QUADRATIC ENHANCEMENTS OF SURFACES: TWO VANISHING RESULTS

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ABSTRACT. This paper records two results which were inexplicably omitted from the paper on Pin structures on low dimensional manifolds in the LMS Lecture Note Series, volume 151, by Kirby and this author. Kirby declined to be listed as a coauthor of this paper.

A Pin^--structure on a surface X induces a quadratic enhancement of the mod 2 intersection form, $q\colon H_1(X;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}$.

Theorem 1.1 says that q vanishes on the kernel of the map in homology to a bounding 3-manifold. This is used by Kreck and Puppe in their paper in Homology, Homotopy and Applications, volume 10. The arXiv version, arXiv:0707.1599 [math.AT], referred to an email from the author to Kreck for the proof. A more polished and public proof seems desirable.

In Section 6 of the paper with Kirby, a Pin^--structure is constructed on a surface X dual to w_2 in an oriented 4-manifold, M^4. Theorem 2.1 says that q vanishes on the Poincaré dual to the image of $H^1(M;\mathbb{Z}/2\mathbb{Z})$ in $H^1(X;\mathbb{Z}/2\mathbb{Z})$.

1. SURFACES BOUNDING 3-MANIFOLDS

Recall that a fixed Pin^-–structure on a surface X defines a function

$$q\colon H_1(X;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}$$

which satisfies the formula

$$q(x + y) = q(x) + q(y) + 2 \cdot (x \bullet y) \text{ for all } x,\, y \in H_1(X;\mathbb{Z}/2\mathbb{Z}).$$

Here $x \bullet y$ denotes the mod–2 intersection of the two classes and $2 \cdot\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ denotes the standard inclusion. Conversely, every such function comes from a unique Pin^-–structure. This is classical, but see [1, Section 3].

Theorem 1.1. *Let M^3 be a 3–manifold with a fixed Pin^-–structure and let X be the boundary of M. Give X the induced Pin^-–structure. Let $x \in H_1(X;\mathbb{Z}/2\mathbb{Z})$ be a class which vanishes in $H_1(M;\mathbb{Z}/2\mathbb{Z})$. Then $q(x) = 0$.*

Proof. We start with two lemmas.

Lemma 1.2. *Let $S^1 \subset X$ be an embedded circle with trivial normal bundle. Fix a Pin^-–structure on X. One can do surgery on this embedding and extend the Pin^-–structure to the trace of the surgery if and only if $q(S^1) = 0$.*

Proof. Pick a point on the circle and orient the tangent space at this point and also orient the circle at the point. This orients, and hence frames, the normal bundle to S^1 in X.

A tubular neighborhood of the circle is now oriented and so a Pin^-–structure on X restricts to a $Spin$–structure on this neighborhood. The framing on the normal bundle induces a stable framing of S^1 and $q(S^1) \in \Omega_1^{Spin} \cong \mathbb{Z}/2\mathbb{Z}$.

This is equivalent to the description in Kirby–Taylor [1] just before Definition 3.5. The definition given there works for all circles, not just ones with trivial normal bundle.

The trace of the surgery is formed by gluing $D^2 \times D^1$ to $S^1 \times D^1$. Since the 1–dimensional framed bordism is $\mathbb{Z}/2\mathbb{Z}$ and maps isomorphically to Ω_1^{Spin}, if $q(S^1) = 0 \in \Omega_1^{Spin}$, then the $Spin$–structure on the circle extends over a disk and hence over $D^2 \times D^1$ and finally over the entire trace.

The "only if" part follows from the next lemma. $\square$

Lemma 1.3. *Let X be a surface bounding a 3–manifold M and suppose M has a Pin^-–structure. Let $S^1 \subset X$ be an embedded circle which bounds an embedded disk in M. Then $q(S^1) = 0$.*

Proof. A tubular neighborhood of the disk in M is trivial. As in the proof of the first lemma, orient a point on the circle and orient the circle. These orientations extend over the neighborhood of the disk and over the disk and hence frame the normal bundle of the disk in M. The Pin^-–structure on M restricts to a $Spin$–structure on the neighborhood of the disk.

Lemma 2.7 of [1] shows that restricting to the disk and then to the bounding circle gives the same $Spin$–structure as restricting to X and then to the circle. The first restriction is obviously 0 and the second is $q(S^1)$. $\square$

We turn now to the proof of the theorem.

Proof. Assume M is connected and hence has a handlebody decomposition with no 0–handles. Divide M into two pieces, Y and M'. The submanifold Y is obtained from X by attaching the 1–handles. The boundary of Y consists of X and another surface F disjoint from X. Let M' be the result of attaching the 2 and 3 handles to F so that $M = Y \cup M'$. Notice that Y is obtained from F by adding 2–handles. The Pin^-–structure on M restricts to one on Y and to one on M' and hence to one on F. Let q_F denote the resulting quadratic enhancement.

