EXTRAPOLATION SPACES FOR C-SEMIGROUPS

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ABSTRACT. Let \( \{T(t)\}_{t \geq 0} \) be a C-semigroup on \( X \). We construct an extrapolation space \( X_s \), such that \( X \) can be continuously densely imbedded in \( X_s \), and \( \{T_s(t)\}_{t \geq 0} \), the extension of \( \{T(t)\}_{t \geq 0} \) to \( X_s \), is strongly uniformly continuous and contractive. Using this enlarged space, we give an answer to the question asked in [M. Li, F. L. Huang, Characterizations of contraction C-semigroups, Proc. Amer. Math Soc. 126 (1998), 1063–1069] in the negative.

1. INTRODUCTION

Let \( X \) be a Banach space, \( B(X) \) the space of all bounded linear operators on \( X \), and \( C \) an injective operator in \( B(X) \). A family of linear bounded operators \( \{T(t)\}_{t \geq 0} \subset B(X) \) is called a C-semigroup if \( T(\cdot) \) is strongly continuous and \( T(0) = C \), \( T(t+s)C = T(t)T(s) \) for \( t, s \geq 0 \). Its generator, \( A \), is defined by

\[
Ax = C^{-1} \left( \lim_{t \to 0} \frac{T(t)x - Cx}{t} \right)
\]

with maximal domain.

A C-semigroup \( \{T(t)\}_{t \geq 0} \) is bounded if there is a constant \( M > 0 \) such that \( \|T(t)\| \leq M \) for all \( t \geq 0 \) and is a contraction C-semigroup if \( \|T(t)x\| \leq \|Cx\| \) for all \( x \in X \) and \( t \geq 0 \).

It is natural that all bounded \( C_0 \)-semigroups are strongly uniformly continuous, while for C-semigroups this is far from obvious. However, we show in this paper that for every bounded C-semigroup \( \{T(t)\}_{t \geq 0} \) on \( X \), an extrapolation space \( X_s \) can be constructed such that the extension of \( \{T(t)\}_{t \geq 0} \) to \( X_s \), \( \{T_s(t)\}_{t \geq 0} \), is a strongly uniformly continuous contraction \( C_s \)-semigroup on \( X_s \), where \( C_s \) is the extension of \( C \) to \( X_s \). Our extrapolation space is smaller than the one given by deLaubenfels [1 2].

Moreover, we take up the open problem asked in [3]. The question was: Suppose that \( A \) is the generator of a contraction C-semigroup on \( X \). Does there exist a restriction of \( A, A' \), which is the generator of a contraction \( C_0 \)-semigroup on \( R(C) \)? If this holds, then \( (\lambda - A')^{-1}R(C) = R(C) \) for all \( \lambda > 0 \). Hence it is crucial that \( R(C) \) be an invariant subspace for \( (\lambda - A)^{-1} \) since \( A' \subseteq A \). So one way to answer the question in the negative is to give a contraction C-semigroup with generator \( A \), in
which $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant. It is easier to construct a bounded $C$-semigroup than a contraction one. Now the extrapolation space is helpful. By making use of it, we can obtain contraction $C$-semigroups from bounded ones.

Throughout this paper, for an operator $A$ on $X$, we write $D(A)$ for its domain, $R(A)$ for its range, and the closure of $R(A)$ is denoted by $\overline{R(A)}$. The $C$-resolvent set of $A$, $\rho_C(A) := \{ \lambda \in \mathbb{C} : \lambda - A$ is injective and $R(C) \subset R(\lambda - A) \}$, and the $C$-resolvent of $A$ is $R_C(\lambda, A) := (\lambda - A)^{-1}C$ for $\lambda \in \rho_C(A)$. For $Y$ a subspace of $X$ and $A$ a linear operator on $X$, we denote by $A|_Y$ the part of $A$ in $Y$, i.e., $A|_Y \subset A$ with maximal domain. For the properties of $C$-semigroups and of contractions, we refer to [2, 3].

2. Main results

First we give a positive answer to the question mentioned above under some additional assumptions. The following result also improves Theorem 3.4 in [1].

