RATIONAL HOMOTOPY OF GAUGE GROUPS

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Abstract. In this brief paper, we observe that basic results from rational homotopy theory provide formulas for the rational homotopy groups of gauge groups of principal bundles $K \to P \to B$ in terms of the rational homotopy groups of $K$ and cohomology groups of $B$ alone.

1. Introduction

Let $K \to P \xrightarrow{\xi} B$ be a continuous principal $K$-bundle, where $K$ is a compact connected Lie group. Denote by $G(\xi)$ the gauge group of $\xi$: that is, the set of all $K$-equivariant self-homeomorphisms of $P$ over $B$. Also, denote by $G_1(\xi)$ the subgroup of $G(\xi)$ consisting of the self-homeomorphisms that preserve the basepoint of $P$. (It is common in the subject to take for $G_1(\xi)$ the self-homeomorphisms that fix a given fibre $K$, but because $G(\xi)$ consists of equivariant maps, this is equivalent to our definition above.) The topology of gauge groups has been considered by many authors; see, for instance, [5], [2] or [11]. Indeed, the study of the homotopy theory of gauge groups (under a different name) goes back to [5].

Now, there is an obvious homeomorphism of groups $G_1(\xi) \cong \text{Map}_*(P, K)_K$, where the subscript $K$ on the mapping space denotes the space of equivariant maps with respect to the free (principal) action of $K$ on $P$ and the conjugation action of $K$ on $K$. In [16], S. Terzić shows that when $B$ is a closed simply connected 4-manifold, there is a formula for the ranks of the homotopy groups $\pi_j(G(\xi))$ and $\pi_j(G_1(\xi))$ in terms of the ranks of the homotopy of $K$ and homology of $B$ alone (see Corollary 3.3).

Also, when $K$ is abelian, we clearly have $G_1(\xi) = \text{Map}_*(P, K)_K = \text{Map}_*(B, K)$. For dim($B$) $\leq 4$, it was shown in [12] that even when $K$ is non-abelian, there is a weak equivalence between $G_1(\xi)$ and $\text{Map}_*(B, K)$. In certain cases, this result can be extended beyond these cases (see Corollary 2.2). Indeed, more recently, in [18, 19], C. Wockel shows that for principal bundles $K \to P \xrightarrow{\xi} S^m$, there is an identification of homotopy types $\text{Map}_*(P, K)_K \simeq \text{Map}_*(S^m, K)$ (and a consequent isomorphism $\pi_q(G_1(\xi)) = \pi_{q+m}(K)$ over $\mathbb{Z}$).

The purpose of this short paper is simply to observe that the philosophy of the preceding paragraph, in the framework of rational homotopy theory, allows the derivation of a general formula (see Theorem 3.1) for the rational homotopy
groups of the gauge groups $G(\xi)$ and $G_1(\xi)$ which, in particular, recovers Wockel’s isomorphism (over the rationals) when the base is a sphere and specializes to the formula of Terzić when the base is a 4-manifold. Because these results are of interest to non-topologists, we have tried to include as many details as possible within the confines of a desire for conciseness.

2. Homotopy type of gauge groups

One basic result of the theory of gauge groups says the following.

**Theorem 2.1.** For a principal bundle $K \to P \xrightarrow{\xi} B$ with classifying map $f: B \to B_K$,

$$G(\xi) = \text{Map}(P, K)_K \simeq \Omega \text{Map}(B, B_K; f)$$

and

$$G_1(\xi) = \text{Map}_*(P, K)_K \simeq \Omega \text{Map}_*(B, B_K; f).$$

This theorem may be proved in several ways. For instance, it was shown in [5, Theorems 5.2 and 5.6, Proposition 4.3] that $G(\xi)$ and $G_1(\xi)$ are the fibres in fibrations with the mapping spaces $\text{Map}(B, B_K; f)$ and $\text{Map}_*(B, B_K; f)$, respectively, as base spaces and with (essentially) contractible total spaces. Also see [2, Theorem 3.3 and Corollary 5.7] and [11, Chapter 2].

