LINEAR ISOMETRIES BETWEEN SPACES OF VECTOR-VALUED LIPSCHITZ FUNCTIONS

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Abstract. In this paper we state a Lipschitz version of a theorem due to Cambern concerning into linear isometries between spaces of vector-valued continuous functions and deduce a Lipschitz version of a celebrated theorem due to Jerison concerning onto linear isometries between such spaces.

1. Introduction

Given a metric space \((X,d)\) and a Banach space \(E\), we denote by \(\text{Lip}(X,E)\) the Banach space of all bounded Lipschitz functions \(f : X \to E\) with the norm \(\|f\| = \max\{L(f), \|f\|_{\infty}\}\), where
\[
L(f) = \sup \left\{ \|f(x) - f(y)\|/d(x,y) : x, y \in X, x \neq y \right\}.
\]
If \(E\) is the field of real or complex numbers, we shall write simply \(\text{Lip}(X)\).

The study of surjective linear isometries between spaces \(\text{Lip}(X)\) was initiated by Roy [9] and Vasavada [10]. In [9, Theorem 1.7], Roy proved that if \((X,d)\) is a compact connected metric space with diameter at most 1, then a map \(T\) is a surjective linear isometry from \(\text{Lip}(X)\) onto itself if and only if there exist a surjective isometry \(\varphi : X \to X\) and a scalar \(\tau\) of modulus 1 such that
\[
T(f)(y) = \tau f(\varphi(y)), \quad \forall y \in Y, \forall f \in \text{Lip}(X).
\]
In [8, Theorem 2], Novinger improved slightly Roy’s result by considering linear isometries from \(\text{Lip}(X)\) onto \(\text{Lip}(Y)\). Vasavada [10] proved it for linear isometries from \(\text{Lip}(X)\) onto \(\text{Lip}(Y)\) when the metric spaces \(X, Y\) are compact with diameter at most 2 and \(\beta\)-connected for some \(\beta < 1\). Weaver [11] developed a technique to remove the compactness assumption on \(X\) and \(Y\) and showed that the above-mentioned characterization holds if \(X, Y\) are complete and 1-connected with diameter at most 2 [11, Theorem D]. The reduction to metric spaces of diameter at most 2 is not restrictive since if \((X,d)\) is a metric space and \(X'\) is the set \(X\) remetrized with the metric \(d'(x,y) = \min\{d(x,y), 2\}\), then the diameter of \(X'\) is at most 2 and \(\text{Lip}(X')\) is isometrically isomorphic to \(\text{Lip}(X)\) [12, Proposition 1.7.1].

We must also mention the complete research carried out on surjective linear isometries between spaces of Hölder functions [2, 3, 6, 7]. We refer the reader to Weaver’s
book *Lipschitz Algebras* [12] for unexplained terminology and more information on the subject. This is essentially the history of the onto scalar-valued case. Recently, into linear isometries (that is, not necessarily surjective) and codimension 1 linear isometries between spaces $\text{Lip}(X)$ have been studied in [5].

In this note we shall go a step further and give a complete description of linear isometries between spaces of vector-valued Lipschitz functions. To our knowledge, little or nothing is known on the matter in the vector-valued case. Our approach to isometries between spaces of vector-valued Lipschitz functions. To our knowledge, strictly convex said to be onto linear isometries between spaces $C(X,E)$ of continuous functions from a compact Hausdorff space $X$ into a Banach space $E$ with the supremum norm. In [4], Jerison extended to the vector case the classical Banach–Stone theorem about from a compact Hausdorff space $X$ into linear isometries (that is, not necessarily surjective) and codimension 1 linear isometries between spaces $C(X,E)$ of continuous functions from a compact Hausdorff space $X$ into a Banach space $E$ with the supremum norm. In [4], Jerison extended to the vector case the classical Banach–Stone theorem about onto linear isometries between spaces $C(X)$, and Jerison’s theorem was generalized by Cambern [1] by considering into linear isometries.

The aim of this paper is to show that Cambern’s and Jerison’s theorems have a natural formulation in the context of Lipschitz functions.

