FINITENESS PROPERTIES
OF LOCAL COHOMOLOGY MODULES
FOR $a$-MINIMAX MODULES

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Abstract. Let $R$ be a commutative Noetherian ring and $a$ an ideal of $R$. In this paper we introduce the concept of $a$-minimax $R$-modules, and it is shown that if $M$ is an $a$-minimax $R$-module and $t$ a non-negative integer such that $H^i_a(M)$ is $a$-minimax for all $i < t$, then for any $a$-minimax submodule $N$ of $H^t_a(M)$, the $R$-module $\text{Hom}_R(R/a, H^t_a(M)/N)$ is $a$-minimax. As a consequence, it follows that the Goldie dimension of $H^t_a(M)/N$ is finite, and so the associated primes of $H^t_a(M)/N$ are finite. This generalizes the main result of Brodmann and Lashgari (2000).

1. Introduction

Let $R$ be a commutative Noetherian ring, $a$ an ideal of $R$, and $M$ a finitely generated $R$-module. An important problem in commutative algebra is determining when the set of associated primes of the $i$th local cohomology module $H^i_a(M)$ of $M$ with support in $V(a)$ is finite (see [11, Problem 4]). A. Singh [21] and M. Katzman [12] have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see [1, 2, 9, 10, 13, 14, 15, 16]. In particular, Brodmann and Lashgari [1, Theorem 2.2] showed that if, for a finitely generated $R$-module $M$ and an integer $t$, the local cohomology modules $H^0_a(M), H^1_a(M), \ldots, H^{t-1}_a(M)$ are finitely generated, then the set $\text{Ass}_R H^t_a(M)/N$ is finite for every finitely generated submodule $N$ of $H^t_a(M)$. For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik’s interesting paper [15].

This paper is concerned with what might be considered a generalization of the above-mentioned result of Brodmann and Lashgari to the class of $a$-minimax modules. More precisely, we shall show that:

**Theorem 1.1.** Let $R$ be a Noetherian ring, $a$ an ideal of $R$ and $M$ an $a$-minimax $R$-module. Let $t$ be a non-negative integer such that $H^i_a(M)$ is $a$-minimax for all
Then for any \( a \)-minimax submodule \( N \) of \( H^i_a(M) \), the \( R \)-module \( \text{Hom}_R(R/a, H^i_a(M)/N) \) is \( a \)-minimax. In particular, the Goldie dimension of \( H^i_a(M)/N \) is finite, and so the set \( \text{Ass}_R H^i_a(M)/N \) is finite.

Recall that an \( R \)-module \( M \) is said to have finite Goldie dimension (written \( G \dim M < \infty \)) if \( M \) does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull \( E(M) \) of \( M \) decomposes as a finite direct sum of indecomposable (injective) submodules. Also, an \( R \)-module \( M \) is said to have finite \( a \)-relative Goldie dimension if the Goldie dimension of the \( a \)-torsion submodule \( \Gamma_a(M) := \bigcup_{n \geq 1} (0 :_M a^n) \) of \( M \) is finite.

We say that an \( R \)-module \( M \) is \( a \)-minimax if the \( a \)-relative Goldie dimension of any quotient module of \( M \) is finite. One of our tools for proving Theorem 1.1 is the following:

**Proposition 1.2.** Let \( R \) be a Noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module and \( N \) an arbitrary \( R \)-module. Let \( i \) be a non-negative integer such that \( \text{Ext}^i_R(M, N) \) is \( a \)-minimax for all \( i \leq t \). Then for any finitely generated \( R \)-module \( L \) with \( \text{Supp} L \subseteq \text{Supp} M \), \( \text{Ext}^i_R(L, N) \) is \( a \)-minimax for all \( i \leq t \).

Throughout this paper, \( R \) will always be a commutative Noetherian ring with non-zero identity, and \( a \) will be an ideal of \( R \). The \( i \)-th local cohomology module of an \( R \)-module \( M \) with respect to \( a \) is defined by

\[
H^i_a(M) = \lim_{n \to \infty} \text{Ext}^i_R(R/a^n, M).
\]

We refer the reader to [7] or [3] for the basic properties of local cohomology.