In general, if $W$ has the homotopy type of an $H$-space (for instance, a topological group or a loop space), then all the components of $\text{Map}(Z, W)$ have the same homotopy type, because multiplication with $f$ provides an equivalence $\text{Map}(Z, W; *) \cong \text{Map}(Z, W; f)$. Furthermore, if $Z = \Sigma X$ is a suspension (or, more generally, if $Z$ is an associative co-H-space), then all components of $\text{Map}_*(Z, W) = \text{Map}_*(\Sigma X, W) \simeq \text{Map}_*(X, \Omega W; f)$. Therefore, under these types of conditions, we have equivalences $\text{Map}_*(Z, W; f) \simeq \text{Map}_*(Z, W; *)$. Furthermore, we also have the general equality $\Omega \text{Map}_*(Z, W; f) \simeq \Omega \text{Map}_*(Z, W; *) = \text{Map}_*(Z, \Omega W)$. Thus, using the fact that $\Omega B_K \simeq K$, we obtain

**Corollary 2.2 ([11, Theorem 2.2.4]).** If all the components of $\text{Map}_*(B, B_K)$ have the same homotopy type, then

$$G_1(\xi) \simeq \text{Map}_*(B, K).$$

The hypothesis of Corollary 2.2 is not always satisfied. For instance, in [8] A. Kono constructs principal $SU(2)$-fibrations $\xi$ and $\xi'$ over $S^4$ with $G(\xi) \neq G(\xi')$. In this case, all the components of $\text{Map}(B, B_K)$ do not have the same homotopy type.

On the other hand, over the rational numbers we can generalize Corollary 2.2 to obtain

**Theorem 2.3.** When $B$ has the homotopy type of a connected finite CW complex, there are rational homotopy equivalences

$$G(\xi) \simeq_\mathbb{Q} \text{Map}(B, K) \quad \text{and} \quad G_1(\xi) \simeq_\mathbb{Q} \text{Map}_*(B, K).$$

**Proof.** The mapping space $\text{Map}(B, B_K; f)$ is a nilpotent space whose rationalization is the space $\text{Map}(B, (B_K)_\mathbb{Q}; f_\mathbb{Q})$ (see [8] Theorems II.2.5 and II.3.11) and [7] for more details on free mapping spaces). Moreover, we know that the rational
cohomology of $B_K$ is a polynomial algebra (see [4] Theorem 1.81 and Example 2.42 for example). More specifically, the cohomology of a compact connected Lie group is an exterior algebra $H^*(K; \mathbb{Q}) = \wedge(u_1, \ldots, u_r)$ with $u_i \in H^{2n_i-1}(K; \mathbb{Q})$, and the Serre spectral sequence for the universal bundle $K \rightarrow E_K \rightarrow B_K$ leads to $H^*(B_K; \mathbb{Q}) = \mathbb{Q}[v_1, \ldots, v_r]$, a polynomial algebra with $v_i \in H^{2n_i}(B_K; \mathbb{Q})$.

Since cohomology corresponds in general to homotopy classes of maps, we have $H^{2n_i}(B_K; \mathbb{Q}) = [B_k, K(\mathbb{Q}, 2n_i)]$, for each $i = 1, \ldots, r$. We then obtain a map

$$B_K \rightarrow \prod_{i=1}^r K(\mathbb{Q}, 2n_i)$$

which clearly induces an isomorphism on rational cohomology (as well as homology) and is, therefore, a rational equivalence. Hence, the rationalization $(B_K)_{\mathbb{Q}}$ is an $H$-space (since a product of $K(\mathbb{Q}, j)$’s clearly is), so $\text{Map}(B, (B_K)_{\mathbb{Q}}; f)$ has the homotopy type of $\text{Map}(B, (B_K)_{\mathbb{Q}}; *)$. Then

$$G(\xi) \simeq \Omega\text{Map}(B, B_K; f) \simeq_\mathbb{Q} \Omega\text{Map}(B, (B_K)_{\mathbb{Q}}; *) \simeq \text{Map}(B, K_{\mathbb{Q}}),$$

since $\Omega B_K \simeq K$.

The exact same argument applies to the based mapping space $\text{Map}_*(B, B_K; f)$ and $G_1(\xi)$.

Remark 2.4. The same argument as in the proof applied to the cohomology algebra $H^*(K; \mathbb{Q}) = \wedge(u_1, \ldots, u_r)$ of a compact connected Lie group $K$ shows that, rationally, $K$ is also a product of Eilenberg-Mac Lane spaces: that is,

$$K_{\mathbb{Q}} \simeq \prod_{i=1}^r K(\mathbb{Q}, 2n_i - 1).$$

3. Rational homotopy of gauge groups

We now use Theorem [2,3] to compute rational homotopy groups of gauge groups for any finite connected base space $B$. Recall that $K \rightarrow P \rightarrow B$ is a continuous principal bundle with $K$ a compact connected Lie group.

