ON THE TOPOLOGY OF POINTWISE CONVERGENCE
ON THE BOUNDARIES OF $L_1$-PREDUALS

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Abstract. In this paper we prove a theorem more general than the following:
"If $(X, \| \cdot \|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and \{ $x_n : n \in \mathbb{N}$ \} is any subset of $X$, then the closure of \{ $x_n : n \in \mathbb{N}$ \} with respect to the topology of pointwise convergence on $B$ is separable with respect to the topology generated by the norm, whenever $\text{Ext}(B_{X^*})$ is weak$^*$ Lindelöf." Several applications of this result are also presented.

1. Introduction

We shall say that a Banach space $(X, \| \cdot \|)$ is an $L_1$-predual if $X^*$ is isometric to $L_1(\mu)$ for some suitable measure $\mu$. Some examples of $L_1$-preduals include $(C(K), \| \cdot \|_{\infty})$, and more generally, the space of continuous affine functions on a Choquet simplex (see [10] for the definition) endowed with the supremum norm (see, [4, Proposition 3.23]). We shall also consider the notion of a boundary. Specifically, for a nontrivial Banach space $X$ over $\mathbb{R}$ we say that a subset $B$ of $B_{X^*}$, the closed unit ball of $X^*$, is a boundary if for each $x \in X$ there exists a $b^* \in B$ such that $b^*(x) = \|x\|$. The prototypical example of a boundary is $\text{Ext}(B_{X^*})$, the set of all extreme points of $B_{X^*}$, but there are many other interesting examples given in [9].

In the recent paper [9] the authors investigate the topology on a Banach space $X$ that is generated by $\text{Ext}(B_{X^*})$ and, more generally, the topology on $X$ generated by an arbitrary boundary of $X$. This paper continues this study.

To be more precise we must first introduce some notation. For a nonempty subset $Y$ of the dual of a Banach space $X$ we shall denote by $\sigma(X,Y)$ the weakest linear topology on $X$ that makes all the functionals from $Y$ continuous. In [9] the authors show (see, [9, Theorem 2.2]) using [3, Lemma 1] that for any compact Hausdorff space $K$, any countable subset $\{ x_n : n \in \mathbb{N} \}$ of $C(K)$ and any boundary $B$ of $(C(K), \| \cdot \|_{\infty})$, the closure of $\{ x_n : n \in \mathbb{N} \}$ with respect to the $\sigma(C(K),B)$ topology is separable with respect to the topology generated by the norm. In this paper we extend this result by showing that if $(X, \| \cdot \|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and $\{ x_n : n \in \mathbb{N} \}$ is any subset of $X$, then the closure of $\{ x_n : n \in \mathbb{N} \}$
in the \(\sigma(X, B)\) topology is separable with respect to the topology generated by the norm whenever \(\text{Ext}(B_X^*)\) is weak* Lindelöf.

We conclude this paper with some applications that indicate the utility of our results.

2. Preliminary results

Let \(X\) be a topological space and let \(F\) be a family of nonempty, closed and separable subsets of \(X\). Then \(F\) is rich if the following two conditions are fulfilled:

(i) for every separable subspace \(Y\) of \(X\), there exist \(Z\in F\) such that \(Y\subseteq Z\);
(ii) for every increasing sequence \((Z_n : n \in \mathbb{N})\) in \(F\), \(\bigcup_{n \in \mathbb{N}} Z_n \in F\).

For any topological space \(X\), the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, namely,

\[ G_X := \{ S \subseteq X : S \text{ is a nonempty, closed and separable subset of } X \} \]

On the other hand, if \(X\) is a separable space, then the partially ordered set has a least element, namely, \(G_{\emptyset} := \{X\}\).

The raison d'être for rich families is revealed next.

Proposition 1. Suppose that \(X\) is a topological space. If \(\{F_n : n \in \mathbb{N}\}\) are rich families of \(X\), then so is \(\bigcap_{n \in \mathbb{N}} F_n\).