**Theorem 2.1.** Let $A = C^{-1}AC$, $\overline{CD(A) = R(C)}$ and $D(A) \subset R(r - A)$ for some $r > 0$. Then the following are equivalent:

(a) $A$ generates a contraction $C$-semigroup on $X$.
(b) $(0, \infty) \subset \rho_C(A)$ and $\lambda \|R_C(\lambda, A)x\| \leq \|Cx\|$ for $\lambda > 0$ and $x \in X$.
(c) $A|_{\overline{R(C)}}$ generates a contraction $C_0$-semigroup on $R(C)$.

**Proof.** (a) $\Rightarrow$ (b) follows from Theorem 3.3 in [1].
(b) $\Rightarrow$ (c). Define $B \subset A$ with $D(B) = CD(A)$. Then $B$ is a densely defined closable operator on $R(C)$. By (b), $\|(\lambda - A)x\| \geq \lambda \|x\|$ for $\lambda > 0$ and $x \in D(B)$; i.e., $B$ is dissipative. This implies that $\overline{B}$ is also dissipative and $R(\lambda - \overline{B})$ is a closed subspace of $R(C)$. To show that $R(\lambda - \overline{B}) = \overline{R(C)}$, let $x \in D(A)$. Since $D(A) \subset R(r - A)$, $x = (r - A)y$ for some $y \in D(A)$ and $ACy = CAy$ due to the assumption $A = C^{-1}AC$,

$$Cx = (r - A)Cy = (r - B)Cy \in R(r - \overline{B}).$$

This implies that $\overline{R(tC)} = \overline{CD(A)} \subset R(r - \overline{B})$, as desired. It now follows from the Lumer-Phillips theorem that $\overline{B}$ generates a contraction $C_0$-semigroup on $R(C)$. It remains to show that $\overline{B} = A|_{\overline{R(C)}}$. It is clear that $\overline{B} \subset A|_{\overline{R(C)}}$, and so $\overline{R(C)} \subset R(r - A|_{\overline{R(C)}})$. Also, the injectivity of $r - A$ implies that of $r - A|_{\overline{R(C)}}$; thus,

$$\overline{B} = A|_{\overline{R(C)}}$$

follows from the identity that $(r - \overline{B})^{-1} = (r - A|_{\overline{R(C)}})^{-1}$.

(c) $\Rightarrow$ (a). Let $T(t) = S(t)C$, where $S(t)$ is the contraction $C_0$-semigroup generated by $A|_{\overline{R(C)}}$ on $R(C)$. It is easy to show that $T(t)$ is a contraction $C$-semigroup; we only need to show that $A$ is the generator. If $x \in D(A)$, then since $ACx = CAx \in R(C)$ by the assumption that $A = C^{-1}AC$, we know that $Cx \in D(A|_{\overline{R(C)}})$ and

$$\frac{T(t)x - Cx}{t} = \frac{S(t)Cx - Cx}{t} \to A|_{\overline{R(C)}}Cx = CA|_{\overline{R(C)}}x = CAx$$

as $t \to 0$, so an extension of $A$ is the generator. Suppose that $\lambda > 0$; if $(\lambda - A)x = 0$, then, since $Cx \in D(A|_{\overline{R(C)}})$,

$$(\lambda - A|_{\overline{R(C)}})Cx = (\lambda - A)Cx = C(\lambda - A)x = 0.$$
Thus $x = 0$; i.e., $\lambda - A$ is injective. Also, for $x \in X$, let $y = R(\lambda, A)C\frac{y}{\|y\|}x$. Then $C\frac{y}{\|y\|}x = (\lambda - A)\frac{y}{\|y\|}x$. This implies that $R(C) \subseteq R(\lambda - A)$ and so $\lambda \in \rho_C(A)$. Then it follows from Corollary 3.12 in [2] that $C^{-1}AC = A$ is the generator.

Now we turn to the construction of the extrapolation space. For simplicity, we only consider bounded $C$-semigroups.