2. A Lipschitz version of Cambern’s theorem

We begin by introducing some notation. Given a Banach space $E$, $B_E$ will denote its unit sphere and $B_E$ its closed unit ball. Let us recall that a Banach space $E$ is said to be strictly convex if every element of $S_E$ is an extreme point of $B_E$. For Banach spaces $E$ and $F$, $L(E,F)$ will stand for the Banach space of all bounded linear operators from $E$ into $F$ with the canonical norm of operators. In the case $E = F$, we shall write $L(E)$ instead of $L(E,F)$. Given a metric space $(X,d)$, we shall denote by $1_X$ the function constantly 1 on $X$ and by $\text{diam}(X)$ the diameter of $X$. If $\varphi : X \to Y$ is a Lipschitz map between metric spaces, $L(\varphi)$ will be its Lipschitz constant.

For any $f \in \text{Lip}(X)$ and $e \in E$, define $f \otimes e : X \to E$ by $(f \otimes e)(x) = f(x) e$. It is easy to check that $f \otimes e \in \text{Lip}(X,E)$ with $\|f \otimes e\|_{\infty} = \|f\|_{\infty} \|e\|$ and $L(f \otimes e) = L(f) \|e\|$, and thus $\|f \otimes e\| = \|f\| \|e\|$.

**Theorem 2.1.** Let $X$ and $Y$ be compact metric spaces and let $E$ be a strictly convex Banach space. Let $T$ be a linear isometry from $\text{Lip}(X,E)$ into $\text{Lip}(Y,E)$ such that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz map $\varphi$ from a closed subset $Y_0$ of $Y$ onto $X$ with $L(\varphi) \leq \max\{1, \text{diam}(X)/2\}$, and a Lipschitz map $y \mapsto T_y$ from $Y$ into $L(E)$ with $\|T_y\| = 1$ for all $y \in Y$, such that $T(f)(y) = T_y(f(\varphi(y)))$, $\forall y \in Y_0$, $\forall f \in \text{Lip}(X,E)$.

**Proof.** For each $x \in X$, define

$$F(x) = \{ f \in \text{Lip}(X,E) : f(x) = \|f\|_{\infty} e \}.$$ 

Clearly, $1_X \otimes e \in F(x)$. For each $\delta > 0$, the map $h_{x,\delta} \otimes e : X \to E$, defined by

$$h_{x,\delta}(z) = \max\{0, 1 - d(z,x)/\delta\} \quad (z \in X),$$

belongs to $F(x)$. Indeed, an easy verification shows that $h_{x,\delta} \in \text{Lip}(X)$ with $\|h_{x,\delta}\|_{\infty} = 1 = h_{x,\delta}(x)$. Hence $h_{x,\delta} \otimes e \in \text{Lip}(X,E)$ with $\|h_{x,\delta} \otimes e\|_{\infty} = 1$ and $(h_{x,\delta} \otimes e)(x) = e$. Then $(h_{x,\delta} \otimes e)(x) = \|h_{x,\delta} \otimes e\|_{\infty} e$ and thus $h_{x,\delta} \otimes e \in F(x)$.

We shall prove the theorem in a series of steps.
Step 1. Let $x \in X$. For each $f \in F(x)$, the set

$$P(f) = \{ y \in Y : T(f)(y) = f(x) \}$$

is nonempty and closed.

Let $f \in F(x)$. If $f = 0$, then $P(f) = Y$ and there is nothing to prove. Suppose $f \neq 0$ and consider $g = \|f\|_{\infty} f + \|f\|^2 (1_X \otimes e)$. Clearly, $g \in \operatorname{Lip}(X,E)$ with $L(g) = \|f\|_{\infty} L(f)$ and $g(x) = \left(\|f\|_{\infty}^2 + \|f\|^2\right) e$. The latter equality implies $g \neq 0$. Since

$$L(g) \leq \|f\|_{\infty} \|f\| \leq \|f\|^2 \|f\|^2 = \|g(x)\| \leq \|g\|_{\infty},$$

it follows that $\|g\| = \|g\|_{\infty}$. Moreover, $\|g\|_{\infty} = \|g(x)\| = \|f\|^2 + \|f\|^2$ since

$$\|g\|_{\infty} = \left\| \|f\|_{\infty} f + \|f\|^2 (1_X \otimes e) \right\|_{\infty} \leq \|f\|_{\infty}^2 + \|f\|^2 = \|g(x)\|.$$