For a proof of this proposition see [2, Proposition 1.1].

Throughout this paper we will be primarily working with Banach spaces, so a natural class of rich families, given a Banach space \(X\), is the family of all closed separable linear subspaces of \(X\), which we denote by \(S_X\). There are however many other interesting examples of rich families that can be found in [2] and [7].

For our first result we will provide another nontrivial example of a rich family, but to achieve this we first need a preliminary result that characterises when a given Banach space is an \(L_1\)-predual.

Lemma 1 ([6, §21, Theorem 7]). For a Banach space \(X\) the following are equivalent:

(i) \(X\) is an \(L_1\)-predual;
(ii) for each weak* continuous convex function \(f\) on \(B_X^*\),

\[ f^*(0) = \frac{1}{2} \max \{ f(x) + f(-x) : x \in B_X^* \}, \]

where \(f^* = \inf \{ h : h \geq f \text{ and } h \text{ is weak* continuous and affine on } B_X^* \} \).

Before proceeding further we shall introduce the following notation. If \(X\) is a normed linear space, then each \(x \in X\) defines a weak* continuous affine function \(\hat{x}\) on \(B_X^*\) via the canonical embedding, that is, \(\hat{x}(x^*) := x^*(x)\) for all \(x^* \in B_X^*\).

Theorem 1. Let \(X\) be an \(L_1\)-predual. Then the set of all closed separable linear subspaces of \(X\) that are themselves \(L_1\)-preduals forms a rich family.

Proof. Let \(\mathcal{L} := \{ Z \in S_X : Z \text{ is an } L_1\text{-predual} \}\). We shall verify that \(\mathcal{L}\) is a rich family. So first let us consider an arbitrary separable closed linear subspace \(Y\) of \(X\). Then by [3, §23, Lemma 1] there exists a closed separable subspace \(Z \in \mathcal{L}\) such that \(Y \subseteq Z\). Next, let us consider an increasing sequence \((Z_n : n \in \mathbb{N})\) in \(\mathcal{L}\).
and let \( Z := \bigcup_{n \in \mathbb{N}} Z_n \). To show that \( Z \in \mathcal{Z} \) we shall appeal to Lemma 2. Let \( f \) be a weak* continuous convex function on \( B_{Z^*} \). Since
\[
\frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} \leq f^*(0),
\]
it is enough to verify that for each \( \varepsilon > 0 \), \( f^*(0) \leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} + \varepsilon \). To this end, suppose that \( \varepsilon > 0 \). Since \( f \) is weak* continuous and convex and \( B_{Z^*} \) is weak* compact, by Corollary I.1.3 there exist \( z_i \in Z \) and \( c_i \in \mathbb{R} \), \( i = 1, \ldots, n \), such that the weak* convex continuous \( g : B_{Z^*} \to \mathbb{R} \) defined by
\[
g := \max \{ \hat{z}_1 + c_1, \hat{z}_2 + c_2, \ldots, \hat{z}_n + c_n \}
\]
satisfies
\[
f(z^*) - \varepsilon < g(z^*) < f(z^*), \quad z^* \in B_{Z^*}.
\]
Since \( \bigcup_{n \in \mathbb{N}} Z_n \) is dense in \( Z \) we may further assume that all the elements \( z_i \) are contained in some fixed \( Z_j \), \( j \in \mathbb{N} \).

