ON THE TOPOLOGY OF POINTWISE CONVERGENCE ON THE BOUNDARIES OF $L_1$-PREDUALS

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Abstract. In this paper we prove a theorem more general than the following: "If $(X, \| \cdot \|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and $\{x_n : n \in \mathbb{N}\}$ is any subset of $X$, then the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the topology of pointwise convergence on $B$ is separable with respect to the topology generated by the norm, whenever $\text{Ext}(B_{X^*})$ is weak$^*$ Lindelöf." Several applications of this result are also presented.

1. Introduction

We shall say that a Banach space $(X, \| \cdot \|)$ is an $L_1$-predual if $X^*$ is isometric to $L_1(\mu)$ for some suitable measure $\mu$. Some examples of $L_1$-preduals include $(C(K), \| \cdot \|_{\infty})$, and more generally, the space of continuous affine functions on a Choquet simplex (see [10] for the definition) endowed with the supremum norm (see, [4, Proposition 3.23]). We shall also consider the notion of a boundary. Specifically, for a nontrivial Banach space $X$ over $\mathbb{R}$ we say that a subset $B$ of $B_{X^*}$, the closed unit ball of $X^*$, is a boundary if for each $x \in X$ there exists a $b^* \in B$ such that $b^*(x) = \|x\|$. The prototypical example of a boundary is $\text{Ext}(B_{X^*})$, the set of all extreme points of $B_{X^*}$, but there are many other interesting examples given in [9].

In the recent paper [9] the authors investigate the topology on a Banach space $X$ that is generated by $\text{Ext}(B_{X^*})$ and, more generally, the topology on $X$ generated by an arbitrary boundary of $X$. This paper continues this study.

To be more precise we must first introduce some notation. For a nonempty subset $Y$ of the dual of a Banach space $X$ we shall denote by $\sigma(X,Y)$ the weakest linear topology on $X$ that makes all the functionals from $Y$ continuous. In [9] the authors show (see, [9] Theorem 2.2]) using [3] Lemma 1) that for any compact Hausdorff space $K$, any countable subset $\{x_n : n \in \mathbb{N}\}$ of $C(K)$ and any boundary $B$ of $(C(K), \| \cdot \|_{\infty})$, the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the $\sigma(C(K), B)$ topology is separable with respect to the topology generated by the norm. In this paper we extend this result by showing that if $(X, \| \cdot \|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and $\{x_n : n \in \mathbb{N}\}$ is any subset of $X$, then the closure of $\{x_n : n \in \mathbb{N}\}$

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in the $\sigma(X,B)$ topology is separable with respect to the topology generated by the norm whenever $\text{Ext}(B_{X^*})$ is weak* Lindelöf.

We conclude this paper with some applications that indicate the utility of our results.

2. Preliminary results

Let $X$ be a topological space and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of $X$. Then $\mathcal{F}$ is rich if the following two conditions are fulfilled:

(i) for every separable subspace $Y$ of $X$, there exists a $Z \in \mathcal{F}$ such that $Y \subseteq Z$;
(ii) for every increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\bigcup_{n \in \mathbb{N}} Z_n \in \mathcal{F}$.

For any topological space $X$, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, namely,

$G_X := \{ S \subseteq X : S is a nonempty, closed and separable subset of X \}$.

On the other hand, if $X$ is a separable space, then the partially ordered set has a least element, namely, $G_\emptyset := \{ X \}$.

The raison d'être for rich families is revealed next.

**Proposition 1.** Suppose that $X$ is a topological space. If $\{F_n : n \in \mathbb{N}\}$ are rich families of $X$, then so is $\bigcap_{n \in \mathbb{N}} F_n$.

For a proof of this proposition see [2, Proposition 1.1].

Throughout this paper we will be primarily working with Banach spaces, so a natural class of rich families, given a Banach space $X$, is the family of all closed separable linear subspaces of $X$, which we denote by $S_X$. There are however many other interesting examples of rich families that can be found in [2] and [7].

For our first result we will provide another nontrivial example of a rich family, but to achieve this we first need a preliminary result that characterises when a given Banach space is an $L_1$-predual.

**Lemma 1** ([6, §21, Theorem 7]). For a Banach space $X$ the following are equivalent:

(i) $X$ is an $L_1$-predual;
(ii) for each weak* continuous convex function $f$ on $B_{X^*}$,

$$f^*(0) = \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{X^*} \},$$

where $f^* = \inf \{ h : h \geq f and h is weak* continuous and affine on B_{X^*} \}$.