Theorem 3.1. If $B$ has the homotopy type of a finite connected CW complex, then for any $q \geq 1$, we have

$$\pi_q(G(\xi)) \otimes \mathbb{Q} \cong \sum_{r \geq 0} H^r(B; \mathbb{Q}) \otimes \pi_{r+q}(K)$$

and

$$\pi_q(G_1(\xi)) \otimes \mathbb{Q} \cong \sum_{r \geq 0} \tilde{H}^r(B; \mathbb{Q}) \otimes \pi_{r+q}(K),$$

where $\tilde{H}$ denotes reduced cohomology.

Remark 3.2. Note that since $H^r(B; \mathbb{Q})$ is a rational vector space, there is no need to write $\pi_{r+q}(K) \otimes \mathbb{Q}$. Also, recall that for path-connected $X$, the term reduced cohomology means that $\tilde{H}^j(X) = H^j(X)$ for $j \geq 1$ and $\tilde{H}^0(X) = 0$. Finally, there has been much work on the rational homotopy groups of mapping spaces. See, for example, [17] [10] [3] [1]. Each of these papers uses the minimal model theory of Sullivan, but in the following proof, the fact that $K$ is very simple over $\mathbb{Q}$ allows us to take a more elementary approach.
Proof. By Theorem 2.3, we have the rational equivalences $G(\xi) \simeq \mathbb{Q} \text{Map}(B, K)$ and $G_1(\xi) \simeq \mathbb{Q} \text{Map}_s(B, K)$. Consider $\pi_q(G_1(\xi)) \otimes \mathbb{Q} = \pi_q(\text{Map}_s(B, K)) \otimes \mathbb{Q}$. Because it is true in general that $\pi_q(\text{Map}_s(X, Y)) = [\Sigma^q X, Y]$, and by the universal property of localization we have

$$\pi_q(\text{Map}_s(B, K)) \otimes \mathbb{Q} = [\Sigma^q B, K]_{\mathbb{Q}} = [\Sigma^q B, K_{\mathbb{Q}}]$$

$$= [\Sigma^q B, \prod_{i=1}^r K(\mathbb{Q}, 2n_i - 1)] \quad \text{by Remark 2.3}$$

$$= \prod_{i=1}^r [\Sigma^q B, K(\mathbb{Q}, 2n_i - 1)]$$

$$= \bigoplus_{i=1}^r H^{2n_i-1-q}(B; \mathbb{Q})$$

$$= \bigoplus_{r \geq 0} \tilde{H}^r(B; \mathbb{Q}) \otimes \pi_{r+q}(K),$$

where, in the last line, we have replaced $2n_i - 1$ by $r + q$ and recognized that the only non-zero terms occur in degrees $j$ where $\pi_j(K) \otimes \mathbb{Q} \neq 0$.

Now, for $H$ an $H$-space, we always have the following relationship between free and based mapping spaces (see [9, Proposition 4.9] or [7] for instance):

$$\text{Map}(X, H; *) \simeq H \times \text{Map}_s(X, H; *).$$

Thus, since all components of $\text{Map}(B, (B_K)_{\mathbb{Q}})$ have the same homotopy type, we can apply this result to see that

$$\pi_q(G(\xi)) \otimes \mathbb{Q} = \pi_q(\text{Map}(B, K_{\mathbb{Q}}; *)) = \pi_q(K) \otimes \mathbb{Q} \oplus \pi_q(\text{Map}_s(B, K_{\mathbb{Q}}; *)),$$

and the formula for $\pi_q(G(\xi)) \otimes \mathbb{Q}$ follows. \hfill $\square$

Of course, if $B = S^m$, then we recover Wockel’s result over $\mathbb{Q}$, $\pi_q(G_1(\xi)) \otimes \mathbb{Q} = \pi_{q+m}(K) \otimes \mathbb{Q}$. Moreover, if the base $B$ of the principal bundle is a closed simply connected 4-manifold, then $H^1(B; \mathbb{Q}) = 0 = H^3(B; \mathbb{Q})$ and $H^4(B; \mathbb{Q}) = \mathbb{Q}$. Hence, from the general formulas above, we obtain