Next, let \( r : B_{Z^*} \to B_{Z_j^*} \) be the restriction mapping (i.e., \( r(z^*) = z^*|_{Z_j} \) for all \( z^* \in B_{Z^*} \)) and let \( h : B_{Z_j^*} \to \mathbb{R} \) be defined by \( h := \max \{ \hat{z}_1 + c_1, \hat{z}_2 + c_2, \ldots, \hat{z}_n + c_n \} \). Then \( h \) is weak* continuous and convex on \( B_{Z_j^*} \) and \( g = h \circ r \). Moreover, by the definition of \( g \) (and the fact that \( r \) is weak*-to-weak* continuous and linear) we have that \( g^*(z^*) \leq h^*(r(z^*)) \) for all \( z^* \in B_{Z^*} \). Now, by the assumption that \( Z_j \) is an \( L_1 \)-predual (and Lemma 1) there exists a \( y^* \in B_{Z_j^*} \) such that
\[
h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)].
\]
Choose \( z^* \in r^{-1}(y^*) \), which is nonempty by the Hahn-Banach extension theorem. Then,
\[
g^*(0) \leq h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)] = \frac{1}{2}[g(z^*) + g(-z^*)] \leq g^*(0).
\]
Therefore,
\[
f^*(0) - \varepsilon = (f - \varepsilon)^*(0) \leq g^*(0) = \frac{1}{2}[g(z^*) + g(-z^*)]
\leq \frac{1}{2}[f(z^*) + f(-z^*)]
\leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \}.
\]
That is, \( f^*(0) \leq \frac{1}{2} \max \{ f(x^*) + f(-x^*) : x^* \in B_{Z^*} \} + \varepsilon \), which completes the proof. \( \square \)

Before we can introduce another class of rich families, we require the following lemma, which is a Banach space version of Theorem 2.10.

**Lemma 2.** Let \( Y \) be a closed separable linear subspace of a Banach space \( X \) and suppose that \( L \subseteq \text{Ext}(B_{X^*}) \) is weak* Lindelöf. Then there exists a closed separable linear subspace \( Z \) of \( X \), containing \( Y \), such that for any \( l^* \in L \) and any \( x^*, y^* \in B_{Z^*} \), if \( l^*|_Z = \frac{1}{2}(x^* + y^*) \), then \( x^*|_Y = y^*|_Y \).

**Proof.** Let \( \mathcal{B} \) be a countable base for the topology on \( (B_Y, \text{weak}^*) \) consisting of closed convex sets. Recall that such a base exists because \( (B_Y, \text{weak}^*) \) is compact, by the Banach-Alaoglu Theorem, and \( (B_Y, \text{weak}^*) \) is metrizable, since \( Y \) is separable. Let:
We now claim that if \( l^* \in L \) and \( f^* \in F \) then \( \sup \{ \tilde{g}(f^*) : f^* \in F \} < \tilde{g}(l^*) \).

Therefore, since \( L \) is weak* Lindelöf for each \( F \in \mathcal{A} \) there exists a countable subset \( C_F \) in \( X \) such that, for each \( l^* \in L \) there exists a \( y \in C_F \) such that \( \sup \{ \tilde{g}(f^*) : f^* \in F \} < \tilde{g}(l^*) \). If we set \( \mathcal{C} := \bigcup \{ C_F : F \in \mathcal{A} \} \) and \( Z := \overline{\text{span}}(C \cup X) \), then \( X \subseteq Z \) and \( Z \) is a closed separable linear subspace of \( X \).

It now only remains to verify that if \( l^* \in L \), \( x^* \), \( y^* \in B_{Z^*} \), and \( l^*|_Z = \frac{1}{2}(x^* + y^*) \), then \( x^*|_Y = y^*|_Y \). So, in order to obtain a contradiction, suppose that for some \( l^* \in L \) and \( x^*, y^* \in B_{Z^*} \), \( l^*|_Z = \frac{1}{2}(x^* + y^*) \) but \( x^*|_Y \neq y^*|_Y \). Then there exists \( B_1, B_2 \in \mathcal{A} \) such that \( x^*|_Y \in B_1 \) and \( y^*|_Y \in B_2 \) and \( B_1 \cap B_2 = \emptyset \). Set \( F_1 := r^{-1}(B_1) \) and \( F_2 := r^{-1}(B_2) \). Then \( F_1, F_2 \in \mathcal{A} \) and \( F_1 \cap F_2 = \emptyset \). Now, by the Hahn-Banach Extension Theorem there exist \( x^*_1 \in B_{X^*} \) and \( y^*_1 \in B_{X^*} \) such that \( x^*_1|_Z = x^* \) and \( y^*_1|_Z = y^* \). Moreover,