We now claim that there exists a point $y \in Y$ such that $T(g/\|g\|)(y) = e$. Contrary to our claim, assume $e \neq T(g/\|g\|)(y)$ for all $y \in Y$. Let $\varepsilon > 0$ and take $h = g/\|g\| + \varepsilon(1_X \otimes e)$. Clearly, $h \in \operatorname{Lip}(X,E)$ and $T(h) = T(g)/\|g\| + \varepsilon(1_Y \otimes e)$. A simple calculation yields

$$L(T(h)) = L(T(g))/\|g\| \leq \|T(g)/\|g\|\| = 1.$$

Next we show that $\|T(h)\|_{\infty} < 1 + \varepsilon$. For any $y \in Y$, we have

$$\|T(h)(y)\| = \|T(g/\|g\|)(y) + \varepsilon e\| \leq 1 + \varepsilon$$

since $\|T(g/\|g\|)(y)\| \leq \|T(g)/\|g\|\| = 1$. Indeed,

$$\|T(g/\|g\|)(y) + \varepsilon e\| < 1 + \varepsilon.$$

Otherwise the vector $u = (1/(1 + \varepsilon))(T(g/\|g\|)(y) + \varepsilon e)$ would be an extreme point of $B_E$ by the strict convexity of $E$, and since $u$ is a convex combination of $T(g/\|g\|)(y)$ and $e$, which are in $B_E$, we infer that $T(g/\|g\|)(y) = e$, a contradiction. Hence $\|T(h)(y)\| < 1 + \varepsilon$ for all $y \in Y$. Since $\|T(h)\|_{\infty} = \|T(h)(y)\|$ for some $y \in Y$, we conclude that $\|T(h)\|_{\infty} < 1 + \varepsilon$. From what we have proved above it is deduced that $\|T(h)\| < 1 + \varepsilon$, but, on the other hand,

$$1 + \varepsilon = \|g(x)/\|g\| + \varepsilon e\| = \|h(x)\| \leq \|h\|_{\infty} \leq \|h\| = \|T(h)\|,$$

which is impossible. This proves our claim.

Now, let $y \in Y$ be such that $T(g/\|g\|)(y) = e$. Since $e = g(x)/\|g\|$, $Tg(y) = g(x)$, that is,

$$\|f\|_{\infty} Tf(y) + \|f\|^2 T(1_X \otimes e)(y) = \left(\|f\|_{\infty}^2 + \|f\|^2\right) e.$$

Since $T(1_X \otimes e) = 1_Y \otimes e$, we have

$$\|f\|_{\infty} Tf(y) + \|f\|^2 e = \left(\|f\|_{\infty}^2 + \|f\|^2\right) e,$$

and thus $T(f)(y) = \|f\|_{\infty} e$, which is $T(f)(y) = f(x)$ since $f \in F(x)$. Hence $P(f) \neq \emptyset$. Moreover, $P(f)$ is closed in $Y$ since $P(f) = T(f)^{-1}\{\{f(x)\}\}$ and $T(f)$ is continuous.

Step 2. For each $x \in X$, the set

$$B(x) = \{ y \in Y : T(f)(y) = f(x), \forall f \in F(x) \}$$

is nonempty and closed.
Let $x \in X$. For each $f \in F(x)$, $P(f)$ is a nonempty closed subset of $Y$ by Step 1. Since $B(x) = \bigcap_{f \in F(x)} P(f)$, $B(x)$ is closed. To prove that $B(x) \neq \emptyset$, since $Y$ is compact and $B(x) = \bigcap_{f \in F(x)} P(f)$, it suffices to check that if $f_1, \ldots, f_n \in F(x)$, then $\bigcap_{j=1}^n P(f_j) \neq \emptyset$.

