SMOOTHNESS OF RADIAL SOLUTIONS TO MONGE-AMPÈRE EQUATIONS

CRISTIAN RIOS AND ERIC T. SAWYER

(Communicated by Matthew J. Gursky)

ABSTRACT. We prove that generalized convex radial solutions to the generalized Monge-Ampère equation \( \det D^2u = f(|x|^2/2, u, |\nabla u|^2/2) \) with \( f \) smooth are always smooth away from the origin. Moreover, we characterize the global smoothness of these solutions in terms of the order of vanishing of \( f \) at the origin.

1. Introduction

It is well known that the radial homogeneous functions \( u = c_{m,n} |x|^{2+2m/n} \) provide nonsmooth solutions to the Monge-Ampère equation \( \det D^2u = |x|^{2m} \) with smooth right-hand side when \( m \in \mathbb{N} \setminus n\mathbb{N} \). This raises the question of when radial solutions \( u \) to the generalized equation

\[
\det D^2u = k(x, u, Du), \quad x \in B_n,
\]

are smooth, given that \( k \) is smooth and nonnegative. When \( u \) is radial, (1.1) reduces to a nonlinear ODE on \([0,1)\) that is singular at the endpoint 0. It is thus easy to prove that \( u \) is always smooth away from the origin, even where \( k \) vanishes, but smoothness at the origin is more complicated and determined by the order of vanishing of \( k \) there.

In fact, Monn \([9]\) proves that if \( k = k(x) \) is independent of \( u \) and \( Du \), then a radial solution \( u \) to (1.1) is smooth if \( k^\pm \) is smooth, and Derridj \([4]\) has extended this criterion to the case when \( k(x, u, Du) = f \left( |x|^2/2, u, |\nabla u|^2/2 \right) \) factors as

\[
f(t, \xi, \zeta) = \kappa(t) \phi(t, \xi, \zeta)
\]

with \( \kappa \) smooth and nonnegative on \([0,1)\), \( \kappa(0) = 0 \), and \( \phi \) smooth and positive on \([0,1) \times \mathbb{R} \times [0, \infty) \). Moreover, Monn also shows that \( u \) is smooth if \( k = k(x) \) vanishes to infinite order at the origin.

These results leave open the case when \( k \) has the general form \( k(x, u, Du) \) and vanishes to infinite order at the origin. The purpose of this paper is to show that radial solutions \( u \) are smooth in this remaining case as well. The following theorem encompasses all of the aforementioned results and applies to generalized convex solutions \( u \) and also with \( f = \kappa \phi \) as in (1.2) but where \( \phi \) is only assumed positive and bounded, not smooth.

Received by the editors April 22, 2008.

2000 Mathematics Subject Classification. Primary 35B65, 35J70; Secondary 35D05, 35D10, 35C15.
Theorem 1.1. Suppose that $u$ is a generalized convex radial solution (in the sense of Alexandrov) to the generalized Monge-Ampère equation (1.1) with

$$k(x, u, Du) = f\left(\frac{|x|^2}{2}, u, \frac{\nabla u^2}{2}\right),$$

where $f$ is smooth and nonnegative on $[0, 1) \times \mathbb{R} \times [0, \infty)$. Then $u$ is smooth in the deleted ball $B \setminus \{0\}$.

Suppose moreover that there are positive constants $c, C$ such that

$$cf(t, 0, 0) \leq f(t, \xi, \zeta) \leq Cf(t, 0, 0)$$

for $(\xi, \zeta)$ near $(0, 0)$. Let $\tau \in \mathbb{Z}_+ \cup \{\infty\}$ be the order of vanishing of $f(t, 0, 0)$ at 0. Then $u$ is smooth at the origin if and only if $\tau \in n\mathbb{Z}_+ \cup \{\infty\}$.

The case when $k = k(x)$ is independent of $u$ and $Du$ is handled by Monn in [9] using an explicit formula for $u$ in terms of $k$:

$$g(t) = C + \left(\frac{n}{2}\right)^\frac{k}{2} \int_0^t \left(\int_0^s w^2 f(w) \frac{dw}{\sqrt{s}}\right)^\frac{n}{2} ds,$$

where $u(x) = g\left(\frac{x^2}{2}\right)$ and $k(x) = f\left(\frac{x^2}{2}\right) \geq 0$ with $r = |x|, x \in \mathbb{R}^n$. In the case that $k$ vanishes to infinite order at the origin, an inequality of Hadamard is used as well. The following scale-invariant version follows from Corollary 5.2 in [9]:

$$\max_{0 \leq \ell \leq x} F^{(\ell)}(t) \leq C_{k, \ell} F(x) \sup_{0 \leq \ell \leq x} \max_{0 \leq \ell \leq x} F^{(k)}(t) \sup_{0 \leq \ell \leq x} F^{(k)}(t), \quad 0 \leq x \leq 1,$$

for all $1 \leq \ell \leq k - 1$ and $k \in \mathbb{N}$ provided $F$ is smooth, nondecreasing on $[0, 1)$ and vanishes to infinite order at 0.