\[
x^*_1|_Y = (x^*_1|_Z)|_Y = x^*|_Y \in B_1 \quad \text{ and } \quad y^*_1|_Y = (y^*_1|_Z)|_Y = y^*|_Y \in B_2.
\]

That is, \( x^*_1 \in F_1 \) and \( y^*_1 \in F_2 \). Therefore, \( \frac{1}{2}(x^*_1 + y^*_1) \in \frac{1}{2}(F_1 + F_2) =: F \). Since \( F \in \mathcal{A} \), by the construction there exists a \( y \in C_F \subseteq C \subseteq Z \) such that \( \sup \{ \tilde{g}(f^*) : f^* \in F \} < \tilde{g}(l^*) \). In particular,

\[
\frac{1}{2}(x^* + y^*)(y) = \tilde{g}\left(\frac{1}{2}(x^*_1 + y^*_1)\right) < l^*(y) = (l^*|_Z)(y).
\]

However, this contradicts the fact that \( \frac{1}{2}(x^* + y^*) = l^*|_Z \). \( \square \)

**Theorem 2.** Let \( X \) be a Banach space and let \( L \subseteq \text{Ext}(B_{X^*}) \) be a weak* Lindelöf subset. Then the set of all \( Z \) in \( S_X \) such that \( \{ l^*|_Z : l^* \in L \} \subseteq \text{Ext}(B_{Z^*}) \) forms a rich family.

**Proof.** Let \( \mathcal{L} \) denote the family of all closed separable linear subspaces \( Z \) of \( X \) such that \( \{ l^*|_Z : l^* \in L \} \subseteq \text{Ext}(B_{Z^*}) \). We shall verify that \( \mathcal{L} \) is a rich family of closed separable linear subspaces of \( X \). So first let us consider an arbitrary closed separable linear subspace \( Y \) of \( X \), with the aim of showing that there exists a subspace \( Z \in \mathcal{L} \) such that \( Y \subseteq Z \). We begin by inductively applying Lemma 2 to obtain an increasing sequence \( \{ Z_n : n \in \mathbb{N} \} \) of closed separable linear subspaces of \( X \) such that \( Y \subseteq Z_1 \) and for any \( l^* \in L \) and any \( x^*, y^* \in B_{Z_{n+1}^*} \), if \( l^*|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*) \), then \( x^*|_{Z_n} = y^*|_{Z_n} \).

We now claim that if \( Z := \bigcup_{n \in \mathbb{N}} Z_n \), then \( l^*|_Z \in \text{Ext}(B_{Z^*}) \) for each \( l^* \in L \). To this end, suppose that \( l^* \in L \) and \( l^*|_Z = \frac{1}{2}(x^* + y^*) \) for some \( x^*, y^* \in B_{Z^*} \). Then for each \( n \in \mathbb{N} \),

\[
l^*|_{Z_{n+1}} = (l^*|_Z)|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)|_{Z_{n+1}} = \frac{1}{2}(x^*|_{Z_{n+1}} + y^*|_{Z_{n+1}})
\]

and \( x^*|_{Z_{n+1}}, y^*|_{Z_{n+1}} \in B_{Z_{n+1}^*} \). Therefore, by construction, \( x^*|_{Z_n} = y^*|_{Z_n} \). Now since \( \bigcup_{n \in \mathbb{N}} Z_n \) is dense in \( Z \) and both \( x^* \) and \( y^* \) are continuous, we may deduce that \( x^* = y^* \), which in turn implies that \( l^*|_Z \in \text{Ext}(B_{Z^*}) \). This shows that \( Y \subseteq Z \) and \( Z \in \mathcal{L} \).
To complete this proof we must verify that for each increasing sequence of closed separable subspaces \( \{Z_n : n \in \mathbb{N}\} \subseteq \mathcal{L} \), \( \bigcup_{n \in \mathbb{N}} Z_n \subseteq \mathcal{L} \). This, however, follows easily from the definition of the family \( \mathcal{L} \).