We can suppose, without loss of generality, that $f_j \neq 0$ for all $j \in \{1, \ldots, n\}$ since $P(f_j) = Y$ if $f_j = 0$. For each $j \in \{1, \ldots, n\}$ define $g_j = \|f_j\|_\infty f_j + \|f_j\|^2(1_X \otimes e)$. As in the proof of Step 1, $g_j \in \text{Lip}(X, E)$ with $g_j(x) = \left(\|f_j\|_\infty + \|f_j\|^2\right)$ and $\|g_j\| = \|f_j\|_\infty^2 + \|f_j\|^2$. Hence $g_j \neq 0$ and we can define $h = (1/n) \sum_{j=1}^n (g_j / \|g_j\|)$. Clearly, $h \in \text{Lip}(X, E)$, $h(x) = e$ and $\|h\|_\infty = 1$. Hence $h(x) = \|h\|_\infty e$ and thus $h \in F(x)$. Then, by Step 1 there exists a point $y \in Y$ such that $T(h)(y) = h(x)$. Since $T(h)(y) = (1/n) \sum_{j=1}^n (T(g_j)(y) / \|g_j\|)$ and $h(x) = e$, it follows that $e = (1/n) \sum_{j=1}^n (T(g_j)(y) / \|g_j\|)$. Since $E$ is strictly convex and $|T(g_j)(y)| / \|g_j\| \leq \|T(g_j)(y) / g_j\| = 1$ for all $j \in \{1, \ldots, n\}$, we infer that $T(g_j)(y) = g_j \otimes e$ for all $j \in \{1, \ldots, n\}$. Reasoning as in Step 1 we obtain $T(f_j)(y) = f_j(x)$ for all $j \in \{1, \ldots, n\}$ and thus $y \in \bigcap_{j=1}^n P(f_j)$.

Step 3. Let $f \in \text{Lip}(X, E)$, $x \in X$ and $y \in B(x)$. If $f(x) = 0$, then $T(f)(y) = 0$.

If $f = 0$, then there is nothing to prove. Suppose $f \neq 0$ and let $\delta = \|f\|_\infty / \|f\|$. Clearly, $L(f) / \|f\|_\infty \leq 1 / \delta$. Consider $h_{x, \delta} \otimes e \in F(x)$. We next prove that $f / \|f\|_\infty + (h_{x, \delta} \otimes e)$ belongs to $F(x)$. Since $f / \|f\|_\infty + (h_{x, \delta} \otimes e) \in \text{Lip}(X, E)$ and $f(x) / \|f\|_\infty + (h_{x, \delta} \otimes e)(x) = e$, it suffices to check that $\|f / \|f\|_\infty + (h_{x, \delta} \otimes e)\|_\infty = 1$. Let $z \in X$. If $d(z, x) \geq \delta$, we have $(h_{x, \delta} \otimes e)(z) = 0$ and so

$$
\|f(z) / \|f\|_\infty + (h_{x, \delta} \otimes e)(z)\| = \|f(z)\| / \|f\|_\infty \leq 1.
$$

If $d(z, x) < \delta$, then $(h_{x, \delta} \otimes e)(z) = (1 - d(z, x) / \delta) e$, and therefore

$$
\|f(z) / \|f\|_\infty + (h_{x, \delta} \otimes e)(z)\| \leq \|f(z)\| / \|f\|_\infty + 1 - d(z, x) / \delta \leq 1,
$$

since

$$
\|f(z)\| / \|f\|_\infty = \|f(z) - f(x)\| / \|f\|_\infty \leq L(f)d(z, x) / \|f\|_\infty \leq d(z, x) / \delta.
$$

Hence $\|f / \|f\|_\infty + (h_{x, \delta} \otimes e)(z)\|_\infty \leq 1$. Since

$$
\|f(x) / \|f\|_\infty + (h_{x, \delta} \otimes e)(x)\| = \|e\| = 1,
$$

we obtain the desired condition.

By the definition of $B(x)$ it follows that

$$
T(f / \|f\|_\infty + (h_{x, \delta} \otimes e))(y) = (f / \|f\|_\infty + (h_{x, \delta} \otimes e))(x),
$$

that is, $T(f)(y) / \|f\|_\infty + T(h_{x, \delta} \otimes e)(y) = e$. Moreover, since $y \in B(x)$ and $h_{x, \delta} \otimes e \in F(x)$, we have $T(h_{x, \delta} \otimes e)(y) = (h_{x, \delta} \otimes e)(x) = e$. Hence $T(f)(y) / \|f\|_\infty + e = e$ and thus $T(f)(y) = 0$.