2. \( a \)-Minimax Modules and Goldie Dimension

For an \( R \)-module \( M \), the **Goldie dimension of \( M \)** is defined as the cardinal of the set of indecomposable submodules of \( E(M) \) which appear in a decomposition of \( E(M) \) into a direct sum of indecomposable submodules. We shall use \( G \dim M \) to denote the Goldie dimension of \( M \). For a prime ideal \( p \), let \( \mu^0(p, M) \) denote the 0-th Bass number of \( M \) with respect to the prime ideal \( p \). It is known that \( \mu^0(p, M) > 0 \) if and only if \( p \in \text{Ass}_R M \). It is clear by the definition of the Goldie dimension that

\[
G \dim M = \sum_{p \in \text{Ass}_R M} \mu^0(p, M).
\]

Also, for any ideal \( a \) of \( R \) and any \( R \)-module \( M \), the **\( a \)-relative Goldie dimension of \( M \)** is defined as

\[
G \dim_a M := \sum_{p \in \text{V}(a)} \mu^0(p, M).
\]

The \( a \)-relative Goldie dimension of an \( R \)-module \( M \) has been studied in [5].

In [24], H. Zöschinger introduced the interesting class of minimax modules, and he has in [24] and [25] given many equivalent conditions for a module to be minimax. The \( R \)-module \( M \) is said to be a **minimax module** if there is a finitely generated submodule \( N \) of \( M \), such that \( M/N \) is Artinian. It was shown by T. Zink [23] and by E. Enochs [4] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. On the other hand, it is known that when \( R \) is a
Noetherian ring, a module is minimax if and only if each of its quotients has finite Goldie dimension, 23 or 25. This motivates the definition:

**Definition 2.1.** Let $a$ be an ideal of $R$. An $R$-module $M$ is said to be **minimax with respect to $a$** or **$a$-minimax** if the $a$-relative Goldie dimension of any quotient module of $M$ is finite; i.e., for any submodule $N$ of $M$, $G \dim_a M/N < \infty$.

**Remark 2.2.** Let $a$ be an ideal of $R$ and let $M$ be an $R$-module.

(i) If $a = 0$, then $M$ is $a$-minimax if and only if $M$ is minimax.

(ii) If $M$ is $a$-torsion, then $M$ is $a$-minimax if and only if $M$ is minimax by [5, Lemma 2.6].

(iii) If $M$ is Noetherian or Artinian, then $M$ is $a$-minimax.

(iv) If $b$ is a second ideal of $R$ such that $a \subseteq b$ and $M$ is $a$-minimax, then $M$ is $b$-minimax. In particular, every minimax module is $a$-minimax.

The following proposition is needed in the proof of the main theorem of this paper.

**Proposition 2.3.** Let $a$ be an ideal of $R$. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $R$-modules. Then $M$ is $a$-minimax if and only if $M'$ and $M''$ are both $a$-minimax.

**Proof.** We may suppose for the proof that $M'$ is a submodule of $M$ and that $M'' = M/M'$. If $M$ is $a$-minimax, then it easily follows from the definition that $M'$ and $M/M'$ are $a$-minimax. Now, suppose that $M'$ and $M/M'$ are $a$-minimax. Let $N$ be an arbitrary submodule of $M$, and let $p \in \text{Ass}(M/N) \cap V(a)$. Then the exact sequence

$$0 \to \frac{M' + N}{N} \to \frac{M}{N} \to \frac{M}{M' + N} \to 0$$

induces the exact sequence

$$0 \to \text{Hom}_R(k(p), \frac{M'}{M'_p \cap N_p}) \to \text{Hom}_R(k(p), \frac{M_p}{N_p}) \to \text{Hom}_R(k(p), \frac{M_p}{M'_p + N_p})$$

where $k(p) = R_p/pR_p$. Moreover, since $\text{Ass}_R M/N \subseteq \text{Ass}_R \frac{M' + N}{N} \cup \text{Ass}_R \frac{M}{M' + N}$, and the sets $\text{Ass}_R \frac{M' + N}{N} \cap V(a)$ and $\text{Ass}_R \frac{M}{M' + N} \cap V(a)$ are finite, it follows that $G \dim_a M/N < \infty$, and so $M$ is $a$-minimax.

