CONSTRUCTIVE DECOMPOSITION OF A FUNCTION OF TWO VARIABLES AS A SUM OF FUNCTIONS OF ONE VARIABLE

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Abstract. Given a compact set \( K \) in the plane, which does not contain any triple of points forming a vertical and a horizontal segment, and a map \( f \in C(K) \), we give a construction of functions \( g, h \in C(\mathbb{R}) \) such that \( f(x, y) = g(x) + h(y) \) for all \((x, y) \in K\). This provides a constructive proof for a part of Sternfeld’s theorem on basic embeddings in the plane. In our proof the set \( K \) is approximated by a finite set of points.

1. Introduction

An embedding \( \varphi: K \to \mathbb{R}^k \) of a compactum (compact metric space) \( K \) in the \( k \)-dimensional Euclidean space \( \mathbb{R}^k \) is called a basic embedding if for each continuous real-valued function \( f \in C(K) \) there exist continuous real-valued functions of a single real variable \( g_1, \ldots, g_k \in C(\mathbb{R}) \) such that \( f(x_1, \ldots, x_k) = g_1(x_1) + \ldots + g_k(x_k) \) for each point \((x_1, \ldots, x_k) \in \varphi(K)\). We also say that the set \( \varphi(K) \) is basically embedded in \( \mathbb{R}^k \).

The question of the existence of basic embeddings is related to Hilbert’s 13th problem [5]: Hilbert conjectured that not all continuous functions of three variables are expressible as superpositions of continuous functions of a smaller number of variables.

Ostrand [10] proved that each \( n \)-dimensional compactum can be basically embedded in \( \mathbb{R}^{2n+1} \) for \( n \geq 1 \). His result is an easy generalization of results of Arnold [1, 3] and Kolmogorov [6, 7].

Sternfeld [14] proved that the \( 2n + 1 \) is the best possible in a very strong sense: namely, no \( n \)-dimensional compactum can be basically embedded in \( \mathbb{R}^{2n} \) for \( n \geq 2 \). Ostrand’s and Sternfeld’s results thus characterize compacta basically embeddable in \( \mathbb{R}^k \) for \( k \geq 3 \). Basic embeddability in the real line is trivially equivalent to embeddability. The remaining problem of the characterization of compacta basically embeddable in \( \mathbb{R}^2 \) was already raised by Arnold [2] and solved by Sternfeld [15]:

**Theorem 1.1** (Sternfeld). Let \( K \) be a compactum and let \( \varphi: K \to \mathbb{R}^k \) be an embedding. Then

(B) \( \varphi \) is a basic embedding
if and only if

(A) there exists an \( m \in \mathbb{N} \) such that the set \( \varphi(K) \) does not contain an array of length \( m \).

**Definition 1.2.** An array is a sequence of points \( \{(x_i, y_i) \in \mathbb{R}^2 \mid i \in I\} \), where \( I = \{1, 2, \ldots, m\} \) or \( I = \mathbb{N} \), such that each two consecutive points \((x_i, y_i), (x_{i+1}, y_{i+1})\) are different and either \( x_{2j-1} = x_{2j} \) and \( y_{2j} = y_{2j+1} \) for all \( j \) or \( y_{2j-1} = y_{2j} \) and \( x_{2j} = x_{2j+1} \) for all \( j \). If \( I = \{1, 2, \ldots, m\} \), then the length of the array is \( m - 1 \).

Using the geometric description (A), Skopenkov [13] gave a characterization of continua basically embeddable in the plane by means of forbidden subsets resembling Kuratowski’s characterization of planar graphs. In a similar way Kurlin [8] characterized finite graphs basically embeddable in \( \mathbb{R} \times T_n \), where \( T_n \) is a star with \( n \)-rays. Repovš and Željko [11] proved a result concerning the smoothness of the functions in the decomposition on basically embedded subsets of the plane and gave a constructive decomposition on finite graphs in the plane which do not contain arrays of arbitrary length (i.e. those satisfying (A) of Theorem 1.1).