Before proceeding further we shall introduce the following notation. If $X$ is a normed linear space, then each $x \in X$ defines a weak* continuous affine function $\hat{x}$ on $B_{X^*}$ via the canonical embedding, that is, $\hat{x}(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$.

**Theorem 1.** Let $X$ be an $L_1$-predual. Then the set of all closed separable linear subspaces of $X$ that are themselves $L_1$-preduals forms a rich family.

**Proof.** Let $\mathcal{L} := \{ Z \in S_X : Z is an L_1$-predual$.\} We shall verify that $\mathcal{L}$ is a rich family. So first let us consider an arbitrary separable closed linear subspace $Y$ of $X$. Then by [6, §23, Lemma 1] there exists a closed separable subspace $Z \in \mathcal{L}$ such that $Y \subseteq Z$. Next, let us consider an increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathcal{L}$.
and let $Z := \bigcup_{n \in \mathbb{N}} Z_n$. To show that $Z \in \mathcal{Z}$ we shall appeal to Lemma 2. Let $f$ be a weak$^*$ continuous convex function on $B_{Z^*}$. Since

$$\frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} \leq f^*(0),$$

it is enough to verify that for each $\varepsilon > 0$, $f^*(0) \leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} + \varepsilon$. To this end, suppose that $\varepsilon > 0$. Since $f$ is weak$^*$ continuous and convex and $B_{Z^*}$ is weak$^*$ compact, by [11, Corollary I.1.3] there exist $z_i \in Z$ and $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that the weak$^*$ convex continuous $g : B_{Z^*} \to \mathbb{R}$ defined by

$$g := \max \{ \tilde{z}_1 + c_1, \tilde{z}_2 + c_2, \ldots, \tilde{z}_n + c_n \}$$

satisfies

$$f(z^*) - \varepsilon < g(z^*) < f(z^*), \quad z^* \in B_{Z^*}.$$ 

Since $\bigcup_{n \in \mathbb{N}} Z_n$ is dense in $Z$ we may further assume that all the elements $z_i$ are contained in some fixed $Z_j$, $j \in \mathbb{N}$.

Next, let $r : B_{Z^*} \to B_{Z_j}$ be the restriction mapping (i.e., $r(z^*) = z^*|Z_j$ for all $z^* \in B_{Z^*}$) and let $h : B_{Z_j} \to \mathbb{R}$ be defined by $h := \max \{ \tilde{z}_1 + c_1, \tilde{z}_2 + c_2, \ldots, \tilde{z}_n + c_n \}$. Then $h$ is weak$^*$ continuous and convex on $B_{Z_j}$ and $g = h \circ r$. Moreover, by the definition of $g$ (and the fact that $r$ is weak$^*$-to-weak$^*$ continuous and linear) we have that $g^*(z^*) \leq h^*(r(z^*))$ for all $z^* \in B_{Z^*}$. Now, by the assumption that $Z_j$ is an $L_1$-predual (and Lemma 1) there exists a $y^* \in B_{Z_j}$ such that

$$h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)].$$

Choose $z^* \in r^{-1}(y^*)$, which is nonempty by the Hahn-Banach extension theorem. Then,

$$g^*(0) \leq h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)] = \frac{1}{2}[g(z^*) + g(-z^*)] \leq g^*(0).$$

Therefore,

$$f^*(0) - \varepsilon = (f - \varepsilon)^*(0) \leq g^*(0) = \frac{1}{2}[g(z^*) + g(-z^*)] \leq \frac{1}{2} [f(z^*) + f(-z^*)] \leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \}.$$

That is, $f^*(0) \leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} + \varepsilon$, which completes the proof.

Before we can introduce another class of rich families, we require the following lemma, which is a Banach space version of [11, Theorem 2.10].

**Lemma 2.** Let $Y$ be a closed separable linear subspace of a Banach space $X$ and suppose that $L \subseteq \text{Ext}(B_{X'})$ is weak$^*$ Lindelöf. Then there exists a closed separable linear subspace $Z$ of $X$, containing $Y$, such that for any $l^* \in L$ and any $x^*, y^* \in B_{Z^*}$, if $l^*|Z = \frac{1}{2}(x^* + y^*)$, then $x^*|_Y = y^*|_Y$.