Let \( X \) be a normed linear space. Then we say that an element \( x^* \in B_X \) is weak\(^*\) exposed if there exists an element \( x \in X \) such that \( y^*(x) < x^*(x) \) for all \( y^* \in B_X \setminus \{x^*\} \). It is not difficult to show that if \( \text{Exp}(B_X) \) denotes the set of all weak\(^*\) exposed points of \( B_X \), then \( \text{Exp}(B_X) \subseteq \text{Ext}(B_X^*) \). However, if \( X \) is a separable \( L_1\)-predual, then the relationship between \( \text{Exp}(B_X^*) \) and \( \text{Ext}(B_X^*) \) is much closer.

**Lemma 3** ([13] Lemma 3.3(b)). If \( X \) is a separable \( L_1\)-predual, then \( \text{Exp}(B_X^*) = \text{Ext}(B_X^*) \).

Let us also pause for a moment to recall that if \( B \) is any boundary of a Banach space \( X \), then

\[
\text{Exp}(B_X^*) \subseteq B \cap \text{Ext}(B_X^*) \subseteq \text{Ext}(B_X^*) \subseteq B^{\text{weak}^*}.
\]

The fact that \( \text{Ext}(B_X^*) \subseteq B^{\text{weak}^*} \) follows from Milman’s theorem, [10] page 8 and the fact that \( B_X^* = \overline{\text{Exp}(B)} \), which in turn follows from a separation argument. Let us also take this opportunity to observe that if \( B_X \) denotes the closed unit ball in \( X \), then \( B_X \) is closed in the \( \varepsilon(X,B) \) topology for any boundary \( B \) of \( X \). Finally, let us end this section with one more simple observation that will turn out to be useful in our later endeavours.

**Proposition 2.** Suppose that \( Y \) is a linear subspace of a Banach space \( (X, \| \cdot \|) \) and \( B \) is any boundary for \( X \). Then for each \( e^* \in \text{Exp}(B_Y^*) \) there exists \( b^* \in B \) such that \( e^* = b^*|_Y \).

**Proof.** Suppose that \( e^* \in \text{Exp}(B_Y^*) \). Then there exists an \( x \in Y \) such that \( y^*(x) < e^*(x) \) for each \( y^* \in B_Y^* \setminus \{e^*\} \). By the fact that \( B \) is a boundary of \( (X, \| \cdot \|) \) there exists a \( b^* \in B \) such that \( b^*(x) = \|x\| \neq 0 \). Then for any \( y^* \in B_Y^* \) we have

\[
y^*(x) = |y^*(x)| = \|y^*\| |x| \leq \|x\| = b^*(x) = (b^*)_Y(x).
\]

In particular, \( e^*(x) \leq (b^*)_Y(x) \). Since \( b^*_Y \in B_Y^* \) and \( y^*(x) < e^*(x) \) for all \( y^* \in B_Y^* \setminus \{e^*\} \), it must be the case that \( e^* = (b^*)_Y \).

This ends our preliminary section.