Sternfeld’s proof of the equivalence \( (A) \Leftrightarrow (B) \) is not direct but uses a reduction to linear operators. In particular, it is not constructive. It is therefore desirable to find a straightforward, constructive proof which will consequently provide an elementary proof of Skopenkov’s and Kurlin’s characterizations. A constructive proof of \( (B) \Rightarrow (A) \) is given in [9]. In this paper we give an elementary construction proving the implication \( (A) \Rightarrow (B) \) provided that \( m = 2 \).

**Theorem 1.3.** Let \( \varphi: K \rightarrow \mathbb{R}^2 \) be an embedding of a compactum \( K \) in the plane such that the set \( \varphi(K) \) does not contain an array of length two. Then for every function \( f \in C(\varphi(K)) \) there exist functions \( g, h \in C(\mathbb{R}) \) such that \( f(x, y) = g(x) + h(y) \) for each point \((x, y) \in \varphi(K)\).

So far, no constructive decomposition of \( f \) as \( g + h \) on compacta in the plane satisfying (A) of Theorem 1.1 has been found, not even in the simplest case when the compactum satisfies (A) with \( m = 2 \). It turns out that even if a set \( \varphi(K) \subset \mathbb{R}^2 \) satisfies this simplest version of condition (A), a constructive decomposition of a function \( f \in C(\varphi(K)) \) is a non-trivial problem. We also believe that our argument can be modified to obtain a constructive proof (in the sense described below) of the implication \( (A) \Rightarrow (B) \) for an arbitrary \( m \in \mathbb{N} \).

The main part of our proof (Theorem 2.2) consists of finding an approximate decomposition of a given function \( f \) as \( g + h \). The functions \( g, h \) are defined on the projections of a finite approximation \( V \) of \( \varphi(K) \). Then they are linearly extended to \( \mathbb{R} \). Apart from two steps where we assert the existence of certain constants, this part of the proof is constructive. The existence of an exact decomposition \( f = g + h \) follows by an elementary iterative procedure (Theorem 2.1).

In [11], the authors give the standard decomposition \( f = g + h \) on finite graphs in the plane. According to the results of [13, 4], a finite graph can be basically embedded in the plane if and only if it can be embedded in a special graph \( R_n \) for some \( n \). The authors of [11] inductively define an embedding \( \varphi: R_n \rightarrow \mathbb{R}^2 \). For a given function \( f \in C(\varphi(R_n)) \), they define the maps \( g, h \in C(\mathbb{R}) \) inductively again, starting from a well chosen subset of \( \varphi(R_n) \). Although the sets in the plane we are dealing with do not contain arrays of length two, they can still be “arbitrarily bad”. In particular, we are not able to choose a suitable subset on which we could start a similar construction of the functions \( g \) and \( h \). However, we do use a similar
approach in the construction of the sequences of approximations which converge to the required functions \( g \) and \( h \).

The proof of our result resembles, in a certain way, the proofs of Arnold [11, 12], Kolmogorov [6, 7] and Ostrand [10]. As in these proofs, we construct a sequence of finite families of squares. However, there the squares (or cubes in higher dimensions) are related only to the dimension of the set in question; here the squares are constructed using the property that the set does not contain an array of length two.

The paper is organized as follows: in Section 2 we formulate and prove the two statements, Theorem 2.1 and Theorem 2.2 which directly imply the main result, Theorem 1.3. Then we give the statements, Lemma 2.5 and Lemma 2.6 required in the proof of Theorem 2.2. Lemma 2.5 is proved in this section, and the proof of Lemma 2.6 is postponed to Section 3. Finally, in Section 4 we roughly explain where the proof breaks down in the case of an arbitrary \( m \).

2. Proof of the main result—Theorem 1.3

It is convenient to omit the embedding \( \varphi \) from the notation \( \varphi(K) \subseteq \mathbb{R}^2 \) and assume that \( K \) is a subset of \( \mathbb{R}^2 \).

We denote points in the plane by \((x, y)\), with \( x, y \in \mathbb{R} \); intervals in \( \mathbb{R} \) by \([x; x']\), \([x; x')\), \((x; x')\), with \( x, x' \in \mathbb{R} \); and segments in \( \mathbb{R}^2 \) by \([z; z']\), with \( z, z' \in \mathbb{R}^2 \).