Let \( \{T(t)\}_{t \geq 0} \) be a bounded $C$-semigroup on $X$ with generator $A$, so there exists some constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. For each $x \in X$, define $\|x\|_s = \sup_{t \geq 0} \|T(t)x\|$. Then

\[
(C.2.1) \quad \|Cx\| \leq \|x\|_s \leq M\|x\|.
\]

Since $C$ is injective, $\| \cdot \|_s$ is a norm on $X$. Denote by $X_s$ the completion of $X$ with respect to the norm $\| \cdot \|_s$. Extend $T(t)$ to $X_s$ by defining $T_s(t)y = \lim_{n \to \infty} T(t)x_n$ for all $t \geq 0$, with the limit taken in $X$, whenever $\{x_n\}$ is a sequence in $X$ converging to $y$, in $X_s$. We also denote by $C_s$ the extension of $C$ to $X_s$. It is not hard to see that $T_s(t)$ is bounded on $X_s$ for each $t \geq 0$, and $C_s$ is injective.

**Theorem 2.2.** Let $X_s$, $T_s(t)$, $C_s$ be as above. Then

(a) For all $t \geq 0$, $R(T_s(t))$ is contained in $R(T(t))$, the closure of $R(T(t))$ in $X$.

In particular, $R(T_s(t)) \subseteq X$ and $R(C_s) \subseteq R(C)$, the closure of $R(C)$ in $X$.

(b) $\{T_s(t)\}_{t \geq 0}$ is a strongly uniformly continuous contraction $C_s$-semigroup.

(c) Suppose that $A_s$ is the generator of $\{T_s(t)\}_{t \geq 0}$. Then

\begin{itemize}
  \item[(c1)] $A \subseteq A_s$;
  \item[(c2)] $A_s = C_s^{-1}AC_s$;
  \item[(c3)] $A = A_s|_X$.
\end{itemize}

**Proof.** (a) follows immediately from the definition of $T_s(t)$.

(b). First, we show that $T_s(t_1 + t_2)C_s = T_s(t_1)T_s(t_2)$ for all $t_1, t_2 \geq 0$. Let $y \in X_s$. Then there exists $\{x_n\} \subset X$ such that $x_n$ converges to $y$ in $X_s$, which means that $T(t)x_n$ converges in $X$ for all $t \geq 0$. Also, by the definition of $T_s(t)$ and (a), we have

\[
C_sT_s(t_1 + t_2)y = \lim_{n \to \infty} T_s(t_1 + t_2)x_n = \lim_{n \to \infty} CT(t_1 + t_2)x_n = \lim_{n \to \infty} T(t_1)T(t_2)x_n = T(t_1)T_s(t_2)y = T_s(t_1)T_s(t_2)y
\]

with the four limits taken in $X$.

Next, for every $x \in X$,

\[
\|T_s(t)x\|_s = \|T(t)x\|_s = \sup_{r \geq 0} \|T(r)T(t)x\| = \sup_{r \geq 0} \|T(r + t)Cx\| \leq \|Cx\|_s = \|C_sx\|_s;
\]

therefore, $\{T_s(t)\}_{t \geq 0}$ is a family of contractions since $X$ is dense in $X_s$.

Finally, we show that $\{T_s(t)\}_{t \geq 0}$ is strongly uniformly continuous. Now let $y \in X_s$. Then there exists a sequence $\{x_n\} \subset X$ satisfying $\|x_n - y\|_s \to 0$ as
n \to \infty. \text{ Thus}
\|T_s(t+h)y - T_s(t)y\|_s
\leq \|T_s(t+h)y - T_s(t+h)x_n\|_s + \|T_s(t+h)x_n - T_s(t)x_n\|_s
+ \|T_s(t)x_n - T_s(t)y\|_s
\leq 2\|C_s(x_n - y)\|_s + \sup_{r \geq 0} \|T(t + r + h)Cx_n - T(t + r)Cx_n\|
\leq 2\|C_s(x_n - y)\|_s + M\|T(h)x_n - Cx_n\|.

We already use the contractivity of \(T_s(t)\) in the above. Note that the right side is independent of \(t\), so \(\{T_s(t)\}_{t \geq 0}\) is strongly uniformly continuous.