### 3. The main results

**Theorem 3.** Let \( B \) be any boundary for a Banach space \( X \) that is an \( L_1\)-predual and suppose that \( \{x_n : n \in \mathbb{N}\} \subseteq X \). Then

\[
\overline{\{x_n : n \in \mathbb{N}\}^{(X,B)}} \subseteq \{x_n : n \in \mathbb{N}\}^{(X,\text{Ext}(B_X^*)}.
\]

**Proof.** In order to obtain a contradiction let us suppose that

\[
\overline{\{x_n : n \in \mathbb{N}\}^{(X,B)}} \not\subseteq \{x_n : n \in \mathbb{N}\}^{(X,\text{Ext}(B_X^*)}.
\]

Choose \( x \in \overline{\{x_n : n \in \mathbb{N}\}^{(X,B)}} \setminus \{x_n : n \in \mathbb{N}\}^{(X,\text{Ext}(B_X^*)} \). Then there exists a finite set \( \{e^*_1, e^*_2, \ldots, e^*_m\} \subseteq \text{Ext}(B_X^*) \) and \( \varepsilon > 0 \) so that

\[
\{y \in X : |e^*_k(x) - e^*_k(y)| < \varepsilon \text{ for all } 1 \leq k \leq m \} \cap \{x_n : n \in \mathbb{N}\} = \emptyset.
\]
Let $Y := \text{span}\{x_n : n \in \mathbb{N}\} \cup \{x\}$, let $F_1$ be any rich family of $L_1$-preduals whose existence is guaranteed by Theorem 1, and let $F_2$ be any rich family such that for every $Z \in F_2$ and every $1 \leq k \leq m$, $e_k^*|_Z \in \text{Ext}(B_{Z^*})$, whose existence is guaranteed by Theorem 2. Next, let us choose $Z \in F_1 \cap F_2$ so that $Y \subseteq Z$. Recall that this is possible because, by Proposition 1, $F_1$ and $F_2$ are rich families. Since $Z$ is a separable $L_1$-predual we have by Lemma 3 that $e_k^*|_Z \in \text{Exp}(B_{Z^*})$ for each $1 \leq k \leq m$. Now, by Proposition 2, for each $1 \leq k \leq m$ there exists a $b_k^* \in B$ such that $e_k^*|_Z = b_k^*|_Z$. Therefore,

$$|b_k^*(x) - b_k^*(x_j)| = |(b_k^*|_Z)(x) - (b_k^*|_Z)(x_j)|$$

$$= |(e_k^*|_Z)(x) - (e_k^*|_Z)(x_j)| = |e_k^*(x) - e_k^*(x_j)|$$

for all $j \in \mathbb{N}$ and all $1 \leq k \leq m$. Thus,

$$\{y \in X : |b_k^*(x) - b_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m\} \cap \{x_n : n \in \mathbb{N}\} = \emptyset.$$ 

This contradicts the fact that $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{(X,B)}$, which completes the proof. 

\[ \square \]

**Corollary 1** ([13, Theorem 1.1(a)]). Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual. Then every relatively countably $\sigma(X,B)$-compact subset is relatively countably $\sigma(X, \text{Ext}(B_{X^*}))$-compact. In particular, every norm bounded, relatively countably $\sigma(X,B)$-compact subset is relatively weakly compact.

**Proof.** Suppose that a nonempty set $C \subseteq X$ is relatively countably $\sigma(X,B)$-compact. Let $\{c_n : n \in \mathbb{N}\}$ be any sequence in $C$. Then by Theorem 3

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} \overline{\{c_k : k \geq n\}}^{(X,B)} \subseteq \bigcap_{n \in \mathbb{N}} \overline{\{c_k : k \geq n\}}^{(X, \text{Ext}(B_{X^*}))}.$$ 

Hence $C$ is relatively countably $\sigma(X, \text{Ext}(B_{X^*}))$-compact. In the case when $C$ is also norm bounded the result follows from either [7] or [8]. \[ \square \]

Recall that a *network* for a topological space $X$ is a family $\mathcal{N}$ of subsets of $X$ such that for any point $x \in X$ and any open neighbourhood $U$ of $x$ there is an $N \in \mathcal{N}$ such that $x \in N \subseteq U$, and a topological space $X$ is said to be $\aleph_0$-monolithic if the closure of every countable set has a countable network.