We define the distance in \( \mathbb{R}^2 \) as \(|(x, y) - (x', y')| = \max\{|x - x'|, |y - y'|\}\) for \((x, y), (x', y') \in \mathbb{R}^2 \). By \( p, q : \mathbb{R}^2 \to \mathbb{R} \), we denote the vertical and horizontal orthogonal projections: \( p(x, y) = x \), \( q(x, y) = y \).

The main result, Theorem 1.3 clearly follows from the following two theorems.

**Theorem 2.1.** Let \( X \subseteq \mathbb{R}^2 \) be a compact subset of the plane. Assume that there exists a positive integer \( k \in \mathbb{N} \) such that for each function \( f \in C(X) \) and each positive real \( \varepsilon > 0 \) there exist functions \( g, h \in C(\mathbb{R}) \) satisfying the following requirements:

1. \(|f(x, y) - g(x) - h(y)| < \varepsilon \) for each point \((x, y) \in X\),
2. \(|g| \leq k||f||, ||h|| \leq k||f||\).

Then there exist functions \( g, h \in C(\mathbb{R}) \) such that \( f(x, y) = g(x) + h(y) \) for each point \((x, y) \in X\).

The above result is obtained by a slight modification of a part of Theorem 4.13 in [12]; we include the proof for the sake of completeness.

**Proof.** Let \( k \) and \( f \) be given. Let \( f_1 = f \). Pick a sequence \( \{\varepsilon_n\}_{n=1}^{\infty} \) of positive reals such that \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \). Assume that \( n \geq 1 \) and \( f_n \) is constructed. Choose functions \( g_n, h_n \in C(\mathbb{R}) \) such that \(||f_n - g_n - h_n|| < \varepsilon_n, ||g_n|| \leq k||f_n||\) and \(||h_n|| \leq k||f_n||\). Let \( f_{n+1} = f_n - g_n - h_n \).

Since \(|g_{n+1}| \leq k||f_{n+1}|| = k||f_n - g_n - h_n|| < k\varepsilon_n\) for each \( n \), the series \( \sum_{n=1}^{\infty} g_n \) uniformly converges to a function \( g \in C(\mathbb{R}) \). Similarly \( \sum_{n=1}^{\infty} h_n = h \in C(\mathbb{R}) \). We can easily see that \( g + h = f_1 = f \).

The proof of the following theorem is postponed to the end of this section.

**Theorem 2.2.** Suppose that a compactum \( K \subseteq \mathbb{R}^2 \) contains no array of length two and that \( f \in C(K) \) is a continuous function on \( K \). Then for each \( \varepsilon > 0 \) there exist functions \( g, h \in C(\mathbb{R}) \) satisfying the following requirements:

1. \(|f(x, y) - g(x) - h(y)| < \varepsilon \) for each point \((x, y) \in K\),
2. \(|g| \leq ||f||, ||h|| \leq 2||f||\).
The construction of the approximations \( g \) and \( h \) in Theorem 2.2 mimics the following construction of the functions \( g, h \) which works for certain types of sets \( K \) (for example finite graphs, considered in [11]). In particular, this idea is used in the proof of Lemma 2.6. Let \( K_x \) be the set of all points \((x, y) \in K\) which have a neighbor in the vertical direction in \( K \), i.e. \( K_x = \{(x, y) \in K | \exists (x, y') \in K, y \neq y'\} \). The set \( K_y \) is defined in a similar way. Assume that both sets \( K_x \) and \( K_y \) are closed.

Since \( K \) does not contain an array of length two, the functions \( p \) and \( q \) are injective on \( K_y \) and \( K_x \), respectively, and the sets \( K_x \) and \( K_y \) are disjoint. For each point \( x \in p(K_y) \) let \( g(x) = f(x, p^{-1}(x)) \), and for each point \( x \in p(K_x) \) let \( g(x) = 0 \). Since \( p(K_x \cup K_y) \) is closed, we can extend \( g \) continuously to \( \mathbb{R} \). For each point \( y \in q(K) \) pick an arbitrary point \((x, y) \in K\) and let \( h(y) = f(x, y) - g(x) \). It is easily seen that \( h \) is continuous on \( q(K) \), and thus it can be extended continuously to \( \mathbb{R} \).