2. Proof of Theorem 1.1

We begin by considering Theorem 1.1 in the case that $u$ is a classical $C^2$ solution to (1.1) and $f$ satisfies (1.2) where $f(t, 0, 0)$ vanishes to finite order $\ell$ at 0. If $k$ is independent of $u$ and $Du$, Monn uses formula (1.4) in [9] to show that $u$ is smooth when $f(w)^\frac{n}{2}$ is smooth. In particular this applies when $\ell \in n\mathbb{Z}_+$. In the general case, we note that (1.3) implies (1.2), the assumption made in [4]. Indeed, using $f^{(k)}(0, \xi, \zeta) = 0$ for $0 \leq k \leq \ell - 1$ we can write

$$f(s, \xi, \zeta) = \int_0^1 (1 - t)^{\ell - 1} \frac{d^\ell}{dt^\ell} f(ts, \xi, \zeta) dt = \int_0^1 s^{\ell} \psi(s, \xi, \zeta) dt,$$

where $\psi(s, \xi, \zeta)$ is smooth and $\psi(0, \xi, \zeta) = \frac{f^{(\ell)}(0, \xi, \zeta)}{\ell!} > 0$. Thus the results of Derridj [4] apply to show that $u$ is smooth for general $k$ when $\ell \in n\mathbb{Z}_+$.

2.1. Generalized Monge-Ampère equations. We now consider radial generalized convex solutions $u$ to the generalized Monge-Ampère equation (1.1), where we assume $k(\cdot, u, q)$ and $k(x, u, \cdot)$ are radial. We first establish that $u \in C^2(B_n) \cap C^\infty(B_n \setminus \{0\})$. We note that results of Guan, Trudinger and Wang in [6] and [8] yield $u \in C^{1,1}(B_n)$ for many $k$ in (1.1), but not in the generality possible in the radial case here. In order to deal with general $k$ it would be helpful to have a formula for $u$ in terms of $k$, but this is problematic. Instead we prove Theorem 1.1...
for general $k$ without solving for the solution explicitly, but using an inductive argument that is based on estimates (1.5) when $k$ vanishes to infinite order at the origin.

Assume that $u$ is a generalized convex solution of (1.1) in the sense of Alexandrov (see [11] and [23] and define $\varphi(t)$ by

$$
\varphi \left( \frac{r^2}{2} \right) = k(x,u(x),Du(x)) = f \left( \frac{|x|^2}{2}, u(x), \frac{\nabla u(x)}{2} \right).
$$

Then $\varphi$ is bounded since $u$ is Lipschitz continuous. It follows that the convex radial function $u$ is continuously differentiable at the origin, since otherwise it would have a conical singularity there and its representing measure $\mu_u$ would have a Dirac component at the origin. Let $g$ be given by formula (1.4) with $\varphi$ in place of $f$, i.e.

$$
g(t) = C_u + \left( \frac{n}{2} \right)^{\frac{1}{2}} \int_0^t \left( \int_0^s w^\frac{n}{2} \varphi(w) \frac{dw}{w} \right)^{\frac{1}{n}} ds,
$$

and with constant $C_u$ chosen so that $u$ and $\tilde{u}$ agree on the unit sphere where

$$
\tilde{u}(x) = g \left( \frac{r^2}{2} \right), \quad 0 < r < 1.
$$

We claim that $\tilde{u}$ is a generalized convex solution to (1.1) in the sense of Alexandrov. To see this we first note that

$$
D^2\tilde{u}(re_1) = \begin{bmatrix} g''r^2 + g' & 0 & \cdots & 0 \\
0 & g' & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & g' \end{bmatrix}
$$

is positive semidefinite; hence $\tilde{u}$ is convex. To prove that the representing measure $\mu_{\tilde{u}}$ of $\tilde{u}$ is $kdx$ it suffices to show, since both $g$ and $f$ are radial, that

$$
\mu_{\tilde{u}}(E) = |B_{\tilde{u}}(E)| = \int_E k
$$

for all annuli $E = \{x \in \mathbb{B}_n : r_1 < |x| < r_2\}$, $0 < r_1 < r_2 < 1$ where

$$
B_{\tilde{u}}(E) = \bigcup_{r_1 < |x| < r_2} \{ \nabla \tilde{u}_3(x) = \left\{ a \in \mathbb{B}_n : \frac{\partial}{\partial r} \tilde{u}(r_1e_1) < |a| < \frac{\partial}{\partial r} \tilde{u}(r_2e_1) \right\} \}.
$$