(c1). Suppose that \(x \in D(A)\). Then by \((2.1)\), we know
\|\frac{T_s(t)x - Csx}{t} - Csx\|_s = \left\|\frac{T(t)x - Cx}{t} - CAx\right\|_s
\leq M\left\|\frac{T(t)x - Cx}{t} - CAx\right\| \to 0 \quad \text{as} \ t \to 0;

it follows that \(x \in D(A_s)\) with \(A_sx = Ax\).

(c2). If \(y \in D(A_s)\), then
\|\frac{T_s(t)y - Csy}{t} - C_sAx\|_s \to 0 \quad \text{as} \ t \to 0.

Since \(R(T_s(t)) \subseteq X\), by the definition of \(\| \cdot \|_s\), we have
\left\|\frac{T(t)(T_s(t)y - Csy)}{t} - T(t)C_sAx\right\| \to 0 \quad \text{as} \ t \to 0

uniformly in \(h\). Set \(h = 0\). Noting that \(C_s\) commutes with \(A_s\) and \(T_s(t)\), we have
\|\frac{T_s(t)Csy - C_sCsy}{t} - C_sAx\|_s \to 0 \quad \text{as} \ t \to 0.

\(C_sy \in D(A_s) \cap X\) and \(A_sC_sy = C_sA_sy \in X\), this means
\left\|\frac{T(t)Csy - C_sCsy}{t} - CA_sCsy\right\| \to 0 \quad \text{as} \ t \to 0,

which implies that \(C_sy \in D(A)\) and \(AC_sy = A_sC_sy = C_sA_sy\), i.e., \(A_sy = C_s^{-1}AC_sy\). So we get \(A_s \subseteq C_s^{-1}AC_s\).

On the other hand, \(C_s^{-1}AC_s \subseteq C_s^{-1}A_sC_s = A_s\) since \(A_s\) is the generator.

(c3). If \(x \in D(A_s) \cap X\) and \(A_sx \in X\), then \(Cx = C_sx \in D(A)\) by (b) and \(ACx = A_sC_sx = C_sA_sx = CA_sx\), which implies that \(A_sx = C^{-1}ACx\). So the claim follows from the fact that \(A = C^{-1}AC\).

\(\square\)

Remark 2.3. (a) It should be mentioned that the extrapolation space, \(W\), of \([1,2]\), is defined only when \(R(C)\) is dense; in \([3]\) it is defined when \(R(C)\) is dense or \(\rho(A)\) contains a half-line. When \(R(C)\) is dense, generating a contraction C-semigroup is equivalent to generating a strongly continuous semigroup of contractions by Theorem 4.6 in \([4]\); thus \(A_s\) of Theorem \((22)\) is such a generator when \(R(C)\) is dense.

(b) Recall the definition of \(W\) in \([1]\) or \([2]\): for \(x \in X\), \(\|x\|_W = \sup_{t \geq 0} \|T(t)x\| = \|x\|_s\). Since \(W\) is a Banach space containing \(X\), and \(X_s\) is the completion of \(X\) under the norm \(\| \cdot \|_s\), it is clear that \(X_s\) is contained in \(W\) when \(R(C)\) is dense and \(W\) is defined.

(c) \(T_s(t)\) from Theorem \((22)\) is a nonincreasing C-semigroup:
\[\|T_s(r)x\|_s = \sup_{t \geq 0} \|T(t)T(r)x\| = \sup_{t \geq r} \|T(t)Cx\|,\]
which is nonincreasing as a function of $r$. This implies that $e^{rA}$, at least formally a strongly continuous semigroup generated by $A$, is a contraction on $\bigcup_{t \geq 0} R(T(t))$, defined by $e^{rA}T(t)x \equiv T(t + r)x$.

(d) As a consequence of (c), when $\bigcup_{t \geq 0} R(T(t))$ is dense, $A_s$ of Theorem 2.2 generates a strongly continuous semigroup of contractions. This is a weaker hypothesis than $R(C)$ being dense.

Now we use the extrapolation space to give a negative answer to the question mentioned in the Introduction.