The function \( g \) was constructed on the set \( p(K_x \cup K_y) \) first, and then, using the fact that \( p(K_x \cup K_y) \) is closed, it was extended continuously to \( \mathbb{R} \). In general, neither \( p(K_x \cup K_y) \) nor \( q(K_x \cup K_y) \) is closed, though, and the set \( K \) can be so “bad” that we cannot find suitable sets like \( K_x \) and \( K_y \) to begin the construction. Therefore we construct approximations of \( K \).

**Definition 2.3.** Let \( \alpha > 0 \) be a positive real. Two points \( z_1, z_2 \in \mathbb{R}^2 \) are said to form an *\( \alpha \)-vertical pair* if \( |p(z_1) - p(z_2)| < \alpha \) and an *\( \alpha \)-horizontal pair* if \( |q(z_1) - q(z_2)| < \alpha \). A sequence of points \( \{z_1, z_2, \ldots, z_k\} \) from \( \mathbb{R}^2 \) such that \( z_i \neq z_j \) for all \( i \neq j \), with the exception that possibly \( z_1 = z_k \), is said to form an *\( \alpha \)-array* if each pair of consecutive points \( z_i, z_{i+1} \) forms either an \( \alpha \)-vertical pair or an \( \alpha \)-horizontal pair or both.

Note that, unlike an array, an \( \alpha \)-array can contain two or more consecutive \( \alpha \)-vertical pairs or two or more consecutive \( \alpha \)-horizontal pairs.

**Definition 2.4.** Let \( \alpha, \beta > 0 \) be positive reals and let \( l \in \mathbb{N} \) be a positive integer. A set \( X \subseteq \mathbb{R}^2 \) is said to be \((\alpha, \beta, l)\)-faithful if for each \( \alpha \)-array \( \{z_0, z_1, \ldots, z_{k+1}\} \) in \( X \) such that \( |p(z_0) - p(z_1)| \geq \beta \) and \( |q(z_k) - q(z_{k+1})| \geq \beta \) we have \( k > l \).

Clearly, if \( X \) is \((\alpha, \beta, l)\)-faithful, then for each positive \( \alpha' \leq \alpha \) each subset \( X' \subseteq X \) is \((\alpha', \beta, l)\)-faithful.

**Lemma 2.5.** Suppose that a compactum \( K \subseteq \mathbb{R}^2 \) contains no array of length two. Then for each \( \beta > 0 \) and \( l \in \mathbb{N} \) there exists an \( \alpha > 0 \) such that \( K \) is \((\alpha, \beta, l)\)-faithful.

**Proof.** Assuming the contrary, there exists a \( \beta > 0 \) and an \( l \in \mathbb{N} \) such that for each \( n \in \mathbb{N} \) the set \( K \) is not \((1/n, \beta, l)\)-faithful. That is, for each \( n \) there is a \( 1/n \)-array \( \{z^n_0, z^n_1, \ldots, z^n_{k+1}\} \subseteq K \) with \( |p(z^n_0) - p(z^n_1)| \geq \beta, |q(z^n_k) - q(z^n_{k+1})| \geq \beta \) and \( k_n \leq l \). Without loss of generality we may assume that \( k_n = l \) for all \( n \). Since \( K \) is compact, there is a subsequence \( \{z^{m_n}_0, z^{m_n}_1, \ldots, z^{m_n}_{l+1}\} \) such that for each \( i \) the sequence \( \{z^{m_n}_i\} \) converges to a point \( z_i \in K \). For each \( i \) either \( p(z_i) = p(z_{i+1}) \) or \( q(z_i) = q(z_{i+1}) \) (or both in case \( z_i \) and \( z_{i+1} \) are equal). Moreover, \( |p(z_0) - p(z_1)| \geq \beta \) and \( |q(z_k) - q(z_{k+1})| \geq \beta \). Hence the set \( \{z_0, z_1, \ldots, z_{l+1}\} \), which is a subset of \( K \), contains an array of length two, which is a contradiction.