Since $\frac{\partial}{\partial r} \tilde{u}(r_1e_1) = g'(\frac{r^2}{2})r_1 = g'(t_1)\sqrt{2t_1}$ with $t_1 = \frac{r^2}{2}$, we thus have

$$
|B_{\tilde{u}}(E)| = \left| \left\{ a \in \mathbb{B}_n : g'(t_1)\sqrt{2t_1} < |a| < g'(t_2)\sqrt{2t_2} \right\} \right|
$$

$$
= \frac{\omega_n}{n} \left\{ g'(t_2)^n(2t_2)^\frac{n}{2} - g'(t_1)^n(2t_1)^\frac{n}{2} \right\}
$$

$$
= \frac{\omega_n}{n} \frac{2^n}{2} \int_{t_1}^{t_2} w^\frac{n}{2} \varphi(w) \frac{dw}{w}
$$

$$
= \omega_n \int_{r_1}^{r_2} r^{n-1} \varphi \left( \frac{r^2}{2} \right) dr = \int_E k.
$$

In particular the convex radial function $\tilde{u}$ must be continuously differentiable, since otherwise there is a jump discontinuity in the radial derivative of $\tilde{u}$ at some distance $r$ from the origin that results in a singular component in $\mu_{\tilde{u}}$ supported on the sphere of radius $r$. 


Now the uniqueness of Alexandrov solutions to the Dirichlet problem (see e.g. \[3\]) yields \( u = \bar{u} \), and hence \( u \in C^1(\mathbb{B}_n) \). Thus \( \varphi \in C[0,1] \), and from (2.3) we have \( u(x) = g\left(\frac{|x|^2}{2}\right) \) and

\[
\varphi(t) = f\left(t, g(t), t g'(t)^2\right),
\]

where using (2.2) we compute that

\[
g'(t) = \left(\frac{n}{2} t^{-\frac{n}{2}} \int_0^t s^{\frac{n}{2} - 1} \varphi(s) \, ds\right)^{\frac{1}{n}}.
\]

In particular \( g' \in C[0,1] \). We now obtain by induction that \( g \in C^{\infty}(0,1) \); hence \( u \in C^{\infty}(\mathbb{B}_n \setminus \{0\}) \). Indeed, if \( g \in C^{\ell}(0,1) \), then (2.4) implies \( \varphi \in C^{\ell-1}(0,1) \) and then (2.2) implies \( g \in C^{\ell+1}(0,1) \).

It will be convenient to use fractional integral operators at this point. For \( \beta > 0 \) and \( f \) continuous define

\[
T_\beta f(s) = \int_0^s \left(\frac{w}{s}\right)^\beta f(w) \frac{dw}{w}, \quad s \neq 0,
\]

\[
T_\beta f(0) = \frac{1}{\beta} f(0)
\]

so that

\[
g(t) = C + \left(\frac{n}{2}\right)^{\frac{1}{n}} \int_0^t \left(T_\frac{2}{n} f(s)\right)^{\frac{1}{n}} ds.
\]

We claim that for \( f \) smooth, nonnegative and of finite type \( \ell, \ell \in \mathbb{Z}_+ \), the same is true of \( T_\beta f \) for all \( \beta > 0 \). This follows immediately from the identity

\[
d_k \frac{d^k}{ds^k} T_\beta f(s) = T_{\beta+k} f^{(k)}(s), \quad k \in \mathbb{N},
\]

and the estimate

\[
T_{\beta+k} f^{(k)}(s) = \frac{1}{\beta+k} f^{(k)}(0) + O(|s|).
\]

When \( k = 1 \), (2.7) follows from differentiating and then integrating by parts, and the general case is then obtained by iteration.