The next corollary generalises [9, Theorem 2.2].

**Corollary 2.** Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$. Then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $X$ is $\aleph_0$-monolithic in the $\sigma(X, \text{Ext}(B_{X^*}))$ topology. In particular, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $\text{Ext}(B_{X^*})$ is weak$^*$ Lindelöf.

**Proof.** From Theorem 3, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X, \text{Ext}(B_{X^*}))}$ since $X$ is $\aleph_0$-monolithic in the $\sigma(X, \text{Ext}(B_{X^*}))$ topology $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X, \text{Ext}(B_{X^*}))}$ has a countable network with respect to the $\sigma(X, \text{Ext}(B_{X^*}))$ topology and hence so does $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$. Let $A(B_{X^*})$ denote the set of all weak$^*$ continuous real-valued affine mappings on $B_{X^*}$. Then the mapping $T : (X, ||\cdot||) \rightarrow (A(B_{X^*}), \|\cdot\|_{\infty})$ defined by $T(x)(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$ is a homeomorphic embedding with respect to both (i) the norm topologies on $X$ and $A(B_{X^*})$ and (ii) the $\sigma(X, \text{Ext}(X^*))$ and $\tau_\text{p}(\text{Ext}(B_{X^*}))$ topologies on $X$ and $A(B_{X^*})$ respectively. The result then follows from [9, Theorem 2.6]. The last claim follows from [9, Theorem 2.14], where
it is shown that if $\text{Ext}(B_{X^*})$ is weak* Lindelöf, then $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology. \hfill \Box

In [9] many conditions are given under which $\sigma(X,\text{Ext}(B_{X^*}))$ is $\aleph_0$-monolithic.

To demonstrate how this last theorem may be applied we shall present some sample applications.

4. Applications

Our first application is to metrizability of compact convex sets. If $K$ is a compact convex set in a real locally convex space, let $\mathcal{A}(K)$ stand for the space of all affine continuous functions on $K$.

Proposition 3. Let $K$ be a Choquet simplex in a separated locally convex space (over $\mathbb{R}$) such that every regular Borel probability measure carried on $\text{Ext}(K)$ is atomic. Then $K$ is metrizable if, and only if, the space $(B_{\mathcal{A}(K)},\sigma(\mathcal{A}(K),B))$ is separable, for some boundary $B$ of $(\mathcal{A}(K),\|\cdot\|_\infty)$.

Proof. This follows directly from Theorem 3 and [9 Theorem 2.19]. \hfill \Box

We remark that there exists a nonmetrizable Choquet simplex $K$ and a boundary $B$ of $(\mathcal{A}(K),\|\cdot\|_\infty)$ such that $(B_{\mathcal{A}(K)},\sigma(\mathcal{A}(K),B))$ is separable. (It is shown in [13] Section 4 that the construction of [31 Example 2.10] yields the required example.)

Our final few results concern automatic continuity. In particular, the next result improves [12 Theorem 6].

Proposition 4. Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $A$ is a separable Baire space. If $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology, then for each continuous mapping $f : A \to (X,\sigma(X,B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$.

Proof. Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup\{U \subseteq A : U \text{ is open and } \|f\|_{-}\text{diam}[f(U)] \leq 2\varepsilon\}.$$ 

We shall show that $O_\varepsilon$ is dense in $A$. To this end, let $W$ be a nonempty open subset of $A$ and let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of $W$. Then by continuity