The proof of the following lemma is postponed until the next section.
Lemma 2.6. Let $f : V \to \mathbb{R}$ be a function defined on a finite set of points $V \subseteq \mathbb{R}^2$. Let $\varepsilon, \delta, \alpha > 0$ be positive reals such that $|f(z) - f(z')| < \varepsilon$ for all pairs $z, z' \in V$ with $|z - z'| < \delta$ and the set $V$ is $(\alpha, \delta, ||f||/\varepsilon)$-faithful. Then there exist functions $G : p(V) \to \mathbb{R}$ and $H : q(V) \to \mathbb{R}$ satisfying the following requirements:

1. $(a) |f(x, y) - G(x) - H(y)| \leq 4\varepsilon$ for each point $(x, y) \in V$, 

2. $G(x_1) - G(x_2) \leq \varepsilon$, for each $\alpha$-vertical pair $(x_1, y_1), (x_2, y_2) \in V$;

3. $|H(y_1) - H(y_2)| \leq 2\varepsilon$ for each $\alpha$-horizontal pair $(x_1, y_1), (x_2, y_2) \in V$;

4. $||G|| \leq ||f||$, $||H|| \leq 2||f||$.

Proof of Theorem 2.2. Since $f$ is continuous on the compact set $K$, the function $f$ is uniformly continuous. Therefore there exists a positive real $\delta > 0$ such that $|f(z) - f(z')| < \varepsilon/8$ for all points $z, z' \in K$ with $|z - z'| < \delta$. According to Lemma 2.5, there exists an $\alpha > 0$ such that the set $K$ is $(\alpha, \delta, ||f||/\varepsilon)$-faithful. Since the set $K$ is $(\alpha', \delta, ||f||/\varepsilon)$-faithful for each $\alpha' \leq \alpha$, we may assume that $\alpha < \delta$. Pick a point from each non-empty intersection $K \cap \{(i\alpha/2; (i+1)\alpha/2) \times [j\alpha/2; (j+1)\alpha/2)]: i, j \in \mathbb{Z}\}$. Denote by $V$ the union of these points. Since $V$ is a subset of $K$, the set $V$ is $(\alpha', ||f||/\varepsilon)$-faithful. According to Lemma 2.6, there exist functions $G : p(V) \to \mathbb{R}$ and $H : q(V) \to \mathbb{R}$ satisfying items 1m, 1c and 2 of that lemma.

Let $p(V) = \{x_1, \ldots, x_k\}$, with $x_i < x_{i+1}$ for all $i$. Define $g$ on $p(V)$ by letting $g(x_i) = G(x_i)$ for each $i$. On each interval $[x_i; x_{i+1}]$ such that $|x_i - x_{i+1}| < \alpha$, extend $g$ linearly between the values $g(x_i)$ and $g(x_{i+1})$. If an interval $[x_i; x_{i+1}]$ is such that $|x_i - x_{i+1}| \geq \alpha$, then there exists an interval of the form $I = [j\alpha/2; (j+1)\alpha/2]$ such that $x_i < j\alpha/2 < (j+1)\alpha/2 \leq x_{i+1}$ and $p^{-1}(I) \cap K = \emptyset$. On the interval $[x_i; x_{i+1}]$ extend $g$ as a constant, equal to $g(x_i)$. If the interval $[j\alpha/2; (j+1)\alpha/2]$ is non-degenerate, i.e. if $(j+1)\alpha/2 < x_{i+1}$, then extend $g$ on this interval as a constant equal to $g(x_{i+1})$. On the interval $[j\alpha/2; (j+1)\alpha/2]$ extend $g$ linearly from $g(x_i)$ to $g(x_{i+1})$. On the intervals $(-\infty; x_1]$ and $[x_k; \infty)$ extend $g$ as a constant.

Let $q(V) = \{y_1, \ldots, y_l\}$ with $y_i < y_{i+1}$ for all $i$. Let $h(y_i) = H(y_i)$ for each $i$. Extend $h$ to $\mathbb{R}$ in a similar way as $g$.

By (1) of Lemma 2.6 and the assumption that $\alpha < \delta$, it follows that $|f(x, y) - g(x) - h(y)| < \varepsilon$. The bounds on the norms of the functions $g$ and $h$ follow from (2) of Lemma 2.6.