Example 2.4. Let $X = c_0(\mathbb{N})$ and $C_0$ be the right shift on $X$, that is,

$$C_0 : (x_1, x_2, x_3, \cdots) \to (0, x_1, x_2, x_3, \cdots).$$

Next let

$$A = \begin{pmatrix} i & C_0^{-1} \\ 0 & -i \end{pmatrix}$$

with $D(A) = X \times R(C_0)$ and

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix}.$$ 

It is not hard to show that $A$ generates a bounded $C$-semigroup on $X \times X$ given by

$$T(t) = \begin{pmatrix} e^{itC_0} & \frac{1}{i}(e^{it} - e^{-it}) \\ 0 & e^{-itC_0} \end{pmatrix},$$

but $\{T(t)\}_{t \geq 0}$ is not contractive. For every $\lambda \neq 0$, if $x_2 \not\in \overline{R(C_0)}$, then

$$\lambda - A)^{-1} C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}C_0x_1 + (\lambda + i)^{-1}(\lambda - i)^{-1}x_2 \\ (\lambda + i)^{-1}C_0x_2 \end{pmatrix} \not\in \overline{R(C)}.$$ 

So $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant. Since $\|C_0x\| = \|x\|$, for all $x \in X$, so $X_s = X$, and $\|\cdot\|_s$ is a topologically equivalent renorming of $X$. Thus $T_s(t) = T(t)$, $(\lambda - A)^{-1} = (\lambda - A_s)^{-1}$. Therefore $\overline{R(C_s)}$ is an invariant space of $\overline{R(C_s)}$.

Thus no restriction of $A_s$ generates a contraction $C_0$-semigroup on $\overline{R(C_s)}$.

Remark 2.5. (a) The result is true for any injective $C_0 \in B(X)$, $X$ an arbitrary complex Banach space, satisfying $\overline{R(C_0)} \neq X$ and $0 \not\in \sigma_s(C_0)$; i.e., $C_0x_n \to 0$ implies $x_n \to 0$.

(b) Although $A_s$ of Example 2.4 does not generate a strongly continuous semigroup on $\overline{R(C)}$, there does exist a subspace, $Y$, between $\overline{R(C)}$ and $X_s$, on which $A_s$ generates a strongly continuous semigroup, namely, $Y = X \times \overline{R(C_0)}$.

We end this paper with some open questions:

1. Is every contraction $C$-semigroup a nonincreasing $C$-semigroup (meaning $t \to \|T(t)x\|$ is nonincreasing, for all $x \in X$)? This is true for $C$ being isometric; that is, $\|Cx\| = \|x\|$ for all $x \in X$, since in this case,

$$\|T(t+s)x\| = \|CT(t+s)x\| = \|T(t+s)Cx\| = \|T(t)T(s)x\| \leq \|CT(s)x\| = \|T(s)x\|.$$ 

We conjecture that it is not true in general cases.

2. If $A$ generates a contraction $C$-semigroup on $X$, does there exist a closed subspace $Y$ such that $\overline{R(C)} \subseteq Y \subseteq X$ and $A|_Y$ generates a strongly continuous semigroup of contractions? Example 2.4 of this paper shows that the answer is no if $Y$ is replaced by $R(C)$, but as remarked above in the section on Example 2.4, the answer is yes (in Example 2.4) with a different choice of $Y$.
3. Does every generator of a bounded $C$-semigroup have an extension, possibly on a larger space, that generates a strongly continuous semigroup of contractions? If 2 is true, then it and Theorem 2.2 would imply the answer is yes; when $R(C)$ is dense or $\rho(A)$ contains a half-line, it is known (1, 2, 3) that the answer is yes.

4. Is there a minimal Banach space in which $X$ is embedded on which an extension of $A$ generates a bounded, strongly continuous semigroup? Even when $R(C)$ is dense, so that an extension as in 3 exists, it is not known if a minimal one exists. In contrast, the interpolation space is maximal (see Chapter V in 2).

5. Does $X_s = W$ always (when $R(C)$ is dense, so that both are defined)? Or is there an example where $W$ is strictly larger than $X_s$?

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References


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