Now suppose that \( f \) satisfies (1.3) and let

\[
\kappa(t) = f(t, 0, 0)
\]

vanish to infinite order at 0. If \( \kappa \) vanishes in a neighbourhood of 0, then so does \( g \), and we have \( g \in C^{\infty}(0,1) \) and \( u \in C^{\infty}(\mathbb{B}_n) \). Thus we will assume \( f_0^t \kappa > 0 \) for \( t > 0 \) in what follows. Note that (2.7) then implies that \( T_2 \kappa \) is smooth and positive on \((0,1)\) and vanishes to infinite order at 0. Since \( g' \in C(0,1) \), it follows that \( \varphi(t) \leq C \kappa(t) \). Thus we have the inequality \( T_2 \varphi(t) \leq C T_2 \kappa(t) \), and from (2.3) we now conclude that \( g'(t) \) also vanishes to infinite order at 0. Now \( \varphi(t) \approx \kappa(t) \) from (1.3), and so also \( T_2 \varphi(t) \approx T_2 \kappa(t) \). From

\[
g''(t) = \frac{\varphi(t)}{2t\left(T_\frac{2}{n} \varphi(t)\right)^{1-\frac{1}{n}}} - \frac{1}{2t} \left(\frac{n}{2} T_\frac{2}{n} \varphi(t)\right)^{\frac{1}{n}},
\]
we then have

$$\left| g''(t) \right| \leq C \frac{\kappa(t)}{2t \left( \frac{n}{2} T_\varphi \kappa(t) \right)^{1 - \frac{n}{2}}} + C \frac{1}{2t} \left( \frac{n}{2} T_\varphi \kappa(t) \right)^{\frac{1}{2}}, \quad 0 < t < 1. \quad (2.9)$$

An application of (1.5) with \( \ell = 1, k > n \) and \( F(t) = \int_0^t s^{\frac{n}{2} - 1} \kappa(s) \, ds \) yields

$$t^n \kappa(t) = F'(t) \leq CF(t)^{1 - \frac{1}{k}}, \quad \text{and so the first term on the right side of (2.9) is}$$

bounded by a multiple of \( t^{\frac{1}{k}} F(t)^{\frac{1}{k} - \frac{1}{k}} \). Thus the right side of (2.9), and hence also \( g''(t) \), vanishes to infinite order at 0. In particular \( g'' \in C[0, 1] \), and we conclude \( u \in C^2(\mathbb{B}_n) \) in this case as well.

Summarizing, we have \( u \in C^\infty(\mathbb{B}_n \setminus \{0\}) \), and in the case that \( f \) satisfies (1.3), we also have \( u \in C^2(\mathbb{B}_n) \). Thus from the above we have that

$$\varphi(t) = f \left( t, g(t), t g'(t)^2 \right) = \kappa(t) \varphi \left( t, g(t), t g'(t)^2 \right),$$

where \( u(x) = g \left( \frac{|x|^2}{2} \right) \in C^2(\mathbb{B}_n) \), \( g \) is given by (2.24) and \( \varphi \in C^1[0, 1] \) by (2.24).

Note that we cannot use (1.5) on the function \( \int_0^t s^{\frac{n}{2} - 1} \varphi(s) \, ds \) here since we have no a priori control on higher derivatives of \( \varphi(s) = f \left( s, g(s), sg'(s)^2 \right) \). Instead we will use (1.5) on the function \( \int_0^t s^{\frac{n}{2} - 1} \kappa(s) \, ds \) together with an inductive argument to control derivatives of \( g \).

From the above we have that \( g'' \in C[0, 1) \cap C^\infty(0, 1) \). Now differentiate (2.8) for \( t > 0 \) using (2.8) to obtain

$$g'''(t) = \frac{1}{2} \left( \frac{n}{2} \right)^{\frac{n}{2} - 1} \left\{ \frac{\varphi(t)}{t^{\frac{n}{2}} \varphi(t)^{1 - \frac{n}{2}}} - \left( \frac{1}{n - 1} \right) \frac{\kappa(t) T_{\varphi}^{\frac{n}{2} + 1} \varphi'(t)}{t^{\frac{n}{2}} \varphi(t)^{2 - \frac{n}{2}}} \right\}$$

$$- \frac{1}{2} \left( \frac{n}{2} \right)^{\frac{n}{2} - 1} \frac{\varphi(t)}{t^{\frac{n}{2}} \varphi(t)^{1 - \frac{n}{2}}} - \frac{1}{2} \left( \frac{n}{2} \right)^{\frac{n}{2}} \left( \frac{1}{n} \right) \frac{T_{\varphi}^{\frac{n}{2} + 1} \varphi'(t)}{t^{\frac{n}{2}} \varphi(t)^{2 - \frac{n}{2}}} - \frac{T_{\varphi} \varphi'(t)}{t^2},$$

and then compute that

$$\phi'(t) = \kappa'(t) \phi \left( t, g(t), t g'(t)^2 \right) + \kappa(t) \varphi_1 \left( t, g(t), t g'(t)^2 \right)$$

$$+ \kappa(t) \varphi_2 \left( t, g(t), t g'(t)^2 \right) g'(t) + \kappa(t) \varphi_3 \left( t, g(t), t g'(t)^2 \right) \left( g'(t)^2 + 2t g'(t) g''(t) \right). \quad (2.11)$$

We will now use \( \varphi \approx \kappa \), (2.10), (2.11) and (1.3) applied with \( F(t) = \int_0^t s^{\frac{n}{2} - 1} \kappa(s) \, ds \), to show that \( g''' \) vanishes to infinite order at 0 and \( g'' \in C[0, 1) \).