3. Proof of Lemma 2.6

We call a pair of points $z_1, z_2 \in V$ distant if $|z_1 - z_2| \geq \delta$; otherwise we call them close. Evidently $|f(z_1) - f(z_2)| < \varepsilon$ holds for every close pair of points $z_1, z_2 \in V$.

In this proof we use the idea given below the statement of Theorem 2.2 in Section 2. The analogue of the set $K_x$ is the set of all points from $V$ which belong to distant $\alpha$-vertical pairs (i.e. it is the set $\{u \in V | \exists w \in V \text{ s.t. } u, w \text{ is a distant } \alpha\text{-vertical pair}\}$), while the analogue of the set $K_y$ is the set of all points from $V$ which belong to distant $\alpha$-horizontal pairs. Accordingly, we define $G$ to be approximately 0 on the first set and approximately $f$ on the second set. In the rest of the points we only require that the difference between the values of $G$ in the vertical projection of each $\alpha$-vertical close pair does not exceed $\varepsilon$. We define $H$ as $f - G$.

For technical reasons we start by defining a function $\gamma : V \to \mathbb{R}$ with the properties listed below. The function $G$ shall be approximately $\gamma$.
(i) (a) $|\gamma(w_1) - \gamma(w_2)| \leq \varepsilon$ for each $\alpha$-vertical close pair and each $\alpha$-horizontal close pair $w_1, w_2 \in V$,
(b) $|f(w) - \gamma(w)| \leq \varepsilon$ for each $w \in V$ which belongs to a distant $\alpha$-horizontal pair,
(c) $|\gamma(w)| \leq \varepsilon$ for each $w \in V$ which belongs to a distant $\alpha$-vertical pair;
(ii) $||\gamma|| \leq ||f||$.

To construct $\gamma$, we define two abstract labeled graphs $(V_+, E_+, l_+)$ and $(V_-, E_-, l_-)$, where both $l_+$ and $l_-$ assign non-negative real values to the edges. The vertex set $V_+$ of the first graph is the set of all points $w \in V$ with $f(w) \geq 0$ with one vertex $w_+$ added, i.e. $V_+ = \{w \in V | f(w) \geq 0\} \cup \{w_+\}$. The vertex set $V_-$ of the second graph is the set of all points $w \in V$ with $f(w) < 0$ with one vertex $w_-$ added, i.e. $V_- = \{w \in V | f(w) < 0\} \cup \{w_-\}$. We define $f(w_+) = ||f|| + \varepsilon$ and $f(w_-) = -||f|| - \varepsilon$.

The labeled edge set $E_+$ consists of edges

- $w_1w_2$, where $w_1, w_2$ is a close $\alpha$-vertical pair or a close $\alpha$-horizontal pair, with label $l_+(w_1w_2) = \varepsilon$
- $w_+w$, where $w$ belongs to a distant $\alpha$-horizontal pair, with label $l_+(w_+, w) = f(w_+) - f(w) = ||f|| + \varepsilon - f(w)$.

The edges $E_-$ are defined analogously. The edges of the form $w_-w$ are labeled by $-f(w_-) + f(w) = ||f|| + \varepsilon + f(w)$, and the remaining edges are labeled by $\varepsilon$. Note that

\[
|f(w_1) - f(w_2)| \leq l_+(w_1w_2) \quad \text{for all } w_1w_2 \in E_+,
|f(w_1) - f(w_2)| \leq l_-(w_1w_2) \quad \text{for all } w_1w_2 \in E_-.\]

Let $d_+: V_+ \to [0; \infty]$ be the function assigning to each vertex connected to $w_+$ by a path in $E_+$ its weighed distance from $w_+$ and assigning to each other vertex the value $\infty$. For each vertex $w \in V_+$ let

$$\gamma(w) = \max\{||f|| + \varepsilon - d_+(w), 0\}.$$ 

The function $d_-: V_- \to [0; \infty]$ is defined analogously, and $\gamma$ on $V_-$ is defined as $\gamma(w) = \min\{||f|| - \varepsilon + d_-(w), 0\}$.

