \mathbb{N} \), a contradiction. This proves our assertion.

Note that \( ||q\xi|| > 0 \) for every \( q \in \mathbb{N} \), because \( \xi \notin \mathbb{Q} \). Here an below \( ||x|| \) stands for the distance from a real number \( x \) to the nearest integer. Fix any \( \varepsilon \) satisfying
\[
0 < 2\varepsilon < \min\{||\xi||, ||2\xi||, \ldots, ||(L-1)\xi||\}.
\]

By our assumption, the sequence \( \{a_n\xi\}_{n=1}^\infty \) has only \( t \) limit points. Hence, for \( n \geq n_0(\varepsilon) \), the fractional part \( \{a_n\xi\} \) must lie in an \( \varepsilon \)-neighborhood of at least one of those \( t \) points. Take an interval \([N+1, N+L] \), where \( N \geq n_0(\varepsilon) \), which contains at least \( t + 1 \) elements of \( A \). (We already proved that this happens for infinitely many \( N \in \mathbb{N} \), so such an interval exists.) By Dirichlet’s box principle, at least two fractional parts, say, \( \{a_{N+u}\xi\} \) and \( \{a_{N+v}\xi\} \), where \( 1 < u < v < L \), lie in an \( \varepsilon \)-neighborhood of the same limit point, say, \( w \). Putting \( r = a_{N+v} - a_{N+u} \), where \( r \in \{1, \ldots, L-1\} \), and using \( ||a_{N+u}\xi - w||, ||a_{N+v}\xi - w|| \leq \varepsilon \), we deduce that
\[
2\varepsilon < ||r\xi|| = ||(a_{N+v} - a_{N+u})\xi|| \leq ||a_{N+u}\xi - w|| + ||w - a_{N+u}\xi|| \leq \varepsilon + \varepsilon = 2\varepsilon,
\]
a contradiction. This completes the proof of the theorem. \( \square \)

By the same argument as above one can prove that if \( \{a_n\}_{n=1}^\infty \) is an increasing sequence of positive integers satisfying \( \liminf_{n \to \infty} (a_{n+1} - a_n) < \infty \) and \( \xi \) is an irrational real number, then the sequence of fractional parts \( \{a_n\xi\}_{n=1}^\infty \) has at least two limit points.

Indeed, the condition \( \liminf_{n \to \infty} (a_{n+1} - a_n) < \infty \) implies that there exists a positive integer \( r \) such that \( a_{n+1} - a_n = r \) for infinitely many \( n \)'s. Suppose that for some real irrational number \( \xi \) we have \( \lim_{n \to \infty} \{a_n\xi\} = w \), where \( 0 \leq w \leq 1 \). Then, for any \( \varepsilon > 0 \) there exists \( n_0(\varepsilon) \) such that \( ||a_n\xi - w|| \leq \varepsilon \) for each \( n \geq n_0(\varepsilon) \). Fix \( \varepsilon > 0 \) satisfying \( 0 < 2\varepsilon < ||r\xi|| \). Take any \( n \geq n_0(\varepsilon) \) for which \( a_{n+1} - a_n = r \). Then
\[
2\varepsilon < ||r\xi|| = ||(a_{n+1} - a_n)\xi|| = ||a_{n+1}\xi - w + w - a_n\xi|| \leq ||a_{n+1}\xi - w|| + ||a_n\xi - w|| \leq 2\varepsilon,
\]
a contradiction.

Acknowledgements

I thank the referee, whose useful remarks made this paper more readable. This research was supported in part by the Lithuanian State Studies and Science Foundation.

References


Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania
E-mail address: arturas.dubickas@mif.vu.lt