Step 4. Let $x, x' \in X$ with $x \neq x'$. Then $B(x) \cap B(x') = \emptyset$.

Suppose $y \in B(x) \cap B(x')$. Let $\delta = d(x, x') > 0$ and consider $h_{x, \delta} \otimes e$. Since $y \in B(x)$ and $h_{x, \delta} \otimes e \in F(x)$, we have $T(h_{x, \delta} \otimes e)(y) = (h_{x, \delta} \otimes e)(x) = e$ by Step 2, but Step 2 also yields $T(h_{x, \delta} \otimes e)(y) = 0$ since $y \in B(x')$ and $(h_{x, \delta} \otimes e)(x') = 0$. So we arrive at a contradiction. Hence $B(x) \cap B(x') = \emptyset$.

Steps 3 and 4 motivate the following:

**Definition 1.** Let $Y_0 = \bigcup_{x \in X} B(x)$. Define $\varphi : Y_0 \to X$ by $\varphi(y) = x$ if $y \in B(x)$. 

Clearly, \( \varphi \) is surjective. Moreover, given \( y \in Y_0 \), there exists \( x \in X \) such that \( y \in B(x) \), and hence \( \varphi(y) = x \) and \( T(f)(y) = f(x) \) for all \( f \in F(x) \).

We shall obtain the representation of \( T \) in terms of the following functions.

**Definition 2.** For each \( y \in Y \), define \( T_y : E \to E \) by \( T_y(u) = T(1_X \otimes u)(y) \).

It is easy to show that \( T_y \in L(E) \) with \( \| T_y \| = 1 = \| T_y(e) \| \) for all \( y \in Y \).

**Step 5.** The map \( y \mapsto T_y \) from \( Y \) into \( L(E) \) is Lipschitz.

Let \( y, z \in Y \). Given \( u \in E \), we have

\[
\| (T_y - T_z)(u) \| \leq L(T(1_X \otimes u))d(y, z)
\]

and thus \( \| T_y - T_z \| \leq d(y, z) \).

**Step 6.** \( T(f)(y) = T_y(f(\varphi(y))) \) for all \( f \in \text{Lip}(X, E) \) and \( y \in Y_0 \).

Let \( f \in \text{Lip}(X, E) \) and \( y \in Y_0 \). Let \( x = \varphi(y) \in X \) and define \( h = f - (1_X \otimes f(x)) \).

Obviously, \( h \in \text{Lip}(X, E) \) with \( h(x) = 0 \). From Step 3 we have \( T(h)(y) = 0 \) and therefore \( T(f)(y) = T(1_X \otimes f(x))(y) = T_y(f(x)) = T_y(f(\varphi(y))) \).

**Step 7.** \( Y_0 \) is closed in \( Y \).

Let \( y \in Y \) and \( \{y_n\} \) be a sequence in \( Y_0 \) which converges to \( y \). Let \( x_n = \varphi(y_n) \) for all \( n \in \mathbb{N} \). Since \( X \) is compact, there exists a subsequence \( \{x_{\sigma(n)}\} \) converging to a point \( x \in X \). Let \( f \in F(x) \). Clearly, \( \{T(f)(y_{\sigma(n)})\} \) converges to \( T(f)(y) \), but also to \( f(x) \) as we see at once. Indeed, for each \( n \in \mathbb{N} \), we have

\[
T(f)(y_{\sigma(n)}) = T_{y_{\sigma(n)}}(f(x_{\sigma(n)})) = T(1_X \otimes f(x_{\sigma(n)}))(y_{\sigma(n)}),
\]

by Step 3 and

\[
f(x) = \|f\|_\infty e = \|f\|_\infty (1_Y \otimes e)(y_{\sigma(n)})
= \|f\|_\infty T(1_X \otimes e)(y_{\sigma(n)}) = T(1_X \otimes f(x))(y_{\sigma(n)}),
\]

since \( f \in F(x) \). We deduce that

\[
\|T(f)(y_{\sigma(n)}) - f(x)\| = \|T(1_X \otimes (f(x_{\sigma(n)}) - f(x)))(y_{\sigma(n)})\|
\leq \|T(1_X \otimes (f(x_{\sigma(n)}) - f(x)))\| = \|T(1_X \otimes (x_{\sigma(n)}) - f(x))\|
= \|f(x_{\sigma(n)}) - f(x)\|
\]

for all \( n \in \mathbb{N} \). Since \( \{f(x_{\sigma(n)})\} \to f(x) \), we conclude that \( \{T(f)(y_{\sigma(n)})\} \to f(x) \).

