THE BOUNDING GENERA AND $w$-INVARIENTS

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Abstract. In this paper, we give an estimate from below of the bounding genera for homology 3-spheres defined by Y. Matsumoto in terms of $w$-invariants. In particular, combining with Matsumoto’s estimates we determine the values of the bounding genera for several infinite families of Brieskorn homology 3-spheres.

1. Introduction

In this paper, we give an estimate from below of the bounding genera for homology 3-spheres defined by Y. Matsumoto in terms of $w$-invariants. In particular, combining with Matsumoto’s estimates we determine the values of the bounding genera for several infinite families of Brieskorn homology 3-spheres.

In 1982, Y. Matsumoto introduced the notion of a bounding genus for integral homology 3-spheres to study the kernel of the Rohlin invariant. Let $\Gamma$ be a nonsingular symmetric bilinear form over $\mathbb{Z}$. A homology 3-sphere $\Sigma$ is said to bound the form $\Gamma$ if and only if $\Sigma$ bounds a compact, oriented, homologically 1-connected smooth 4-manifold $W$ whose intersection form defined on $H_2(W)$ is isomorphic to $\Gamma$. Here a topological space $X$ is said to be homologically 1-connected if it is connected and $H_1(X) = \{0\}$. Let $H$ be the hyperbolic form, i.e., the intersection form of $S^2 \times S^2$. Then the bounding genus is defined as follows.

Definition 1.1 (Y. Matsumoto [11]). Let $\Sigma$ be a homology 3-sphere. Then the bounding genus $|\Sigma|$ of $\Sigma$ is defined to be

$$|\Sigma| := \begin{cases} \min \{ n \mid \Sigma \text{ bounds } nH \}, & \mu(\Sigma) = 0, \\ +\infty, & \mu(\Sigma) = 1, \end{cases}$$

where $\mu(\Sigma)$ is the Rohlin invariant of $\Sigma$.

Remark 1.2. If the Rohlin invariant $\mu(\Sigma)$ of the homology 3-sphere $\Sigma$ vanishes, then $\Sigma$ bounds a smooth spin 4-manifold $W$ with signature $\text{Sign}(W)$ divisible by 16. By taking the connected sum with several copies of $K$3 surfaces or the $K$3 surface with reversed orientation, if necessary, we may assume that $\text{Sign}(W) = 0$ and hence $W$ is an indefinite spin 4-manifold. It is known that the intersection form of indefinite spin 4-manifolds is isomorphic to the direct sum of several copies of the hyperbolic form $H$.
Remark 1.3. The bounding genus $|\Sigma|$ gives a homology cobordism invariant; i.e., it gives a map $|\cdot|: \Theta^H_3 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ from the homology cobordism group $\Theta^H_3$ of homology 3-spheres.

Remark 1.4. The bounding genus $|\Sigma|$ satisfies the triangle inequality $|\Sigma + \Sigma'| \leq |\Sigma| + |\Sigma'|$ and in fact gives a distance in $\Theta^H_3$ allowing the value to be infinity.

Remark 1.5. The notion of 1-connected bounding genus $\|\Sigma\|$ is also defined by replacing “homological 1-connectedness” by ordinary “1-connectedness” in the definition of the bounding genus $|\Sigma|$. Clearly the inequality $|\Sigma| \leq \|\Sigma\|$ holds.

Matsumoto gave upper estimates on the bounding genera for several families of homology 3-spheres using Dehn-Kirby calculus. For example, he gave the following estimates.

Proposition 1.6 (Y. Matsumoto [11, §4, Proposition 4.4]). $|\Sigma(2,7,14m - 1)| \leq 3$ for any positive odd integer $m$.

For example, the bounding genus of the Brieskorn homology 3-sphere $\Sigma(2,7,13)$ satisfies $|\Sigma(2,7,13)| \leq 3$. Matsumoto called this estimate “hard-to-improve”. In fact, R. Kirby proved that $\Sigma(2,7,13)$ bounds the plumbed 4-manifold $P(\Gamma_{16})$ associated to the intersection form $\Gamma_{16}$. Hence if $\Sigma(2,7,13)$ bounds $2 \cdot H$, then the closed 4-manifold $M = -P(\Gamma_{16}) \cup |2 \cdot H|$ obtained by gluing $-P(\Gamma_{16})$ and $|2 \cdot H|$ along the boundary $\Sigma(2,7,13)$ leads to the inequality

$$\frac{11}{8} |\text{Sign}(M)| = \frac{11}{8} \cdot |\Sigma(2,7,13)| \geq 16 > 16 = b_2 (M)$$

which violates the following $11/8$-conjecture proposed by Y. Matsumoto [11].

Conjecture 1.7 (Y. Matsumoto [11]). Let $M$ be a closed spin 4-manifold. Then the following inequality holds:

$$\frac{11}{8} |\text{Sign}(M)| \leq b_2 (M).$$

To determine the bounding genera we need an estimate from below. In fact, M. Furuta proved an inequality called the $10/8$-inequality close to the $11/8$-conjecture by using the finite-dimensional approximation of the Seiberg-Witten monopole equation on closed spin 4-manifolds.