**Corollary 3.3 (Terzić’s Formula [16, Propositions 1 and 2]).** If $K \to P \to B$ is a principal bundle (as above) with $B$ a closed simply connected 4-manifold with second Betti number $b_2(B)$, then

$$\text{rank}(\pi_q(G(\xi))) = b_2(B) \cdot \text{rank}(\pi_{q+2}(K)) + \text{rank}(\pi_{q+4}(K)) + \text{rank}(\pi_q(K))$$

and

$$\text{rank}(\pi_q(G_1(\xi))) = b_2(B) \cdot \text{rank}(\pi_{q+2}(K)) + \text{rank}(\pi_{q+4}(K)).$$

Note that because the formula only involves ranks of homotopy groups, it is in fact a result about rational homotopy groups. Also, since the non-zero rational homology of $B$ occurs only in even-degrees and the non-zero rational homotopy of $K$ occurs only in odd degrees, all even-degree rational homotopy of the gauge groups vanishes. For the same reasons, this will also be true whenever $B$ has $H^{\text{odd}}(B; \mathbb{Q}) = 0$. In particular, we have
Example 3.4. If $K \to P \xrightarrow{\xi} \mathbb{C}P^m$ is a principal bundle, then $\pi_q(G(\xi)) \otimes \mathbb{Q} = 0 = \pi_q(G_1(\xi)) \otimes \mathbb{Q}$ for $q$ even and, for $q$ odd,

$$\pi_q(G(\xi)) \otimes \mathbb{Q} = \bigoplus_{i=0}^m \pi_{q+2i}(K) \otimes \mathbb{Q} \quad \text{and} \quad \pi_q(G_1(\xi)) \otimes \mathbb{Q} = \bigoplus_{i=1}^m \pi_{q+2i}(K) \otimes \mathbb{Q}.$$ 

These observations can be put in a wider context. A space $B$ is said to be *rationally elliptic* if its rational homotopy and rational homology are both finite dimensional. For instance, spheres and homogeneous spaces are rationally elliptic. If a rationally elliptic $B$ also has positive Euler characteristic, $\chi(B) > 0$, then it is known that $H^{\text{odd}}(B; \mathbb{Q}) = 0$ (see [13] Theorem 2.75 for instance). Therefore, by the discussion above,

$$\pi_{\text{even}}(G(\xi)) \otimes \mathbb{Q} = 0 = \pi_{\text{even}}(G_1(\xi)) \otimes \mathbb{Q}$$

for any principal bundle $K \to P \xrightarrow{\xi} B$. The connection to geometry arises from two sources (see [4] Section 6.4 for a fuller discussion). First, there is the conjecture of Raoul Bott that compact manifolds of positive sectional curvature are rationally elliptic. Second, there is the conjecture of Heinz Hopf that even-dimensional compact manifolds of positive sectional curvature have positive Euler characteristics. If both these conjectures are true, then by what we have said above, the even-degree rational homotopy groups of gauge groups vanish. This elicits the following question:

**Question 3.5.** Let $K \to P \xrightarrow{\xi} B$ be a principal bundle. If $B^{2m}$ is a compact manifold with positive sectional curvature (in some metric), then is it true that

$$\pi_{\text{even}}(G(\xi)) \otimes \mathbb{Q} = 0 = \pi_{\text{even}}(G_1(\xi)) \otimes \mathbb{Q} ?$$

4. **The gauge group of the universal bundle**

Because it classifies all principal $K$-bundles, the most important principal bundle is the universal bundle $\xi_u : K \to E_K \to B_K$. Thus, its gauge group is of interest. Unfortunately, in the proof of Theorem 3.1, we needed the base space of the bundle to be a finite complex in order to be able to localize mapping spaces. Of course, this is not the case for $B_K$. Nevertheless, we can still compute the rational homotopy groups of $G(\xi_u)$ by making use of the more algebraic framework of rational homotopy theory (see [4] for instance) and, in particular, a theorem of Smith [13].

Let $\text{aut}_1(X) = \text{Map}(X, X; 1_X)$, the monoid of self-homotopy equivalences of $X$ homotopic to the identity $1_X$. There is a classifying space $\text{Baut}_1(X)$ with the usual property that $\Omega\text{Baut}_1(X) = \text{aut}_1(X)$. There is a general way to study the rational homotopy type of $\text{Baut}_1(X)$ from the viewpoint of commutative differential graded algebras and differential graded Lie algebras (see, for instance, [14] [15]). This viewpoint equates the rational homotopy groups with the homology of the complex of degree-lowering derivations on the minimal model of $X$. We will not go into details about models since the following special case is all that we need.