**Proof.** Let $\mathcal{B}$ be a countable base for the topology on $(B_{Y'}, \text{weak}^*)$ consisting of closed convex sets. Recall that such a base exists because $(B_{Y'}, \text{weak}^*)$ is compact, by the Banach-Alaoglu Theorem, and $(B_{Y'}, \text{weak}^*)$ is metrizable, since $Y$ is separable. Let:
\[(i) \mathcal{F} := \{r^{-1}(B) : B \in \mathcal{B}\}, \text{ where } r : B_{X^*} \to B_{Y^*} \text{ is the restriction mapping;}
\]
\[(ii) \mathcal{R} := \{\frac{1}{2}(F_1 + F_2) : F_1, F_2 \in \mathcal{F} \text{ and } F_1 \cap F_2 = \emptyset\}.
\]
By construction \(\bigcup \mathcal{R} \subseteq B_{X^*} \setminus \text{Ext}(B_{X^*})\) and so \(L \cap \bigcup \mathcal{R} = \emptyset\). Furthermore, for each \(l^* \in L\) and \(F \in \mathcal{R}\) there exists a \(y \in X\) such that

\[\text{sup}\{\hat{y}(f^*) : f^* \in F\} < \hat{y}(l^*).\]

Therefore, since \(L\) is weak* Lindelöf for each \(F \in \mathcal{R}\) there exists a countable subset \(C_F\) in \(X\) such that, for each \(l^* \in L\) there exists a \(y \in C_F\) such that \(\text{sup}\{\hat{y}(f^*) : f^* \in F\} < \hat{y}(l^*)\). If we set \(C := \bigcup\{C_F : F \in \mathcal{R}\}\) and \(Z := \overline{\text{span}}(C \cup X)\), then \(X \subseteq Z\) and \(Z\) is a closed separable linear subspace of \(X\).

It now only remains to verify that if \(l^* \in L\), \(x^*, y^* \in B_{Z^*}\), and \(l^*|Z = \frac{1}{2}(x^* + y^*)\), then \(x^*|_Y = y^*|_Y\). So, in order to obtain a contradiction, suppose that for some \(l^* \in L\) and \(x^*, y^* \in B_{Z^*}\), \(l^*|Z = \frac{1}{2}(x^* + y^*)\) but \(x^*|_Y \neq y^*|_Y\). Then there exists \(B_1, B_2 \in \mathcal{R}\) such that \(x^*|_Y \in B_1\) and \(y^*|_Y \in B_2\) and \(B_1 \cap B_2 = \emptyset\). Set \(F_1 := r^{-1}(B_1)\) and \(F_2 := r^{-1}(B_2)\). Then \(F_1, F_2 \in \mathcal{F}\) and \(F_1 \cap F_2 = \emptyset\). Now, by the Hahn-Banach Extension Theorem there exist \(x^*_1 \in B_{X^*}\) and \(y^*_1 \in B_{X^*}\) such that \(x^*_1|_Z = x^*\) and \(y^*_1|_Z = y^*\). Moreover,

\[x^*_1|_Y = (x^*_1|_Z)|_Y = x^*|_Y \in B_1 \quad \text{and} \quad y^*_1|_Y = (y^*_1|_Z)|_Y = y^*|_Y \in B_2.
\]

That is, \(x^*_1 \in F_1\) and \(y^*_1 \in F_2\). Therefore, \(\frac{1}{2}(x^*_1 + y^*_1) \in \frac{1}{2}(F_1 + F_2) =: F\). Since \(F \in \mathcal{R}\), by the construction there exists a \(y \in C_F \subseteq C \subseteq Z\) such that \(\text{sup}\{\hat{y}(f^*) : f^* \in F\} < \hat{y}(l^*)\). In particular,

\[\frac{1}{2}(x^* + y^*)(y) = \hat{y}\left(\frac{1}{2}(x^*_1 + y^*_1)\right) < l^*(y) = (l^*|Z)(y).
\]

However, this contradicts the fact that \(\frac{1}{2}(x^* + y^*) = l^*|_Z\). \(\square\)

**Theorem 2.** Let \(X\) be a Banach space and let \(L \subseteq \text{Ext}(B_{X^*})\) be a weak* Lindelöf subset. Then the set of all \(Z\) in \(S_X\) such that \(\{l^*|_Z : l^* \in L\} \subseteq \text{Ext}(B_{Z^*})\) forms a rich family.