Let us show that the function $\gamma$ satisfies (i) and (ii).

\[\text{(ia)}\] Let $w_1, w_2$ be a close $\alpha$-vertical pair or a close $\alpha$-horizontal pair in $V$. If both $w_1$ and $w_2$ are in $V_+$ or both $w_1$ and $w_2$ are in $V_-$, then it follows directly from the definition of $\gamma$ that $|\gamma(w_1) - \gamma(w_2)| \leq \varepsilon$. So, let $w_1 \in V_+$ and $w_2 \in V_-$. By an inductive argument we show that

\[
0 \leq \gamma(w) \leq f(w) \quad \text{for all } \ w \in V_+,
0 \leq f(w) \leq \gamma(w) \leq 0 \quad \text{for all } \ w \in V_-.
\]

The points $w_1, w_2$ form a close pair; therefore $|f(w_1) - f(w_2)| \leq \varepsilon$. So $|\gamma(w_1) - \gamma(w_2)| = |\gamma(w_1) - f(w_1) - f(w_2)| \leq f(w_1) - f(w_2) < \varepsilon$.

\[\text{(ib)}\] Let $w$ belong to a distant $\alpha$-horizontal pair in $V$. For example, let $w \in V_+$. Since $w$ is connected to $w_+$ by an edge labeled by $||f|| + \varepsilon - f(w)$, this expression is an upper bound on $d_+(w)$. On the other hand, equation (3.1) implies that the sum of the labels on each path connecting $w_+$ to $w$ is at least $||f|| + \varepsilon - f(w)$, and $d_+(w)$ follows.

\[\text{(ic)}\] Let $w$ belong to a distant $\alpha$-vertical pair in $V$. For instance, let $w \in V_+$. If $w$ is not connected to $w_+$ by a path, then, by definition, $\gamma(w) = 0$. So, let $w_+w_1w_2 \ldots w_k$ with $w_k = w$ be a path such that $d_+(w)$ is equal to the sum of its
labels. (Note that except for \(w,w_1\) this path does not contain any other edge of the form \(w,w\) because this would yield a cycle.) By the definition of \(E_+\), the vertex \(w_1\) belongs to some distant \(\alpha\)-horizontal pair \(w_0, w_1\) in \(V\). By assumption the vertex \(w_k = w\) belongs to some distant \(\alpha\)-vertical pair \(w_k, w_{k+1}\). The path \(w_1w_2\ldots w_k\) is such that each of its edges corresponds to a close \(\alpha\)-vertical or \(\alpha\)-horizontal pair in \(V\). Hence the vertices \(w_0, w_1, \ldots, w_{k+1}\) form an \(\alpha\)-array which satisfies the requirements of Definition 2.4. By assumption the set \(V\) is \((\alpha, \delta, ||f||/\varepsilon)|\)-faithful, so \(k > ||f||/\varepsilon\). Each of the edges \(w_1w_2, \ldots, w_{k-1}w_k\) is labeled by \(\varepsilon\), so \(d_+(w) = d_+(w_1, w) + (k - 1)\varepsilon > ||f|| - f(w_1) + \varepsilon + (||f||/\varepsilon - 1)\varepsilon \geq ||f||\). Hence \(||f|| + \varepsilon - d_+(w) < \varepsilon\), so \(|\gamma(w)| < \varepsilon\).

Item (i) follows directly from equation (3.2).

The functions \(G\) and \(H\) are constructed in the following way. For each point \(x \in p(V)\) fix an arbitrary point \((x, y_x) \in V\) and define \(G(x) = \gamma(x, y_x)\). For each point \(y \in q(V)\) fix an arbitrary point \((x_y, y) \in V\) and let \(H(y) = f(x_y, y) - \gamma(x_y, y)\).

In order to prove item (i) of Lemma 2.6 consider an arbitrary point \((x, y) \in V\).

Then
\[
(3.3) \quad |f(x, y) - G(x) - H(y)| = |f(x, y) - \gamma(x, y_x) - f(x_y, y) + \gamma(x_y, y)|.
\]

The points \((x, y), (x, y_x)\) form a \(\alpha\)-vertical pair (in fact a vertical pair), and the points \((x, y), (x_y, y)\) form an \(\alpha\)-horizontal pair (in fact a horizontal pair).

If both pairs are close pairs, then item (ia) and the fact that \(|f(x, y) - f(x_y, y)| < \varepsilon\) imply that the expression in equation (3.3) is at most \(4\varepsilon\).