Hence \( T(f)(y) = f(x) \) and thus \( y \in B(x) \subset Y_0 \).

**Step 8.** The map \( \varphi : Y_0 \to X \) is Lipschitz and \( L(\varphi) \leq \max\{1, \text{diam}(X)/2\} \).

Let \( y, z \in Y_0 \) be such that \( \varphi(y) \neq \varphi(z) \) and put \( \delta = d(\varphi(y), \varphi(z))/2 \). Define \( f_{y, z} = \delta(h_{\varphi(y), \varphi(z)} - h_{\varphi(z), \varphi(y)}) \) on \( X \). It is easy to see that \( f_{y, z} \in \text{Lip}(X) \) and \( \|f_{y, z}\| \leq k := \max\{1, \text{diam}(X)/2\} \). Since \( T \) is an isometry, \( \|T(f_{y, z} \otimes e)\| \leq k \). This inequality implies \( L(T(f_{y, z} \otimes e)) \leq k \). It follows that

\[
\|T(f_{y, z} \otimes e)(y) - T(f_{y, z} \otimes e)(z)\| \leq kd(y, z).
\]

Using Step 3 we get

\[
T(f_{y, z} \otimes e)(y) = T_y((f_{y, z} \otimes e)(\varphi(y))) = T_y(\delta e) = \delta e,
T(f_{y, z} \otimes e)(z) = T_z((f_{y, z} \otimes e)(\varphi(z))) = T_z(-\delta e) = -\delta e.
\]

We conclude that \( d(\varphi(y), \varphi(z)) \leq kd(y, z) \). \( \square \)
The condition in Theorem 2.1 $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$, is not too restrictive if we analyse the known results in the scalar case. In this case our condition means $T(1_X) = 1_Y$; notice that the connectedness assumptions on the metric spaces in [9, Lemma 1.5] and [11, Lemma 6] yield a similar condition, namely, that $T(1_X)$ is a constant function.

3. A Lipschitz version of Jerison’s theorem

Recall that a map between metric spaces $\varphi : X \to Y$ is said to be a Lipschitz homeomorphism if $\varphi$ is bijective and $\varphi$ and $\varphi^{-1}$ are both Lipschitz.

**Theorem 3.1.** Let $X, Y$ be compact metric spaces and let $E$ be a strictly convex Banach space. Let $T$ be a linear isometry from Lip($X, E$) onto Lip($Y, E$) such that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \text{diam}(X)/2\}$ and $L(\varphi^{-1}) \leq \max\{1, \text{diam}(Y)/2\}$, and a Lipschitz map $y \mapsto T_y$ from $Y$ into $L(E)$ where $T_y$ is an isometry from $E$ onto itself for all $y \in Y$ such that $T(f)(y) = T_y(f(\varphi(y)))$, $\forall y \in Y$, $\forall f \in \text{Lip}(X, E)$.

**Proof.** Let $Y_0$ and $\varphi$ be as in Theorem 2.1 Since $T^{-1} : \text{Lip}(Y, E) \to \text{Lip}(X, E)$ is a linear isometry and $T^{-1}(1_Y \otimes e) = 1_X \otimes e$, applying Theorem 2.1 we have $T^{-1}(g)(x) = (T^{-1})_x(g(\psi(x)))$, $\forall x \in X_0$, $\forall g \in \text{Lip}(Y, E)$, where $\psi$ is a Lipschitz map from a closed subset $X_0$ of $X$ onto $Y$ with $L(\psi) \leq \max\{1, \text{diam}(Y)/2\}$, and $x \mapsto (T^{-1})_x$ is a Lipschitz map from $X$ into $L(E)$. Namely, $X_0 = \bigcup_{y \in Y} B(y)$ where, for each $y \in Y$, $B(y) = \{ x \in X : T^{-1}(g)(x) = g(y), \forall g \in F(y) \}$ with $F(y) = \{ g \in \text{Lip}(Y, E) : g(y) = \| g \|_\infty e \}$, and $\psi : X_0 \to Y$ is the Lipschitz map defined by $\psi(x) = y$ if $x \in B(y)$. Moreover, using the same arguments as in Step 3 the following can be proved:

**Claim 1.** Let $g \in \text{Lip}(Y, E)$, $y \in Y$ and $x \in B(y)$. If $g(y) = 0$, then $T^{-1}(g)(x) = 0$.

After this preparation we proceed to prove the theorem. Fix $x \in X$ and let $y \in B(x)$. We first prove that $x \in B(y)$. Suppose that $x \notin B(y)$. Since $B(y) \neq \emptyset$, there exists $x' \in B(y)$ with $x' \neq x$. Take $f \in \text{Lip}(X, E)$ for which $f(x) = 0$ and $f(x') \neq 0$. Since $y \in B(x)$ and $f(x) = 0$, we have $T(f)(y) = 0$ by Step 3. Then $T^{-1}(T(f))(x') = 0$ since $x' \in B(y)$ by Claim 1 and thus $f(x') = 0$, a contradiction. Therefore $x \in B(y) \subseteq X_0$ and thus $X_0 = X$. Next we see that $Y_0 = Y$. Let $y \in Y$. We can take a point $x \in B(y)$. As above it is proved that $y \in B(x)$ and thus $y \in Y_0$.

To see that $\varphi$ is a Lipschitz homeomorphism, let $y \in Y$. Then $y \in B(x)$ for some $x \in X$, that is, $\varphi(y) = x$. Moreover, by what we have proved above, $x \in B(y)$ and so $\psi(x) = y$. As a consequence, $\psi(\varphi(y)) = y$. Since $\varphi$ was surjective, $\varphi$ is bijective with $\varphi^{-1} = \psi$ and thus $\varphi$ is a Lipschitz homeomorphism.

To check that $T_y$ is an isometry from $E$ into itself for every $y \in Y$, we first show that $T$ sends nonvanishing functions of Lip($X, E$) into nonvanishing functions of Lip($Y, E$). Assume there exists $f \in \text{Lip}(X, E)$ such that $f(x) \neq 0$ for all $x \in X$, but $T(f)(y) = 0$ for some $y \in Y$. By the surjectivity of $\psi$, there is a point $x \in X_0$ such that $\psi(x) = y$, that is, $x \in B(y)$. Since $T(f)(y) = 0$, by Claim 1
we have \( f(x) = T^{-1}(T(f))(x) = 0 \), a contradiction. Hence \( T \) maps nonvanishing functions into nonvanishing functions. If, for some \( y \in Y \), \( T_y \) is not an isometry, then there exists a \( u \in \mathcal{S}_E \) such that \( \|T_y(u)\| = \|T(1_X \otimes u)(y)\| < 1 \). Since \( T \) is surjective, there is an \( f \in \text{Lip}(X, E) \) such that \( T(f) = 1_Y \otimes T(1_X \otimes u)(y) \). Thus \( \|f\|_\infty \leq \|f\| = \|T(f)\| = \|T(1_X \otimes u)(y)\| < 1 \) and \( 1_X \otimes u - f \) never vanishes on \( X \). As \( T(1_X \otimes u)(y) = T(f)(y) \), we arrive at a contradiction.

Next we prove that \( T_y : E \to E \) is surjective for every \( y \in Y \). Fix \( y \in Y \) and let \( v \in E \). Since \( T \) is surjective, there exists \( f \in \text{Lip}(X, E) \) such that \( T(f) = 1_Y \otimes v \). Let \( u = (f \circ \varphi)(y) \in E \). Using Step 6, we have \( T_y(u) = T_y(f(\varphi(y))) = T(f)(y) = v \). Hence \( T_y \) is surjective.