**Proof.** Let \(\mathcal{L}\) denote the family of all closed separable linear subspaces \(Z\) of \(X\) such that \(\{l^*|_Z : l^* \in L\} \subseteq \text{Ext}(B_{Z^*})\). We shall verify that \(\mathcal{L}\) is a rich family of closed separable linear subspaces of \(X\). So first let us consider an arbitrary closed separable linear subspace \(Y\) of \(X\), with the aim of showing that there exists a subspace \(Z \in \mathcal{L}\) such that \(Y \subseteq Z\). We begin by inductively applying Lemma 2 to obtain an increasing sequence \((Z_n : n \in \mathbb{N})\) of closed separable linear subspaces of \(X\) such that: \(Y \subseteq Z_1\) and for any \(l^* \in L\) and any \(x^*, y^* \in B_{Z^*}\), if \(l^*|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)\), then \(x^*|_{Z_n} = y^*|_{Z_n}\).

We now claim that if \(Z := \bigcup_{n \in \mathbb{N}} Z_n\), then \(l^*|_Z \in \text{Ext}(B_{Z^*})\) for each \(l^* \in L\). To this end, suppose that \(l^* \in L\) and \(l^*|_Z = \frac{1}{2}(x^* + y^*)\) for some \(x^*, y^* \in B_{Z^*}\). Then for each \(n \in \mathbb{N}\),

\[l^*|_{Z_{n+1}} = (l^*|_Z)|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)|_{Z_{n+1}} = \frac{1}{2}(x^*|_{Z_{n+1}} + y^*|_{Z_{n+1}})
\]

and \(x^*|_{Z_{n+1}}, y^*|_{Z_{n+1}} \in B_{Z_{n+1}}\). Therefore, by construction, \(x^*|_{Z_n} = y^*|_{Z_n}\). Now since \(\bigcup_{n \in \mathbb{N}} Z_n\) is dense in \(Z\) and both \(x^*\) and \(y^*\) are continuous, we may deduce that \(x^* = y^*\), which in turn implies that \(l^*|_Z \in \text{Ext}(B_{Z^*})\). This shows that \(Y \subseteq Z\) and \(Z \in \mathcal{L}\).
To complete this proof we must verify that for each increasing sequence of closed separable subspaces \( (Z_n : n \in \mathbb{N}) \) in \( L : \bigcup_{n \in \mathbb{N}} Z_n \subseteq L \). This, however, follows easily from the definition of the family \( L \).

Let \( X \) be a normed linear space. Then we say that an element \( x^* \in B_X \) is \textit{weak* exposed} if there exists an element \( x \in X \) such that \( y^*(x) < x^*(x) \) for all \( y^* \in B_X \setminus \{ x^* \} \). It is not difficult to show that if \( \text{Exp}(B_X) \) denotes the set of all weak* exposed points of \( B_X \), then \( \text{Exp}(B_X) \subseteq \text{Ext}(B_X) \). However, if \( X \) is a separable \( L_1 \)-predual, then the relationship between \( \text{Exp}(B_X) \) and \( \text{Ext}(B_X) \) is much closer.

**Lemma 3** ([13] Lemma 3.3(b)). If \( X \) is a separable \( L_1 \)-predual, then \( \text{Exp}(B_X) = \text{Ext}(B_X) \).

Let us also pause for a moment to recall that if \( B \) is any boundary for a Banach space \( X \), then

\[
\text{Exp}(B_X) \subseteq B \cap \text{Ext}(B_X) \subseteq \text{Ext}(B_X) \subseteq \overline{B}^{\text{weak*}}.
\]

The fact that \( \text{Ext}(B_X) \subseteq \overline{B}^{\text{weak*}} \) follows from Milman’s theorem, [10, page 8] and the fact that \( B_X = \overline{\text{Exp}^w(B)} \), which in turn follows from a separation argument. Let us also take this opportunity to observe that if \( B_X \) denotes the closed unit ball in \( X \), then \( B_X \) is closed in the \( e(X,B) \) topology for any boundary \( B \) of \( X \). Finally, let us end this section with one more simple observation that will turn out to be useful in our later endeavours.

**Proposition 2.** Suppose that \( Y \) is a linear subspace of a Banach space \( (X, \| \cdot \|) \) and \( B \) is any boundary for \( X \). Then for each \( e^* \in \text{Exp}(B_Y) \) there exists \( b^* \in B \) such that \( e^* = b^*|_Y \).

