A SHORT PROOF OF PITT'S COMPACTNESS THEOREM

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Abstract. We give a short proof of Pitt's theorem that every bounded linear operator from $\ell_p$ or $c_0$ into $\ell_q$ is compact whenever $1 \leq q < p < \infty$.

A bounded linear operator between two Banach spaces $X$ and $Y$ is said to be compact if it maps the closed unit ball of $X$ into a relatively compact subset of $Y$.

Theorem (Pitt; see for example [1], p. 175). Let $1 \leq q < p \leq +\infty$, and put $X_p = \ell_p$ if $p < +\infty$ and $X_\infty = c_0$. Then every bounded linear operator from $X_p$ into $\ell_q$ is compact.

Proof. Let $T : X_p \to \ell_q$ be a norm-one operator. As $1 < p$, the dual of $X_p$ is separable. Hence every bounded sequence in $X_p$ has a weakly Cauchy subsequence. Thus, for proving the compactness of $T$, it is enough to show that $T$ is weak-to-norm continuous. So, let us consider a weakly null sequence $(h_n)$ in $X_p$. We have to show that $\lim_{n \to \infty} ||T(h_n)|| = 0$. We claim that

1. for every $x \in c_0$ and for every weakly null sequence $(w_n)$ in $c_0$,
   $$\limsup_{n \to \infty} ||x + w_n|| = \max(||x||, \limsup_{n \to \infty} ||w_n||),$$

2. for every $x \in \ell_r$, $1 \leq r < \infty$, and for every weakly null sequence $(w_n)$ in $\ell_r$,
   $$\limsup_{n \to \infty} ||x + w_n||^r = ||x||^r + \limsup_{n \to \infty} ||w_n||^r.$$

Indeed this is obvious when $x$ is finitely supported, because the coordinates of $(w_n)$ along the support of $x$ tend to 0 in norm. The general case is true by the density of finitely supported elements in $X_p$ and since the norm is a Lipschitzian function.

Fix $0 < \varepsilon < 1$. By definition of the norm of $T$, there exists $x_\varepsilon \in X_p$ such that $||x_\varepsilon|| = 1$ and $1 - \varepsilon \leq ||T(x_\varepsilon)|| \leq 1$. Moreover, for all $n \in \mathbb{N}$ and for all $t > 0$

$$||T(x_\varepsilon) + T(th_n)|| \leq ||x_\varepsilon + th_n||.$$

In the left-hand side of (0), we apply claim (2) in $\ell_q$, with $x = T(x_\varepsilon)$ and the weakly null sequence $(T(th_n))$.

First, assume $p < +\infty$. We apply claim (2) to the right-hand side of (0) with $r = p$, $x = x_\varepsilon$ and the weakly null sequence $(th_n)$ to obtain

$$\left[||T(x_\varepsilon)||^p + t^p \limsup_{n \to \infty} ||T(h_n)||^p \right]^\frac{1}{p} \leq \left[||x_\varepsilon||^p + t^p \limsup_{n \to \infty} ||h_n||^p \right]^\frac{1}{p}.$$
Recall that $\|x_\epsilon\| = 1, 1 - \epsilon \leq \|T(x_\epsilon)\| \leq 1$ and that $(h_n)$ is weakly convergent, thus bounded by some $M > 0$. This gives

$$\limsup_{n \to \infty} \|T(h_n)\|^q \leq \frac{1}{t^q} \left( (1 + t^p M^p)^{q/p} - (1 - \epsilon)^q \right).$$

Taking $t = \epsilon^{\frac{1}{2q}}$ here, we get

$$\limsup_{n \to \infty} \|T(h_n)\|^q \leq \frac{1}{\epsilon^{q/p}} \left[ 1 + \frac{q}{p} M^p \epsilon - (1 - q \epsilon) + o(\epsilon) \right].$$

Now, letting $\epsilon \to 0$ here, we get that $\limsup_{n \to \infty} \|T(h_n)\|^q \leq 0$, and therefore the sequence $(T(h_n))$ norm-converges to 0.

Second, assume $p = +\infty$. We apply claim (1) to the right-hand side of (0) to obtain

$$\limsup_{n \to \infty} \|T(h_n)\|^q \leq \frac{1}{t^q} \left[ \max \left( 1, t^q M^q \right) - (1 - \epsilon)^q \right].$$

Considering here any $0 < \epsilon < M^{-2q}$ and then taking $t = \epsilon^{\frac{1}{2q}}$, we get that

$$\limsup_{n \to \infty} \|T(h_n)\|^q \leq \frac{1}{\epsilon^{q/2}} \left[ 1 - (1 - \epsilon)^q \right].$$

Now, letting $\epsilon \to 0$ here, we get as before that the sequence $(T(h_n))$ norm-converges to 0.

The framework of this paper was inspired by [2]. The proof given in [2], devoted to the case $p < +\infty$, uses Stegall’s variational principle.

**References**