**Corollary 2.4.** Let $a$ be an ideal of $R$. Then any quotient of an $a$-minimax module, as well as any finite direct sum of $a$-minimax modules, is $a$-minimax.

**Proof.** The assertion follows from the definition and Proposition 2.3.

**Corollary 2.5.** Let $a$ be an ideal of $R$. Let $M$ be a finitely generated $R$-module and $N$ an $a$-minimax $R$-module. Then $\text{Ext}_R^i(M, N)$ and $\text{Tor}_R^i(M, N)$ are $a$-minimax modules for all $i$. In particular, the $R$-modules $\text{Ext}_R^i(R/a, N)$ and $\text{Tor}_R^i(R/a, N)$ are $a$-minimax for all $i$. 


Proposition 2.6. Let \( \mathfrak{a} \) be an ideal of \( R \). Let \( M \) be an \( \mathfrak{a} \)-minimax \( R \)-module such that \( \text{Ass}_R M \subseteq V(\mathfrak{a}) \). Then \( H^i_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \).

Proof. If \( i = 0 \), then \( H^0_\mathfrak{a}(M) = \Gamma_\mathfrak{a}(M) \) is a submodule of \( M \), and so by Proposition 2.3, \( \Gamma_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax. As \( \text{Ass}_R M/\Gamma_\mathfrak{a}(M) \subseteq \text{Ass}_R M \), it easily follows from \( \text{Ass}_R M \subseteq V(\mathfrak{a}) \) that \( M = \Gamma_\mathfrak{a}(M) \). Consequently, by [3, Corollary 2.1.7(ii)], \( H^i_\mathfrak{a}(M) = 0 \) for all \( i > 0 \), and so \( H^i_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \), as required.

We are now ready to state and prove the main result of this section, which will be used in the main result of Section 4.

Theorem 2.7. Let \( \mathfrak{a} \) be an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module and \( N \) an arbitrary \( R \)-module. Let \( t \) be a non-negative integer such that \( \text{Ext}^j_R(M,N) \) is \( \mathfrak{a} \)-minimax for all \( i \leq t \). Then for any finitely generated \( R \)-module \( L \) with \( \text{Supp} L \subseteq \text{Supp} M \), \( \text{Ext}^j_R(L,N) \) is \( \mathfrak{a} \)-minimax for all \( i \leq t \).

Proof. Since \( \text{Supp} L \subseteq \text{Supp} M \), according to Gruson’s Theorem [22, Theorem 4.1], there exists a chain

\[
0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,
\]

such that the factors \( L_j/L_{j-1} \) are homomorphic images of a direct sum of finitely many copies of \( M \). Now consider the exact sequences

\[
0 \to K \to M^n \to L_1 \to 0
\]

\[
0 \to L_1 \to L_2 \to L_2/L_1 \to 0
\]

\[
\vdots
\]

\[
0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0,
\]

for some positive integer \( n \).

Now from the long exact sequence

\[
\cdots \to \text{Ext}^i_R(L_{j-1},N) \to \text{Ext}^i_R(L_j/L_{j-1},N) \to \text{Ext}^i_R(L_j,N) \to \text{Ext}^{i+1}_R(L_{j-1},N) \to \cdots
\]

and an easy induction on \( k \), it suffices to prove the case when \( k = 1 \).

Thus there is an exact sequence

\[
(*) \quad 0 \to K \to M^n \to L \to 0
\]

for some \( n \in \mathbb{N} \) and some finitely generated \( R \)-module \( K \).

Now, we use induction on \( t \). First, \( \text{Hom}_R(L,N) \) is a submodule of \( \text{Hom}_R(M^n,N) \); hence in view of the assumption and Corollary 2.4, \( \text{Ext}^0_R(L,N) \) is \( \mathfrak{a} \)-minimax. So assume that \( t > 0 \) and that \( \text{Ext}^j_R(L',N) \) is \( \mathfrak{a} \)-minimax for every finitely generated
Example 3.2. Let \( R \) be an ideal of \( R \), and let \( t \) be a non-negative integer. Then, for any \( R \)-module \( M \) the following conditions are equivalent:

(i) \( \text{Ext}^i_R(R/a, M) \) is \( a \)-minimax for all \( i \leq t \).