**Proof.** Suppose that \( e^* \in \text{Exp}(B_Y) \). Then there exists an \( x \in Y \) such that \( y^*(x) < e^*(x) \) for each \( y^* \in B_Y \setminus \{ e^* \} \). By the fact that \( B \) is a boundary of \( (X, \| \cdot \|) \) there exists a \( b^* \in B \) such that \( b^*(x) = \| x \| \neq 0 \). Then for any \( y^* \in B_Y \) we have

\[
y^*(x) \leq b^*(x) \leq \| y^* \| \| x \| \leq \| x \| = b^*(x) = (b^*|_Y)(x).
\]

In particular, \( e^*(x) \leq b^*|_Y(x) \). Since \( b^*|_Y \in B_Y \) and \( y^*(x) < e^*(x) \) for all \( y^* \in B_Y \setminus \{ e^* \} \), it must be the case that \( e^* = b^*|_Y \).

This ends our preliminary section.

3. The main results

**Theorem 3.** Let \( B \) be any boundary for a Banach space \( X \) that is an \( L_1 \)-predual and suppose that \( \{ x_n : n \in \mathbb{N} \} \subseteq X \). Then

\[
\{ x_n : n \in \mathbb{N} \}^{e(X,B)} \cap \{ x_n : n \in \mathbb{N} \}^{e(X,\text{Ext}(B_X))}.
\]

**Proof.** In order to obtain a contradiction let us suppose that

\[
\{ x_n : n \in \mathbb{N} \}^{e(X,B)} \nsubseteq \{ x_n : n \in \mathbb{N} \}^{e(X,\text{Ext}(B_X))}.
\]

Choose \( x \in \{ x_n : n \in \mathbb{N} \}^{e(X,B)} \setminus \{ x_n : n \in \mathbb{N} \}^{e(X,\text{Ext}(B_X))} \). Then there exists a finite set \( \{ e_1^*, e_2^*, \ldots, e_m^* \} \subseteq \text{Ext}(B_X) \) and \( \varepsilon > 0 \) so that

\[
\{ y \in X : |e_k^*(x) - e_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m \} \cap \{ x_n : n \in \mathbb{N} \} = \emptyset.
\]
Let $Y := \text{span} \{ x_n : n \in \mathbb{N} \} \cup \{ x \}$, let $F_1$ be any rich family of $L_1$-preduals whose existence is guaranteed by Theorem 1, and let $F_2$ be any rich family such that for every $Z \in F_2$ and every $1 \leq k \leq m$, $e_k^*|Z \in \text{Ext}(B_{Z^*})$, whose existence is guaranteed by Theorem 2. Next, let us choose $Z \in F_1 \cap F_2$ so that $Y \subseteq Z$. Recall that this is possible because, by Proposition 1, $F_1 \cap F_2$ is a rich family. Since $Z$ is a separable $L_1$-predual we have by Lemma 3 that $e_k^*|Z \in \text{Exp}(B_{Z^*})$ for each $1 \leq k \leq m$. Now, by Proposition 2, for each $1 \leq k \leq m$ there exists a $b_k^* \in B$ such that $e_k^*|Z = b_k^*|Z$. Therefore,

$$|b_k^*(x) - b_k^*(x_j)| = |(b_k^*|Z)(x) - (b_k^*|Z)(x_j)|$$

$$= |(e_k^*|Z)(x) - (e_k^*|Z)(x_j)| = |e_k^*(x) - e_k^*(x_j)|$$

for all $j \in \mathbb{N}$ and all $1 \leq k \leq m$. Thus,

$$\{ y \in X : |b_k^*(x) - b_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m \} \cap \{ x_n : n \in \mathbb{N} \} = \emptyset.$$ 

This contradicts the fact that $x \in \overline{\{ x_n : n \in \mathbb{N} \}}^{(X,B)}$, which completes the proof.

**Corollary 1** ([13, Theorem 1.1(a)]). Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual. Then every relatively countably $\sigma(X,B)$-compact subset is relatively countably $\sigma(X,\text{Ext}(B_{X^*}))$-compact. In particular, every norm bounded, relatively countably $\sigma(X,B)$-compact subset is relatively weakly compact.

