A SHORT PROOF OF PITT'S COMPACTNESS THEOREM

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Abstract. We give a short proof of Pitt's theorem that every bounded linear operator from \( \ell_p \) or \( c_0 \) into \( \ell_q \) is compact whenever \( 1 \leq q < p < \infty \).

A bounded linear operator between two Banach spaces \( X \) and \( Y \) is said to be compact if it maps the closed unit ball of \( X \) into a relatively compact subset of \( Y \).

Theorem (Pitt; see for example [1], p. 175). Let \( 1 \leq q < p \leq +\infty \), and put \( X_p = \ell_p \) if \( p < +\infty \) and \( X_\infty = c_0 \). Then every bounded linear operator from \( X_p \) into \( \ell_q \) is compact.

Proof. Let \( T : X_p \to \ell_q \) be a norm-one operator. As \( 1 < p \), the dual of \( X_p \) is separable. Hence every bounded sequence in \( X_p \) has a weakly Cauchy subsequence. Thus, for proving the compactness of \( T \), it is enough to show that \( T \) is weak-to-norm continuous. So, let us consider a weakly null sequence \( (h_n) \) in \( X_p \). We have to show that \( \lim_{n \to \infty} \|T(h_n)\| = 0 \). We claim that

1. for every \( x \in c_0 \) and for every weakly null sequence \( (w_n) \) in \( c_0 \),
   \[
   \limsup_{n \to \infty} \|x + w_n\| = \max(\|x\|, \limsup_{n \to \infty} \|w_n\|),
   \]
2. for every \( x \in \ell_r \), \( 1 \leq r < \infty \), and for every weakly null sequence \( (w_n) \) in \( \ell_r \),
   \[
   \limsup_{n \to \infty} \|x + w_n\|^r = \|x\|^r + \limsup_{n \to \infty} \|w_n\|^r.
   \]

Indeed this is obvious when \( x \) is finitely supported, because the coordinates of \( (w_n) \) along the support of \( x \) tend to 0 in norm. The general case is true by the density of finitely supported elements in \( X_p \) and since the norm is a Lipschitzian function.

Fix \( 0 < \varepsilon < 1 \). By definition of the norm of \( T \), there exists \( x_\varepsilon \in X_p \) such that \( \|x_\varepsilon\| = 1 \) and \( 1 - \varepsilon \leq \|T(x_\varepsilon)\| \leq 1 \). Moreover, for all \( n \in \mathbb{N} \) and for all \( t > 0 \)

\[
(0) \quad \|T(x_\varepsilon) + T(th_n)\| \leq \|x_\varepsilon + th_n\|.
\]

In the left-hand side of \( (0) \), we apply claim (2) in \( \ell_q \), with \( x = T(x_\varepsilon) \) and the weakly null sequence \( (T(th_n)) \).

First, assume \( p < +\infty \). We apply claim (2) to the right-hand side of \( (0) \) with \( r = p \), \( x = x_\varepsilon \) and the weakly null sequence \( (h_n) \) to obtain

\[
\left( \left\| T(x_\varepsilon) \right\|^p + t^p \limsup_{n \to \infty} \|T(h_n)\| \right)^\frac{1}{p} \leq \left( \|x_\varepsilon\|^p + t^p \limsup_{n \to \infty} \|h_n\|^p \right)^\frac{1}{p}.
\]

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Recall that \( \| x_\varepsilon \| = 1, 1 - \varepsilon \leq \| T(x_\varepsilon) \| \leq 1 \) and that \((h_n)\) is weakly convergent, thus bounded by some \( M > 0 \). This gives
\[
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{t^q} \left[ (1 + t^p M^p)^{q/p} - (1 - \varepsilon)^q \right].
\]
Taking \( t = \varepsilon^{\frac{1}{p}} \) here, we get
\[
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{\varepsilon^{q/p}} \left[ 1 + \frac{2}{p} M^p \varepsilon - (1 - q^q) + o(\varepsilon) \right].
\]
Now, letting \( \varepsilon \to 0 \) here, we get that \( \limsup_{n \to \infty} \| T(h_n) \|^q \leq 0 \), and therefore the sequence \((T(h_n))\) norm-converges to 0.

Second, assume \( p = +\infty \). We apply claim (1) to the right-hand side of (0) to obtain
\[
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{t^q} \left[ \max (1, t^q M^q) - (1 - \varepsilon)^q \right].
\]
Considering here any \( 0 < \varepsilon < M^{-2q} \) and then taking \( t = \varepsilon^{\frac{1}{q}} \), we get that
\[
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{\varepsilon^{1/2}} \left[ 1 - (1 - \varepsilon)^q \right].
\]
Now, letting \( \varepsilon \to 0 \) here, we get as before that the sequence \((T(h_n))\) norm-converges to 0. □

The framework of this paper was inspired by [2]. The proof given in [2], devoted to the case \( p < +\infty \), uses Stegall’s variational principle.

REFERENCES


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