(ii) For any ideal \( b \) of \( R \) with \( b \supseteq a \), \( \text{Ext}^i_R(R/b, M) \) is \( a \)-minimax for all \( i \leq t \).

(iii) For any finitely generated \( R \)-module \( N \) with \( \text{Supp} N \subseteq V(a) \), \( \text{Ext}^i_R(N, M) \) is \( a \)-minimax for all \( i \leq t \).

(iv) For any minimal prime ideal \( p \) over \( a \), \( \text{Ext}^i_R(R/p, M) \) is \( a \)-minimax for all \( i \leq t \).

Proof. In view of Theorem 2.7 it is enough to show that (iv) implies (i). To do this, let \( p_1, \ldots, p_n \) be the minimal primes of \( a \). Then, by assumption, the \( R \)-modules \( \text{Ext}^i_R(R/p_j, M) \) are \( a \)-minimax for all \( j = 1, 2, \ldots, n \). Hence by Corollary 2.4 \( \bigoplus_{j=1}^n \text{Ext}^i_R(R/p_j, M) \cong \text{Ext}^i_R(\bigoplus_{j=1}^n R/p_j, M) \) is \( a \)-minimax. Since \( \text{Supp}(\bigoplus_{j=1}^n R/p_j) = \text{Supp} R/a \), it follows from Theorem 2.7 that \( \text{Ext}^i_R(R/a, M) \) is \( a \)-minimax, as required. \( \square \)

3. \( a \)-Cominimax modules and local cohomology

Let \( R \) be a Noetherian ring, \( a \) an ideal of \( R \) and \( M \) an \( R \)-module. Recall that \( M \) is said to be \( a \)-cofinite if \( M \) has support in \( V(a) \) and \( \text{Ext}^i_R(R/a, M) \) is a finitely generated \( R \)-module for each \( i \) (see [8]). This motivates the following definition:

**Definition 3.1.** Let \( R \) be a Noetherian ring and \( a \) an ideal of \( R \). We say that an \( R \)-module \( M \) is \( a \)-cominimax if the support of \( M \) is contained in \( V(a) \) and \( \text{Ext}^i_R(R/a, M) \) is \( a \)-minimax for all \( i \geq 0 \).

**Example 3.2.** (i) Let \( a \) be an ideal of \( R \) and let \( M \) be an \( a \)-minimax \( R \)-module such that \( \text{Supp} M \subseteq V(a) \). Then it follows from Corollary 2.5 that \( M \) is \( a \)-cominimax.

In particular, every \( a \)-cominimax \( R \)-module with support in \( V(a) \) is \( a \)-cominimax.

(ii) Let \( a \) be an ideal of \( R \). Then every \( a \)-cofinite \( R \)-module is \( a \)-cominimax. In particular, any Noetherian module with support in \( V(a) \) is \( a \)-cominimax.

(iii) Let \( a \) be an ideal of \( R \) and let \( N \) be a pure submodule of an \( R \)-module \( M \). Then \( M \) is \( a \)-cominimax if and only if \( N \) and \( M/N \) are \( a \)-cominimax. In fact, P.M. Cohn’s characterization of purity (see [20] Theorem 3.65) implies that the sequence

\[ 0 \rightarrow \text{Ext}^i_R(R/a, N) \rightarrow \text{Ext}^i_R(R/a, M) \rightarrow \text{Ext}^i_R(R/a, M/N) \rightarrow 0 \]

is exact for all \( i \) (see also the proof of [18, Proposition 2.7]). Hence the result follows from Proposition 2.3.
Proposition 3.3. Let $a$ be an ideal of $R$. Let
$$0 \to M' \to M \to M'' \to 0$$
be an exact sequence of $R$-modules such that two of the modules are $a$-cominimax. Then so is the third one.