$$f(W) \subseteq \overline{\{f(a_n) : n \in \mathbb{N}\}}^{\sigma(X,B)},$$

which is norm separable by Corollary 2. Therefore there exists a countable set $\{x_n : n \in \mathbb{N}\}$ in $X$ such that $f(W) \subseteq \bigcup_{n \in \mathbb{N}}(x_n + \varepsilon B_X)$. For each $n \in \mathbb{N}$, let $C_n := f^{-1}(x_n + \varepsilon B_X)$. Since each $x_n + \varepsilon B_X$ is closed in the $\sigma(X,B)$ topology each set $C_n$ is closed in $A$ and, moreover, $W \subseteq \bigcup_{n \in \mathbb{N}}C_n$. Since $W$ is of the second Baire category in $A$ there exist a nonempty open set $U \subseteq W$ and a $k \in \mathbb{N}$ such that $U \subseteq C_k$. Then $U \subseteq O_\varepsilon \cap W$ and $O_\varepsilon$ is indeed dense in $A$. Hence $f$ is $\|\cdot\|_{-}$-continuous at each point of $\bigcap_{n \in \mathbb{N}}O_{1/n}$. \hfill \Box

Theorem 4. Suppose that $A$ is a topological space with countable tightness that possesses a rich family $\mathcal{F}$ of Baire subspaces and suppose that $X$ is an $L_1$-predual. Then for any boundary $B$ of $X$ and any continuous function $f : A \to (X,\sigma(X,B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$ provided $X$ is $\aleph_0$-monolithic in the $\sigma(X,\text{Ext}(B_{X^*}))$ topology.
Proof. In order to obtain a contradiction let us suppose that \( f \) does not have a dense set of points of continuity with respect to the norm topology on \( X \). Since \( A \) is a Baire space (by \([7]\) Theorem 3.3)), this implies that for some \( \varepsilon > 0 \) the open set
\[
O_\varepsilon := \bigcup \{ U \subseteq A : U \text{ is open and } \| \cdot \| \cdot \text{-diam}[f(U)] \leq 2\varepsilon \}
\]
is not dense in \( A \). That is, there exists a nonempty open subset \( W \) of \( A \) such that \( W \cap O_\varepsilon = \emptyset \). For each \( x \in A \), let \( F_x := \{ y \in A : \| f(y) - f(x) \| > \varepsilon \} \). Then \( x \in F_x \) for each \( x \in W \). Moreover, since \( A \) has countable tightness, for each \( x \in W \), there exists a countable subset \( C_x \) of \( F_x \) such that \( x \in \overline{C_x} \).

Next, we inductively define an increasing sequence of separable subspaces \( (F_n : n \in \mathbb{N}) \) of \( A \) and countable sets \( (D_n : n \in \mathbb{N}) \) in \( A \) such that:

(i) \( W \cap F_1 \neq \emptyset \);
(ii) \( \bigcup\{ C_x : x \in D_n \cap W \} \cup F_n \subseteq F_{n+1} \in \mathcal{F} \) for all \( n \in \mathbb{N} \), where \( D_n \) is any countable dense subset of \( F_n \).

Note that since the family \( \mathcal{F} \) is rich, this construction is possible.

Let \( F := \bigcup_{n \in \mathbb{N}} F_n \) and \( D := \bigcup_{n \in \mathbb{N}} D_n \). Then \( \overline{F} = F \in \mathcal{F} \) and \( \| \cdot \| \cdot \text{-diam}[f(U)] \geq \varepsilon \) for every nonempty open subset \( U \) of \( F \cap W \). Therefore, \( f|_F \) has no points of continuity in \( F \cap W \) with respect to the \( \| \cdot \| \cdot \text{-topology} \). This, however, contradicts Proposition 4. \( \square \)

Our final result improves \([7]\) Theorem 4.7).

Corollary 3. Suppose that \( A \) is a topological space with countable tightness that possesses a rich family of Baire subspaces and suppose that \( K \) is a compact Hausdorff space. Then for any boundary of \( (C(K), \| \cdot \|_\infty) \) and any continuous function \( f : A \to (C(K), \sigma(C(K), B)) \) there exists a dense subset \( D \) of \( A \) such that \( f \) is continuous with respect to the \( \| \cdot \|_\infty \)-topology at each point of \( D \).

References


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