Theorem 1.8 (M. Furuta [9]). For any closed spin 4-manifold $M$ with $\text{Sign}(M) \neq 0$, the following inequality holds:

$$\frac{10}{8} |\text{Sign}(M)| + 2 \leq b_2 (M).$$

If we apply this inequality to $M = -P(\Gamma_{16}) \cup |2 \cdot H|$, then we have the inequality

$$\frac{10}{8} |\Sigma(2,7,13)| + 2 = 22 > 20 = 16 + 4$$

violating the $10/8$-inequality above and hence $|\Sigma(2,7,13)| = 3$. For other Brieskorn homology 3-spheres $\Sigma$, we need to find “good” spin 4-manifolds such as $P(\Gamma_{16})$ which $\Sigma$ bounds.

In a joint work with M. Furuta [7], we used a $V$-manifold version of the $10/8$-inequality to define a homology cobordism invariant for a class of homology 3-spheres which we call the $w$-invariant. The notion of $V$-manifold is defined by I. Satake [15] as a generalization of manifolds which allows neighborhoods to be...
the quotients of Euclidean spaces divided by finite group actions. The \( w \)-invariant can be considered as the Seiberg-Witten theory counterpart of the invariant \([3]\) of R. Fintushel and R. Stern defined by using the Donaldson theory. In fact, the \( w \)-invariant is defined for a triple \((\Sigma, X, c)\) composed of a homology 3-sphere \(\Sigma\), a compact smooth spin 4-V-manifold \(X\) with boundary \(\Sigma\), and a \(V\)-spin\(^c\) structure \(c\) on \(X\), and it takes values in the integers, \(w(\Sigma, X, c) \in \mathbb{Z}\). If the \(V\)-spin\(^c\) structure \(c\) comes from the \(V\)-spin structure on \(X\), then the value \(w(\Sigma, X, c)\) modulo 2 is equal to Rohlin’s \(\mu\)-invariant.

By using this invariant \(w(\Sigma, X, c)\), we give the following estimate on bounding genera from below whose proof will be given in Section 2.

**Theorem 1.9.** Let \(\Sigma\) be an integral homology 3-sphere bounding a compact smooth spin 4-V-manifold \(X\) with \(V\)-spin\(^c\) structure \(c\) which comes from a \(V\)-spin structure on \(X\). Then the following inequalities hold:

1. If \(w(\Sigma, X, c) > 0\), then \(|\Sigma| \geq w(\Sigma, X, c) - b_2^+ (X) + 1\).
2. If \(w(\Sigma, X, c) < 0\), then \(|\Sigma| \geq -w(\Sigma, X, c) - b_2^- (X) + 1\).

As in the case of smooth manifolds, we need to find a “good” spin 4-V-manifold \(X\) to give an efficient estimate. However, for Seifert homology 3-spheres \(\Sigma = \Sigma(a_1, \ldots, a_n)\), we can take \(X\) to be the canonical \(D^2\)-\(V\)-bundle \(X \to S^2\) over \(S^2\) associated to the Seifert fibration \(\Sigma \to S^2\). Then \(X\) is a 4-V-manifold with \(n\)-singular points which are cones over lens spaces and with \(b_2^+ (X) = 0\), \(b_2^- (X) = 1\). If one of the \(a_i\)'s is even, then \(X\) admits a unique \(V\)-spin structure \(c\) on \(X\). For example, the value of the \(w\)-invariant of the Brieskorn homology 3-sphere \(\Sigma(2, 7, 13)\) is \(w(\Sigma(2, 7, 13), X, c) = 2 > 0\). Hence by Theorem 1.9 we see that \(|\Sigma(2, 7, 13)| \geq 2 - 0 + 1 = 3\). Therefore the bounding genus of \(\Sigma(2, 7, 13)\) is certainly \(|\Sigma(2, 7, 13)| = 3\). In Section 3 we will prove the following:

**Proposition 1.10.** \(|\Sigma(2, 7, 14m - 1)| = 3\) for any positive odd integer \(m\).

R. Fintushel and R. Stern defined the invariant \(R(a_1, \ldots, a_n)\) for Seifert homology 3-spheres \(\Sigma(a_1, \ldots, a_n)\) by using the Donaldson theory and proved that if \(R(a_1, \ldots, a_n) > 0\), then \(\Sigma(a_1, \ldots, a_n)\) cannot be the boundary of an acyclic 4-manifold \([3]\). Hence if \(R(a_1, \ldots, a_n) > 0\), then we can show that \(|\Sigma(a_1, \ldots, a_n)| \geq 1\). Matsumoto proved for example that \(|\Sigma(2, 3, 12k - 1)| \leq 1\), whereas the \(w\)-invariant of \(\Sigma(2, 3, 11)\) is zero, and hence the above Theorem 1.9 cannot be applied. However we see that \(R(2, 3, 11) > 0\) and therefore \(|\Sigma(2, 3, 11)| = 1\). In Section 3 we will prove the following:

**Proposition 1.11.** \(|\Sigma(2, 3, 12k - 1)| = 1\) for any non-negative integer \(k\).

2. Bounding genera and \(w\)-invariants

First we recall the \(V\)-manifold version of the 10/8-inequality.