Proof. The exact sequence
$$0 \to M' \to M \to M'' \to 0$$
induces a long exact sequence
$$\cdots \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, M') \to \text{Ext}^{i+1}_R(R/a, M'' \to \cdots.$$ 
Now the result follows easily from Proposition 2.3.

Corollary 3.4. Let $a$ be an ideal of $R$. Let $f : M \to N$ be a homomorphism between two $a$-cominimax modules such that one of the three modules $\ker f$, $\text{Im} f$ and $\text{Coker} f$ is $a$-cominimax. Then all three of them are $a$-cominimax.

Proof. The result follows from Proposition 3.3 and the following exact sequences:
$$0 \to \ker f \to M \to \text{Im} f \to 0,$n \to \text{Im} f \to N \to \text{Coker} f \to 0.$$ 

Proposition 3.5. Let $a$ be an ideal of $R$. Let $M$ be an $R$-module such that $\text{Supp} M \subseteq V(a)$ and $0 : M a$ has finite Goldie dimension. Then $M$ has finite Goldie dimension.

Proof. Since $0 : M a$ has finite Goldie dimension and $\text{Supp} M \subseteq V(a)$, it follows from Bourbaki’s Theorem (see [4, Exercise 1.2.27]) that $\text{Ass} M$ is finite. On the other hand, for any $p \in \text{Ass} M$ we have
$$\text{Hom}_{R_p}(k(p), M_p) \cong \text{Hom}_{R_p}(k(p), 0 : M_p aR_p),$$
as $k(p)$-vector spaces, where $k(p) = R_p/pR_p$. Therefore $\mu^0(p, M)$ is finite, and so $G \dim M < \infty$.

Corollary 3.6. Let $a$ be an ideal of $R$, and let $M$ be an $a$-cominimax $R$-module. Then $M$ has finite Goldie dimension. In particular the set of associated primes of $M$ is finite.

Proof. This is immediate from Proposition 3.5.

Proposition 3.7. Let $a$ be an ideal of $R$. Let $M$ be an $R$-module such that $H^i_a(M)$ is $a$-cominimax for all $i$. Then $\text{Ext}^i_R(R/a, M)$ is $a$-minimax for all $i$.

Proof. The case $i = 0$ is clear, so let $i > 0$ and do induction on $i$. We first reduce to the case $\Gamma_a(M) = 0$. To do this, let $\tilde{M} = M/\Gamma_a(M)$. Then we have the long exact sequence
$$\cdots \to \text{Ext}^i_R(R/a, \Gamma_a(M)) \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, \tilde{M}) \to \cdots$$
and the isomorphism $H^i_a(M) \cong H^i_a(\tilde{M})$ for $i > 0$. So in view of Proposition 2.3 we may assume that $\tilde{M}$ is $a$-torsion free. Let $E$ be the injective envelope of $\tilde{M}$ and put $L = E/\tilde{M}$. Then $\text{Hom}_R(R/a, E) = 0$, and we therefore get the isomorphisms $H^i_a(L) \cong H^{i+1}_a(M)$ and $\text{Ext}^i_R(R/a, L) \cong \text{Ext}^{i+1}_R(R/a, M)$ for all $i \geq 0$. Now the assertion follows by induction.
Proposition 3.8. Let $a$ be an ideal of $R$. Let $M$ be an $R$-module such that $\text{Ext}^i_R(R/a, M)$ is $a$-minimax for all $i$. If $t$ is a non-negative integer such that $H^i_a(M)$ is $a$-cominimax for all $i \neq t$, then $H^t_a(M)$ is $a$-cominimax.

Proof. We use induction on $t$. Let $\bar{M} = M/\Gamma_a(M)$. Then $H^i_a(M) \cong H^i_a(\bar{M})$ for all $i > 0$. If $t = 0$, then $H^0_a(M)$ is $a$-cominimax for all $i$. Hence by Proposition 3.7, $\text{Ext}^i_R(R/a, M)$ is $a$-minimax for all $i$. It follows that $\Gamma_a(M)$ is $a$-cominimax. So let $t > 0$ and suppose that the result has been proved for $t - 1$. Since $\Gamma_a(M)$ is $a$-cominimax, the exact sequence