Finally, as a direct consequence of Theorem 3.1, we obtain the following:

**Corollary 3.2.** Let \( X, Y \) be compact metric spaces with diameter at most 2 and let \( E \) be a strictly convex Banach space. Then every surjective linear isometry \( T \) from \( \text{Lip}(X, E) \) into \( \text{Lip}(Y, E) \) satisfying that \( T(1_X \otimes e) = 1_Y \otimes e \) for some \( e \in \mathcal{S}_E \), can be expressed as \( T(f)(y) = T_y(f(\varphi(y))) \) for all \( y \in Y \) and \( f \in \text{Lip}(X, E) \), where \( \varphi : Y \to X \) is a surjective isometry and \( y \mapsto T_y \) is a Lipschitz map from \( Y \) into \( L(E) \) such that \( T_y \) is an isometry from \( E \) onto \( E \) for all \( y \in Y \).

In the special case that \( E \) is a Hilbert space, Theorems 2.1 and 3.1 can be improved as follows. For a Hilbert space \( E \), let us recall that a unitary operator is a linear map \( \Phi : E \to E \) that is a surjective isometry.

**Corollary 3.3.** Let \( X \) and \( Y \) be compact metric spaces and let \( E \) be a Hilbert space. Let \( T \) be a linear isometry from \( \text{Lip}(X, E) \) into \( \text{Lip}(Y, E) \) such that \( T(1_X \otimes e) \) is a constant function for some \( e \in \mathcal{S}_E \). Then there exists a Lipschitz map \( \varphi \) from a closed subset \( Y_0 \) of \( Y \) onto \( X \) with \( L(\varphi) \leq \max\{1, \text{diam}(X)/2\} \) and a Lipschitz map \( y \mapsto T_y \) from \( Y \) into \( L(E) \) with \( \|T_y\| = 1 \) for all \( y \in Y \) such that

\[
T(f)(y) = T_y(f(\varphi(y))), \quad \forall y \in Y_0, \forall f \in \text{Lip}(X, E).
\]

If, in addition, \( T \) is surjective, then \( Y_0 = Y \), \( \varphi \) is a Lipschitz homeomorphism with \( L(\varphi^{-1}) \leq \max\{1, \text{diam}(Y)/2\} \) and, for each \( y \in Y \), \( T_y \) is a unitary operator.

**Proof.** Assume that \( T(1_X \otimes e) = 1_Y \otimes u \) for some \( u \in E \). Obviously, \( \|u\| = 1 \). Since \( E \) is a Hilbert space, we can construct a unitary operator \( \Phi : E \to E \) such that \( \Phi(u) = e \). Define \( S : \text{Lip}(Y, E) \to \text{Lip}(Y, E) \) by

\[
S(g)(y) = \Phi(g(y)), \quad \forall y \in Y, \forall g \in \text{Lip}(Y, E).
\]

It is easy to prove that \( S \) is a surjective linear isometry satisfying that \( S(1_Y \otimes u) = 1_Y \otimes e \). Hence \( R = S \circ T \) is a linear isometry from \( \text{Lip}(X, E) \) into \( \text{Lip}(Y, E) \) with \( R(1_X \otimes e) = 1_Y \otimes e \). Then Theorem 2.1 guarantees the existence of a Lipschitz map \( \varphi \) from a closed subset \( Y_0 \) of \( Y \) onto \( X \) with \( L(\varphi) \leq \max\{1, \text{diam}(X)/2\} \) and a Lipschitz map \( y \mapsto R_y \) from \( Y \) into \( L(E) \) with \( \|R_y\| = 1 \) for all \( y \in Y \) such that

\[
R(f)(y) = R_y(f(\varphi(y))), \quad \forall y \in Y_0, \forall f \in \text{Lip}(X, E).
\]

For each \( y \in Y \), consider \( T_y = \Phi^{-1} \circ R_y \in L(E) \). It is easily seen that the map \( y \mapsto T_y \) from \( Y \) into \( L(E) \) is Lipschitz with \( \|T_y\| = 1 \) for all \( y \in Y \). Moreover, for any \( y \in Y_0 \) and \( f \in \text{Lip}(X, E) \), we have

\[
T(f)(y) = \Phi^{-1}(R_y(f(\varphi(y)))) = T_y(f(\varphi(y))).
\]

If, in addition, \( T \) is surjective, the rest of the corollary follows by applying Theorem 3.1 to \( R \).
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REFERENCES