To see this, we first note that \( F \) is smooth, nonnegative and vanishes to infinite order at 0 since the same is true of \( \kappa \). Next, for any \( \ell \geq 1 \) and \( \varepsilon > 0 \), (1.3) with \( k \) large enough yields

$$\sup_{0 < s \leq t} \left| F^{(\ell)}(s) \right| \leq C_{\ell, \varepsilon} F(t)^{1 - \varepsilon}. \quad (2.12)$$
Moreover we have
\begin{equation}
|\beta T_\beta h (t)| \leq \sup_{0 < s \leq t} |h (s)|, \quad F (t) = t^{\frac{2}{n}} T_\frac{2}{n} \kappa (t), \quad T_2 \varphi (t) \approx T_\frac{2}{n} \kappa (t).
\end{equation}

Now using
\[
F' (t) = t^{\frac{2}{n} - 1} \kappa (t), \\
F'' (t) = t^{\frac{2}{n} - 1} \kappa' (t) + \left( \frac{n}{2} - 1 \right) t^{\frac{2}{n} - 2} \kappa (t)
\]
we obtain
\[
\left| \kappa' (t) \phi \left( t, g (t), tg' (t)^2 \right) \right| \leq C |\kappa' (t)| = C t^{1 - \frac{2}{n}} F'' (t) - \left( \frac{n}{2} - 1 \right) t^{\frac{2}{n} - 2} F' (t),
\]
and an application of (2.12) gives
\[
\left| \kappa' (t) \phi \left( t, g (t), tg' (t)^2 \right) \right| \leq C t^{-\frac{2}{n}} F (t)^{1 - \varepsilon}.
\]

We obtain similar estimates for the remaining terms in (2.11), and all together this yields
\[
|\varphi' (t)| \leq C \varepsilon t^{-\alpha} F (t)^{1 - \varepsilon}, \quad \text{for some } \alpha > 0.
\]

Using the second and third lines in (2.13) we now show that the first term in braces in (2.10) satisfies
\[
\left| \frac{\varphi' (t)}{t T_2 \varphi (t)^{1 - \varepsilon}} \right| \leq C t^{-\alpha} F (t)^{1 - \varepsilon} \approx C \varepsilon t^{-\frac{2}{n} (1 - \varepsilon) - \alpha - 1} T_2 \kappa (t)^{\frac{2}{n} - \varepsilon},
\]
which vanishes to infinite order at 0 if 0 < \varepsilon < \frac{1}{n}. Similar arguments, using (2.11) and the first line in (2.13) to estimate \(T_2 + 1\varphi' (t)\), apply to the remaining terms in (2.10), and this completes the proof that \(g''\) vanishes to infinite order at 0 and \(g''' \in C [0, 1]\).

We now observe that we can
- continue to differentiate (2.10) to obtain a formula for \(g^{(\ell)}\) involving only appropriate powers of \(T_2 \varphi (t) \approx T_2 \kappa (t)\) in the denominator and derivatives of \(\varphi\) of order at most \(\ell - 2\) in the numerator,
- and continue to differentiate (2.11) to obtain a formula for \(\varphi^{(\ell - 2)}\) involving derivatives of \(g\) of order at most \(\ell - 1\).

It is now clear that the above arguments apply to prove that derivatives of \(g (t)\) of all orders vanish to infinite order at 0 and are continuous on \([0, 1]\). This shows that \(g\) is smooth on \([0, 1]\) and thus that \(u\) is smooth on \(B_n\).

**References**

[1] A. D. Alexandrov, Dirichlet’s problem for the equation \(\text{Det} \|z_{ij}\| = \Phi (z_1, ..., z_n, z, x_1, ..., x_n), I, Vestnik Leningrad Univ. Ser. Mat. Mekh. Astr. 13 (1958), 5-24.


Department of Mathematics, University of Calgary, Calgary, Alberta, Canada
E-mail address: crios@math.ucalgary.ca

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada
E-mail address: saw6453cdn@aol.com