**Theorem 4.1 ([13] Theorem 2 and Corollary 2]).** If $X$ is a rational $H$-space of finite type, then

$$\pi_q(\text{aut}_1(X)) \otimes \mathbb{Q} = \pi_q(\Omega\text{Baut}_1(X)) \otimes \mathbb{Q} = \text{Der}^q(H^*(X; \mathbb{Q})),$$

where $\text{Der}^q(H^*(X; \mathbb{Q}))$ is the vector space of derivations on the cohomology algebra which lower degree by $q$. 

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Recall that $X$ is a rational $H$-space if its $\mathbb{Q}$-localization is an $H$-space. In fact, we have already used the fact that $B_K$ is a rational $H$-space of finite type in the proof of Theorem 2.3, so Theorem 4.1 applies to $B_K$. Indeed, as we described in the proof of Theorem 2.3, $H^*(B_K; \mathbb{Q}) = \mathbb{Q}[v_1, \ldots, v_r]$, a polynomial algebra with $v_i \in H^{2n_i}(B_K; \mathbb{Q})$; so Hopf’s classification says that $B_K \simeq \prod_i K(\mathbb{Q}, 2n_i)$, with the $v_i$’s corresponding to a basis for the rational homotopy groups. Therefore $B_K$ is an $H$-space after rationalization.

Now, by Theorem 2.1 the gauge group of the universal bundle is given by $G(\xi_u) = \Omega \text{Map}(B_K, B_K; 1) = \Omega(\text{aut}_1 B_K)$. By Theorem 4.1 we can then compute the gauge group from the derivations of cohomology $\text{Der}^*(H^*(B_K; \mathbb{Q})) = \text{Der}^*(\mathbb{Q}[u_\alpha])$. The derivations of this algebra are particularly easy to understand. In particular, a basis for the derivations which lower degree by $q + 1$ consists of those derivations that are non-zero on a single generator $u_t$ (in degree $t$, say) and have image any element in degree $t - q - 1$. Since the $u_\alpha$ generate $\pi_\alpha(B_K)$ and $\mathbb{Q}[u_\alpha] = H^*(B_K; \mathbb{Q})$, we can make the identification

$$\text{Der}^{q+1}(\mathbb{Q}[u_\alpha]) = \bigoplus_{t \geq 0} H^{t-q-1}(B_K; \mathbb{Q}) \otimes \pi_t(B_K).$$

We then obtain the same formula as in Theorem 3.1 but now for the universal bundle having infinite-dimensional base $B_K$.

**Theorem 4.2.**

$$\pi_q(G(\xi_u)) \otimes \mathbb{Q} = \bigoplus_{r \geq 0} H^r(B_K; \mathbb{Q}) \otimes \pi_{q+r}(K).$$

**Proof.**

$$\pi_q(G(\xi_u)) \otimes \mathbb{Q} = \pi_q(\Omega(\text{aut}_1 B_K, 1)) \otimes \mathbb{Q} = \pi_{q+1}(\text{aut}_1 B_K, 1) \otimes \mathbb{Q} = \text{Der}^{q+1}(\mathbb{Q}[u_\alpha]) = \bigoplus_t H^{t-q-1}(B_K; \mathbb{Q}) \otimes \pi_t(B_K) = \bigoplus_r H^r(B_K; \mathbb{Q}) \otimes \pi_{q+r}(K),$$

where we have used the general facts that $\pi_j(\Omega X) = \pi_{j+1}(X)$ and $\pi_j(K) = \pi_{j+1}(B_K)$. This is, of course, the same result as in Theorem 3.1. \qed

The gauge group has many equivalent definitions (see [11, Chapter 2]). For the universal bundle, the most homotopically interesting one is

$$G(\xi_u) = \text{Map}(E_K, K)_K,$$

the mapping space of equivariant maps $E_K \to K$, where the action on $K$ is by conjugation. In homotopy theory, this is exactly the definition of the homotopy fixed set $K^{hK}$ of the conjugation action. In general, it is very difficult to obtain explicit information about $K^{hK}$. Here, as a byproduct, we find

**Corollary 4.3.** Let $K$ act on itself by conjugation. Then the rational homotopy groups of the homotopy fixed set are given by

$$\pi_q(K^{hK}) \otimes \mathbb{Q} = \bigoplus_{r \geq 0} H^r(B_K; \mathbb{Q}) \otimes \pi_{q+r}(K).$$
The homotopy fixed set $K^hK$ (and $G(\xi_a)$) may also be identified with the space of sections of the associated fibre bundle $K_K \overset{def}{=} E_K \times_K K \to B_K$, where again the action of $K$ on itself is by conjugation. Yet one more tantalizing connection arises from the following folklore identification. Because we cannot find a reference, we give a brief outline of the proof.