$$\cdots \to \text{Ext}^i_R(R/a, \Gamma_a(M)) \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, \bar{M}) \to \cdots$$

allows us to assume that $M$ is $a$-torsion free. Let $E$ be the injective envelope of $M$ and put $L = E/M$. Then $\text{Hom}_R(R/a, E) = 0$ and $\Gamma_a(E) = 0$, and we therefore get the isomorphisms $H^i_a(L) = H^{i+1}_a(M)$ and $\text{Ext}^i_R(R/a, L) \cong \text{Ext}^{i+1}_R(R/a, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Corollary 3.9. Let $a$ be an ideal of $R$ and $M$ an $a$-minimax $R$-module. If $t$ is a non-negative integer such that $H^i_a(M)$ is $a$-cominimax for all $i \neq t$, then $H^t_a(M)$ is $a$-cominimax.

Proof. This follows from Corollary 2.5 and Proposition 3.8 □

Corollary 3.10. Let $a$ be a principal ideal of $R$ and $M$ an $a$-minimax $R$-module. Then $H^i_a(M)$ is $a$-cominimax for all $i \geq 0$.

Proof. Since $H^0_a(M)$ is a submodule of $M$, it turns out that $H^0_a(M)$ is $a$-cominimax by Proposition 2.3 and Example 3.2(i). Also $H^i_a(M) = 0$ for all $i > 1$. Therefore, the result follows from Corollary 3.9 □

4. Finiteness of Associated Primes

It will be shown in this section that the subjects of the previous sections can be used to prove a finiteness result about local cohomology modules. In fact, we will generalize the main result of Brodman and Lashgari to $a$-minimax modules. The main result is Theorem 4.2. The following theorem will serve to shorten the proof of the main theorem.

Theorem 4.1. Let $a$ be an ideal of $R$ and let $M$ be an $R$-module. Let $t$ be a non-negative integer such that $H^i_a(M)$ is $a$-cominimax for all $i \neq t$, and $\text{Ext}^t_R(R/a, M)$ is $a$-minimax. Then for any $a$-minimax submodule $N$ of $H^t_a(M)$ and for any finitely generated $R$-module $L$ with $\text{Supp} L \subseteq V(a)$, the $R$-module $\text{Hom}_R(L, H^t_a(M)/N)$ is $a$-minimax.

Proof. The exact sequence

$$0 \to N \to H^t_a(M) \to H^t_a(M)/N \to 0$$

provides the following exact sequence:

$$\text{Hom}_R(L, H^t_a(M)) \to \text{Hom}_R(L, H^t_a(M)/N) \to \text{Ext}^1_R(L, N) \to \cdots$$

Since by Corollary 2.5, $\text{Ext}^1_R(L, N)$ is $a$-minimax, so in view of Proposition 2.3 it is thus sufficient for us to show that the $R$-module $\text{Hom}_R(L, H^t_a(M))$ is $a$-minimax. To this end, in view of Corollary 2.8, it is enough for us to show that the $R$-module $\text{Hom}_R(R/a, H^t_a(M))$ is $a$-minimax.
We use induction on $t$. When $t = 0$, the $R$-module $\text{Hom}_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax, by assumption. Since

$$\text{Hom}_R(R/\mathfrak{a}, H^0_\mathfrak{a}(M)) \cong \text{Hom}_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)) \cong \text{Hom}_R(R/\mathfrak{a}, M),$$

it follows that $\text{Hom}_R(R/\mathfrak{a}, H^0_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax.

Now suppose, inductively, that $t > 0$ and that the result has been proved for $t - 1$. Since $\Gamma_\mathfrak{a}(M)$ is $\mathfrak{a}$-cominimax, it follows that $\text{Ext}^i_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax for all $i \geq 0$. On the other hand, the exact sequence

$$0 \rightarrow \Gamma_\mathfrak{a}(M) \rightarrow M \rightarrow M/\Gamma_\mathfrak{a}(M) \rightarrow 0$$

induces the exact sequence

$$\text{Ext}^i_R(R/\mathfrak{a}, M) \rightarrow \text{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_\mathfrak{a}(M)) \rightarrow \text{Ext}^{i+1}_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)).$$