**Proof.** Suppose that a nonempty set $C \subseteq X$ is relatively countably $\sigma(X,B)$-compact. Let $\{c_n : n \in \mathbb{N}\}$ be any sequence in $C$. Then by Theorem 3

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} \overline{\{ c_k : k \geq n \}}^{(X,B)} \subseteq \bigcap_{n \in \mathbb{N}} \overline{\{ c_k : k \geq n \}}^{(X,\text{Ext}(B_{X^*}))}.$$ 

Hence $C$ is relatively countably $\sigma(X,\text{Ext}(B_{X^*}))$-compact. In the case when $C$ is also norm bounded the result follows from either [7] or [8].

Recall that a network for a topological space $X$ is a family $\mathcal{N}$ of subsets of $X$ such that for any point $x \in X$ and any open neighbourhood $U$ of $x$ there is an $N \in \mathcal{N}$ such that $x \in N \subseteq U$, and a topological space $X$ is said to be $\aleph_0$-monolithic if the closure of every countable set has a countable network.

The next corollary generalises [9, Theorem 2.2].

**Corollary 2.** Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $\{ x_n : n \in \mathbb{N} \} \subseteq X$. Then $\overline{\{ x_n : n \in \mathbb{N} \}}^{(X,B)}$ is norm separable whenever $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology. In particular, $\overline{\{ x_n : n \in \mathbb{N} \}}^{(X,B)}$ is norm separable whenever $\text{Ext}(B_{X^*})$ is weak$^*$ Lindelöf.

**Proof.** From Theorem 3, $\overline{\{ x_n : n \in \mathbb{N} \}}^{(X,B)} \subseteq \overline{\{ x_n : n \in \mathbb{N} \}}^{(X,\text{Ext}(B_{X^*}))}$. Since $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology $\overline{\{ x_n : n \in \mathbb{N} \}}^{(X,\text{Ext}(B_{X^*}))}$ has a countable network with respect to the $\sigma(X,\text{Ext}(B_{X^*}))$ topology and hence so does $\overline{\{ x_n : n \in \mathbb{N} \}}^{(X,B)}$. Let $\mathcal{A}(B_{X^*})$ denote the set of all weak$^*$ continuous real-valued affine mappings on $B_{X^*}$. Then the mapping $T : (X, \| \cdot \|) \to (\mathcal{A}(B_{X^*}), \| \cdot \|_{\infty})$ defined by $T(x)(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$ is a homeomorphic embedding with respect to both (i) the norm topologies on $X$ and $\mathcal{A}(B_{X^*})$ and (ii) the $\sigma(X,\text{Ext}(X^*))$ and $\tau_{\text{pol}}(\text{Ext}(B_{X^*}))$ topologies on $X$ and $\mathcal{A}(B_{X^*})$ respectively. The result then follows from [9, Theorem 2.6]. The last claim follows from [9, Theorem 2.14], where
it is shown that if $\text{Ext}(B_{X^*})$ is weak$^*$ Lindelöf, then $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology. □

In [9] many conditions are given under which $\sigma(X,\text{Ext}(B_{X^*}))$ is $\aleph_0$-monolithic.

To demonstrate how this last theorem may be applied we shall present some sample applications.

4. Applications

Our first application is to metrizability of compact convex sets. If $K$ is a compact convex set in a real locally convex space, let $\mathcal{A}(K)$ stand for the space of all affine continuous functions on $K$.

**Proposition 3.** Let $K$ be a Choquet simplex in a separated locally convex space (over $\mathbb{R}$) such that every regular Borel probability measure carried on $\text{Ext}(K)$ is atomic. Then $K$ is metrizable if, and only if, the space $(B_{\mathcal{A}(K)},\sigma(\mathcal{A}(K),B))$ is separable, for some boundary $B$ of $(\mathcal{A}(K),\|\cdot\|_\infty)$.

**Proof.** This follows directly from Theorem 3 and [9, Theorem 2.19]. □

We remark that there exists a nonmetrizable Choquet simplex $K$ and a boundary $B$ of $(\mathcal{A}(K),\|\cdot\|_\infty)$ such that $(B_{\mathcal{A}(K)},\sigma(\mathcal{A}(K),B))$ is separable. (It is shown in [13, Section 4] that the construction of [11, Example 2.10] yields the required example.)

Our final few results concern automatic continuity. In particular, the next result improves [12, Theorem 6].

**Proposition 4.** Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $A$ is a separable Baire space. If $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology, then for each continuous mapping $f : A \to (X,\sigma(X,B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$.