**Lemma 4.4.** The bundle $K_K = E_K \times_K K \to B_K$ is homotopy equivalent to the free loop space fibration $B_K^S^1 \to B_K$.

**Proof.** We outline the proof in steps.

**Step 1.** First define an action $(K \times K) \times K \to K$ by $(g, h) \cdot k = g \cdot k \cdot h^{-1}$. Then it is straightforward to show that $\phi: (E_K \times E_K)/K \to K_{(K \times K)}$, $\phi([x, y]) = [x, y, e]$, is a homeomorphism, where $K$ acts on $E_K \times E_K$ diagonally (on the right in both factors), $K_{(K \times K)}$ is the Borel construction for the action defined above and $e$ is the identity of $K$. An inverse is given by $\phi^{-1}([x, y, e]) = [x, y]$.

**Step 2.** Define a map $\theta: (E_K \times E_K)/K \to B_K$ by composing $\phi$ with $K_{(K \times K)} \to B_{K_{(K \times K)}} \simeq B_K \times B_K \to B_K$, where the last map is projection onto the first factor. The fibre is $F = \{[\bar{x}, \bar{y}] \mid [x] = [\bar{x}]\}$, where $[x]$ is fixed in $B_K$. Define maps $\beta: E_K \to F$ and $\gamma: E_K \to K$ by $\beta(y) = [x, y]$ and $\gamma([\bar{x}, \bar{y}]) = \bar{y} \cdot k$ where $k$ is the unique element of $K$ such that $\bar{x} = x \cdot k$. Notice that we are using the fact that $K$ acts freely on $E_K$.

We then see that $F = E_K$ and, since $\theta$ is a fibration, $(E_K \times E_K)/K \simeq B_K$. Note that a homotopy inverse to $\theta$ is given by $\sigma: B_K \to (E_K \times E_K)/K$, $\sigma([x]) = [x, x]$. Furthermore, note that the following triangle commutes (where $\Delta$ is the diagonal, $\Delta(z) = (z, z)$).

$$
\begin{array}{ccc}
B_K & \overset{\phi \circ \sigma}{\longrightarrow} & K_{(K \times K)} \\
\Delta \downarrow & & \downarrow p \\
B_K \times B_K & & B_K \times B_K
\end{array}
$$

since $p \circ \sigma([x]) = p([x, x, e]) = [x, x] = ([x], [x])$.

**Step 3.** Note that $K_K$ consists of elements $[x, k]$ with $[x, k] = [\bar{x}, \bar{k}]$ if and only if there is $h \in K$ such that $x \cdot h = \bar{x}$ and $hkh^{-1} = \bar{k}$. Define $\psi: K_K \to K_{(K \times K)}$ by $\psi([x, k]) = [x, x, k]$. Then the following square is a pullback (and a homotopy pullback since $K_{(K \times K)} \to B_K \times B_K$ is a fibration).

$$
\begin{array}{ccc}
K_K & \overset{\psi}{\longrightarrow} & K_{(K \times K)} \\
\phi \downarrow & & \downarrow p \\
B_K & \overset{\Delta}{\longrightarrow} & B_K \times B_K
\end{array}
$$

Since it is a homotopy pullback, we can replace $K_{(K \times K)} \to B_K \times B_K$ by the homotopy equivalent $\Delta: B_K \to B_K \times B_K$ as shown in Step 2. We therefore obtain $K_K$ as the homotopy pullback of $\Delta$ over itself. But this is well-known (see [**4** Theorem 5.11]); specifically, we have the following homotopy commutative diagram where the left square is a homotopy pullback (and the vertical maps are the usual
Thus, the free loop space \( B^{S_1}_K \) also arises as the homotopy pullback of \( \Delta \) over itself. Hence, \( B^{S_1}_K \simeq K \). \( \square \)

Note that the free loop space has played and continues to play important roles in both geometry and homotopy theory (see \[4\] for example). Finally, we have the identification of the gauge group of the universal bundle with the space of sections of the free loop fibration on \( B_K \),

\[
G(\xi_u) = \Gamma(B^{S_1}_K \to B_K).
\]

By Theorem 4.2, we then know the rational homotopy groups of this intriguing space of sections.

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