Let \((x, y), (x, y_x)\) be a distant pair. From (ia) it follows that \(|\gamma(x, y)| \leq \varepsilon\) and \(|\gamma(x, y_x)| \leq \varepsilon\).

If \((x, y), (x, y_x)\) is a close pair, then \(|f(x, y) - f(x_y, y)| < \varepsilon\) and, by (ia), it follows that \(|\gamma(x, y) - \gamma(x, y_x)| \leq \varepsilon\). Therefore \(|\gamma(x, y)| \leq 2\varepsilon\) and (3.3) is at most \(4\varepsilon\).

If \((x, y), (x, y_x)\) is a distant pair, then, by (ib), we have \(|f(x, y) - \gamma(x, y)| \leq \varepsilon\) and \(|f(x, y) - \gamma(x, y_x)| \leq \varepsilon\), so \(|f(x, y)| \leq 2\varepsilon\). Hence (3.3) is at most \(4\varepsilon\).

If the pair \((x, y), (x, y_x)\) is close and the pair \((x, y), (x_y, y)\) is distant, then (3.3) can be rewritten as \(|(f(x, y) - \gamma(x, y)) + (\gamma(x, y) - \gamma(x, y_x)) + (-f(x_y, y) + \gamma(x_y, y))|\) and we obtain the bound \(3\varepsilon\).

In order to prove (ib), consider an \(\alpha\)-vertical pair \((x, y), (u, v)\). In this case
\[
(3.4) \quad |G(x) - G(u)| = |\gamma(x, y_x) - \gamma(u, v_u)|.
\]

The points \((x, y_x), (u, v_u)\) form an \(\alpha\)-vertical pair as well. If it is a close pair, then the expression in equation (3.4) is bounded by \(\varepsilon\), by (ia); and if it is a distant pair, then it is bounded by \(\varepsilon\) as well, by (ic).

Item (ib) is shown similarly, using the expression
\[
|H(y) - H(v)| = |f(x_y, y) - \gamma(x_y, y) - f(u_v, v) + \gamma(u_v, v)|.
\]

Item (ii) follows directly from (i).

4. The Case of an Arbitrary \(m\)

Let us roughly explain where the proof of the implication (A) \(\Rightarrow\) (B) of Theorem 1.1 breaks down in the case of an arbitrary \(m\). The analogue of the faithfulness property in the case when a given set \(K\) contains no array of length \(m, m > 2\), is the following. A set \(X \subseteq \mathbb{R}^2\) is \((m, \alpha, \beta, l)|\)-faithful if for each \(\alpha\)-array \(\{z_0, z_1, \ldots, z_{k+1}\}\) in \(X\) with the property that there are indices \(0 < i_1 < j_1 < i_2 < j_2 \leq \ldots \leq i_m < j_m < k+1\) such that

\[
|p(z_{i_1}) - p(z_{j_1})| \geq \beta, |q(z_{i_2}) - q(z_{j_2})| \geq \beta, |p(z_{i_k}) - p(z_{j_k})| \geq \beta,
\]

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..., |q(z_{i_n}) - q(z_{j_m})| \geq \beta for an even m and |p(z_{i_n}) - p(z_{j_m})| \geq \beta for an odd m, or a similar series of inequalities but starting with |q(z_{i}) - q(z_{j_1})| \geq \beta, we have k > l.

However, we cannot prove an analogue of Lemma 2.5 not even in the case when the set K contains no array of length m = 3. For instance, there can be a sequence of 1/n-arrays \{\{z_0^n, z_1^n, \ldots, z_m^n\}\}_n, where \|p(z_0^n) - p(z_1^n)\| \geq \beta, \|q(z_0^n) - q(z_1^n)\| \geq \beta, \|p(z_2^n) - p(z_3^n)\| \geq \beta, and z_0^n \to z_0, z_1^n \to z_1, z_2^n \to z_2, z_3^n \to (z_1 + z_2)/2, z_4^n \to z_1, z_5^n \to z_0 as n \to \infty. Hence this sequence converges to an array of length two and not three. So this does not lead to a contradiction as in the proof of Lemma 2.5.

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