**Proof.** Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup\{U \subseteq A : U \text{ is open and } \| \cdot \| -\text{diam}(f(U)) \leq 2\varepsilon\}.$$  

We shall show that $O_\varepsilon$ is dense in $A$. To this end, let $W$ be a nonempty open subset of $A$ and let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of $W$. Then by continuity

$$f(W) \subseteq \overline{\{f(a_n) : n \in \mathbb{N}\}}^{\sigma(X,B)},$$

which is norm separable by Corollary 2. Therefore there exists a countable set $\{x_n : n \in \mathbb{N}\}$ in $X$ such that $f(W) \subseteq \bigcup_{n \in \mathbb{N}}(x_n + \varepsilon B_X)$. For each $n \in \mathbb{N}$, let $C_n := f^{-1}(x_n + \varepsilon B_X)$. Since each $x_n + \varepsilon B_X$ is closed in the $\sigma(X,B)$ topology each set $C_n$ is closed in $A$ and, moreover, $W \subseteq \bigcup_{n \in \mathbb{N}}C_n$. Since $W$ is of the second Baire category in $A$ there exist a nonempty open set $U \subseteq W$ and a $k \in \mathbb{N}$ such that $U \subseteq C_k$. Then $U \subseteq O_\varepsilon \cap W$ and $O_\varepsilon$ is indeed dense in $A$. Hence $f$ is $\| \cdot \|$-continuous at each point of $\bigcap_{n \in \mathbb{N}}O_{1/n}$. □

**Theorem 4.** Suppose that $A$ is a topological space with countable tightness that possesses a rich family $\mathcal{F}$ of Baire subspaces and suppose that $X$ is an $L_1$-predual. Then for any boundary $B$ of $X$ and any continuous function $f : A \to (X,\sigma(X,B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$ provided $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology.
Proof. In order to obtain a contradiction let us suppose that \( f \) does not have a dense set of points of continuity with respect to the norm topology on \( X \). Since \( A \) is a Baire space (by \([7\) Theorem 3.3]), this implies that for some \( \varepsilon > 0 \) the open set
\[
O_\varepsilon := \bigcup \{ U \subseteq A : \text{\( U \) is open and } \| \cdot \|_{\text{diam}}[f(U)] \leq 2\varepsilon \}
\]
is not dense in \( A \). That is, there exists a nonempty open subset \( W \) of \( A \) such that \( W \cap O_\varepsilon = \emptyset \). For each \( x \in A \), let \( F_x := \{ y \in A : \| f(y) - f(x) \| > \varepsilon \} \). Then \( x \in F_x \) for each \( x \in W \). Moreover, since \( A \) has countable tightness, for each \( x \in W \), there exists a countable subset \( C_x \) of \( F_x \) such that \( x \in C_x \).

Next, we inductively define an increasing sequence of separable subspaces \( (F_n : n \in \mathbb{N}) \) of \( A \) and countable sets \( (D_n : n \in \mathbb{N}) \) in \( A \) such that:
\[
\begin{align*}
(1) & \quad W \cap F_1 \neq \emptyset; \\
(2) & \quad F_n := \bigcup \{ C_x : x \in D_n \cap W \} \cup F_n \subseteq F_{n+1} \in \mathcal{F} \text{ for all } n \in \mathbb{N}, \text{ where } D_n \text{ is any countable dense subset of } F_n.
\end{align*}
\]
Note that since the family \( \mathcal{F} \) is rich, this construction is possible.

Let \( F := \bigcup_{n \in \mathbb{N}} F_n \) and \( D := \bigcup_{n \in \mathbb{N}} D_n \). Then \( F \in \mathcal{F} \) and \( \| \cdot \|_{\text{diam}}[f(U)] \geq \varepsilon \) for every nonempty open subset \( U \) of \( F \cap W \). Therefore, \( f|_F \) has no points of continuity in \( F \cap W \) with respect to the \( \| \cdot \| \)-topology. This, however, contradicts Proposition 4.

Our final result improves \([7\ Theorem 4.7]).

Corollary 3. Suppose that \( A \) is a topological space with countable tightness that possesses a rich family of Baire subspaces and suppose that \( K \) is a compact Hausdorff space. Then for any boundary of \((C(K), \| \cdot \|_{\infty})\) and any continuous function \( f : A \to (C(K), \alpha(C(K), B)) \) there exists a dense subset \( D \) of \( A \) such that \( f \) is continuous with respect to the \( \| \cdot \|_{\infty} \)-topology at each point of \( D \).

References


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