Let $x \in H_1(X; \mathbb{Z}/2\mathbb{Z})$ be a class that vanishes in $H_1(M; \mathbb{Z}/2\mathbb{Z})$. The class x can always be represented by disjoint embedded circles using the usual trick for removing transverse intersections. Let $\kappa \in H_2(M, X; \mathbb{Z}/2\mathbb{Z})$ be the resulting relative class. By excision, $H_2(M, X; \mathbb{Z}/2\mathbb{Z}) = H_2(M', F; \mathbb{Z}/2\mathbb{Z})$, so let $x_1 \in H_1(F; \mathbb{Z}/2\mathbb{Z})$ be the boundary of κ after excision. It follows from the construction that x_1 vanishes in $H_1(M'; \mathbb{Z}/2\mathbb{Z})$ and that there is a relative class $\lambda \in H_2(Y, X \perp\!\!\!\perp F; \mathbb{Z}/2\mathbb{Z})$ with boundary $x + x_1$.

Every homology class in F which dies in M' can be represented after handle slides and additions by the boundary of a 2–handle. But this is a surgery, so by Lemma 1.2, $q_F(x_1) = 0$.

By adding all the 1–handles in a small disk in X, we see F as a connected sum of X and some tori and Klein bottles. In particular, there is a class $\bar{x}$ in $H_1(F; \mathbb{Z}/2\mathbb{Z})$ which can be joined to $x \in H_1(X; \mathbb{Z}/2\mathbb{Z})$ by an embedded cylinder in Y. Then

$\bar{x} + x_1$ vanishes in $H_1(Y; \mathbb{Z}/2\mathbb{Z})$ and again Y has no 0 or 1–handles when built from F, so $q_F(\bar{x} + x_1) = 0$.

Make the cylinder from x to $\bar{x}$ transverse to a surface spanning x and x_1 representing λ. The intersection will be some circles and some arcs and the usual "arc has two ends" argument shows $\bar{x} \bullet x_1 = x \bullet x$. But since x bounds in M, $x \bullet x = 0$. Hence $0 = q_F(\bar{x} + x_1) = q_F(\bar{x}) + q_F(x_1) = q_F(\bar{x})$.

Since we added the 1–handles in a small disk $D^2 \subset X$, we see an embedding $X' \times [0, 1] \subset Y$, where $X' = X - D^2$. Furthermore, this embedding is the inclusion $F' \subset F$ at $X' \times 0$ and is the inclusion $X' \subset X$ at $X' \times 1$. Here $X' \subset X$ is the inclusion whose complement is the connected sum of tori and Klein bottles. The cylinder between our representatives of x and $\bar{x}$ lies in $X' \times [0, 1]$.

Since the $Spin$–structure on the circles representing x or $\bar{x}$ can be computed from the Pin^-–structure on a neighborhood of these circles [1, Definition 3.5], we can work in $X' \times [0, 1]$ with its induced Pin^-–structure.

The Pin^-–structures on $X' \times 0$ and $X' \times 1$ are equivalent and since equivalent Pin^-–structures induce the same restriction to an $S^1 \subset X'$, $q_F(\bar{x}) = q(x)$ and hence $q(x) = 0$. $\qquad\square$

2. The dual to w_2

Let M^4 be an oriented 4–manifold and let $X \subset M$ be dual to w_2. In [1, Section 6] we used a $Spin$–structure on $M - X$ to construct a Pin^-–structure on X.

Theorem 2.1. *The composition*

$$H^1(M; \mathbb{Z}/2\mathbb{Z}) \to H^1(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cap [X]} H_1(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{q} \mathbb{Z}/4\mathbb{Z}$$

is trivial.

Remark 2.2. This says that the image of $H^1(M; \mathbb{Z}/2\mathbb{Z})$ in $H^1(X; \mathbb{Z}/2\mathbb{Z})$ can be at most half–dimensional.

Proof. The set of $Spin$–structures on $M - X$ is an $H^1(M; \mathbb{Z}/2\mathbb{Z})$–torsor. The set of Pin^-–structures on X is an $H^1(X; \mathbb{Z}/2\mathbb{Z})$–torsor. By Lemma 6.2 of [1], if the $Spin$–structure is changed by $x \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, then the Pin^-–structure changes by $i^*(x)$, where $i \colon X \to M$ denotes the embedding.

Associated to q there is an element $\beta(q) \in \mathbb{Z}/8\mathbb{Z}$ called the Brown–Arf invariant. The only properties needed of this invariant will be recalled below. Theorem 6.3 of [1] says that $2 \cdot \beta(X) = X \bullet X - \text{sign}(M) \bmod 16$, where $X \bullet X \in \mathbb{Z}$ is the self–intersection and $\text{sign}(M)$ is the signature.

Lemma 3.7 of [1] says that if the Pin^-–structure is changed by $y \in H^1(X; \mathbb{Z}/2\mathbb{Z})$, then β changes by $2 \cdot q(y \cap [X])$.

If the $Spin$–structure on $M - X$ is changed by $x \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, then the right–hand side of $2 \cdot \beta(X) = X \bullet X - \text{sign}(M)$ does not change, and hence neither can the left. Therefore $2 \cdot q\big(i^*(x) \cap [X]\big) = 0$. But since $2 \cdot \colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ is injective, the result follows. $\qquad\square$

References

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[2] Matthias Kreck and Volker Puppe, *Involutions on* 3-*manifolds and self-dual, binary codes,* Homology, Homotopy Appl. **10** (2008), no. 2, 139–148 (electronic).

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