Hence, by Proposition 2.5 and the assumption, the $R$-module $\text{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax. Also since $H^0_\mathfrak{a}(M/\Gamma_\mathfrak{a}(M)) = 0$ and $H^i_\mathfrak{a}(M/\Gamma_\mathfrak{a}(M)) \cong H^i_\mathfrak{a}(M)$ for all $i \geq 0$, it follows that $H^i_\mathfrak{a}(M/\Gamma_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax for all $i < t$. Therefore we may assume that $M$ is $\mathfrak{a}$-torsion free. Let $E$ be an injective envelope of $M$ and put $M_1 = E/M$. Then also $\Gamma_\mathfrak{a}(E) = 0$ and $\text{Hom}_R(R/\mathfrak{a}, E) = 0$. Consequently, $\text{Ext}^i_R(R/\mathfrak{a}, M_1) \cong \text{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $H^i_\mathfrak{a}(M_1) \cong H^{i+1}_\mathfrak{a}(M)$ for all $i \geq 0$ (including the case $i = 0$). The induction hypothesis applied to $M_1$ yields that $\text{Hom}_R(R/\mathfrak{a}, H^{i-1}_\mathfrak{a}(M_1))$ is $\mathfrak{a}$-minimax. Hence $\text{Hom}_R(R/\mathfrak{a}, H^i_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax.

Now we are prepared to prove the main theorem of this section, which is a generalization of the main result of Brodmann and Lashgari.

**Theorem 4.2.** Let $\mathfrak{a}$ be an ideal of $R$ and let $M$ be an $\mathfrak{a}$-minimax $R$-module. Let $t$ be a non-negative integer such that $H^i_\mathfrak{a}(M)$ is $\mathfrak{a}$-minimax for all $i < t$. Then for any $\mathfrak{a}$-minimax submodule $N$ of $H^i_\mathfrak{a}(M)$, the $R$-module $\text{Hom}_R(R/\mathfrak{a}, H^i_\mathfrak{a}(M)/N)$ is $\mathfrak{a}$-minimax. In particular, the Goldie dimension of $H^i_\mathfrak{a}(M)/N$ is finite, and so the set $\text{Ass}_R H^i_\mathfrak{a}(M)/N$ is finite.

**Proof.** Apply Theorem 4.1 and Corollary 2.5.

Nhan, in [19, Proposition 5.5], established the following corollary in the case $R$ is local. The following result provides a slight generalization of [19, Proposition 5.5] and [11, Theorem 2.2].

**Corollary 4.3.** Let $R$ be a Noetherian ring, $\mathfrak{a}$ an ideal of $R$ and $M$ a finitely generated $R$-module. Let $\text{Obj}(\mathcal{N})$ (resp. $\text{Obj}(\mathcal{A})$) denote the category of all Noetherian (resp. Artinian) $R$-modules and $R$-homomorphisms. Let $t$ be a non-negative integer such that $H^i_\mathfrak{a}(M) \in \text{Obj}(\mathcal{N}) \cup \text{Obj}(\mathcal{A})$ for all $i < t$. Then the $R$-module $\text{Hom}_R(R/\mathfrak{a}, H^i_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax, and so the set $\text{Ass}_R H^i_\mathfrak{a}(M)$ is finite.

**Proof.** Apply Theorem 4.1 and the fact that the class of $\mathfrak{a}$-minimax modules contains all Noetherian and Artinian modules.

**Corollary 4.4.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, $\mathfrak{a}$ an ideal of $R$ and $M$ a finitely generated $R$-module. Assume that $\mathfrak{a}$ contains an $M$-filter regular sequence of length $t$. Then $H^i_\mathfrak{a}(M)$ has finite Goldie dimension.

**Proof.** According to Melkersson [17, Theorem 3.1], $H^i_\mathfrak{a}(M)$ is Artinian for all $i < t$. Hence, it follows from Corollary 4.3 that $\text{Hom}_R(R/\mathfrak{a}, H^i_\mathfrak{a}(M))$ is $\mathfrak{a}$-minimax, and so $G \dim H^i_\mathfrak{a}(M)$ is finite.
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