**Theorem 2.1 (7).** Let \(X\) be a closed smooth spin 4-V-manifold. Fix a Riemannian \(V\)-metric on \(X\) and let \(D(X)\) be the positive chiral Dirac operator. Suppose the \(V\)-index of the Dirac operator is positive: \(\text{ind}_V D(X) > 0\). Then the following inequality holds:

\[
\text{ind}_V D(X) + 1 \leq b_2^+ (X).
\]

The \(w\)-invariant is defined as follows.
Definition 2.2. Let \((\Sigma, X, c)\) be a triple composed of a homology 3-sphere \(\Sigma\), a compact smooth spin 4-V-manifold \(X\) with boundary \(\partial X \cong \Sigma\), and a \(V\)-spin\(^c\) structure \(c\) on \(X\). Then we define
\[
w(\Sigma, X, c) := \text{ind}_V D(X \cup_\Sigma W) + \frac{\text{Sign}(W)}{8},
\]
where \(W\) is a smooth spin 4-manifold with boundary \(\partial W \cong -\Sigma\).

Remark 2.3. \(w(\Sigma, X, c)\) does not depend on the choice of \(W\) and its spin structure by the excision properties of \(V\)-indices and the fact that the \(L\)-genus is \((-8)\)-times the \(\hat{A}\)-genus. Moreover, if the \(V\)-spin\(^c\) structure \(c\) comes from a \(V\)-spin structure, then \(w(\Sigma, X, c)\) may depend on the choice of \(X\) and \(c\), but Theorem 2.1 implies a homology cobordism invariance of \(w(\Sigma, X, c)\) in a certain class of homology 3-spheres \(\Sigma\) including the set of all Seifert homology 3-spheres [7].

By using this Theorem 2.1, we give a proof of Theorem 1.9.

Proof of Theorem 1.9. Let \(m = |\Sigma|\) be the bounding genus of \(\Sigma\). Then \(\Sigma\) bounds a homologically 1-connected compact oriented smooth spin 4-manifold \(W_m\) with intersection form \(mH\). This implies that \(b_2^+(W_m) = m\) and \(\text{Sign} W_m = 0\). On the other hand, let \(X\) be a closed spin 4-V-manifold \(X\) with \(V\)-spin structure \(c\) with boundary \(\partial X \cong \Sigma\). Let \(Z\) be a closed spin 4-V-manifold obtained by gluing \(X\) and \(-W_m\) along the boundary \(\Sigma\). Then we have \(b_2^+(Z) = b_2^+(X) + m\). Note that
\[
w(\Sigma, X, c) = \text{ind}_V D(Z) + \frac{\text{Sign} W_m}{8} = \text{ind}_V D(Z).
\]
Suppose that \(w(\Sigma, X, c) > 0\). Then by Theorem 2.1 we have
\[w(\Sigma, X, c) = \text{ind}_V D(Z) \leq b_2^+(Z) - 1 = b_2^+(X) + m - 1.\]
Similarly, if \(w(\Sigma, X, c) < 0\), then we apply Theorem 2.1 by replacing \(X\) with \(-X\) and by noting \(b_2^+(-Z) = b_2^-(Z)\) and \(\text{ind}_V D(-Z) = -\text{ind}_V D(Z)\) to get
\[-w(\Sigma, X, c) = -\text{ind}_V D(Z) = \text{ind}_V D(-Z) \leq b_2^+(-Z) - 1 = b_2^-(Z) - 1 = b_2^-(X) + m - 1.
\]
Hence the assertion follows. \(\square\)

3. Bounding genera of Brieskorn homology 3-spheres

In this section, we calculate \(w\)-invariants of Brieskorn homology 3-spheres to give estimates of bounding genera from below, and combining with Matsumoto’s result we determine the bounding genera for several examples of Brieskorn homology 3-spheres.

The explicit formula of the \(w\)-invariant for the Brieskorn homology 3-spheres is given in a joint work with M. Furuta [7], and more generally, the invariant for the homology 3-spheres of plumbing type [13] is calculated in [6] by using the Kawasaki \(V\)-index formula [10]. In fact, the \(w\)-invariant of plumbed homology 3-spheres \(\Sigma(\Gamma)\) is essentially equal to the \(\mu\)-invariant defined by W. Neumann [12] and L. Siebenmann [17] as follows.
Theorem 3.1 (N. Saveliev [16], cf. Y. Fukumoto-M. Furuta-M. Ue [8], [5]). Let \( \Sigma(\Gamma) \) be a plumbed homology 3-sphere associated to a weighted tree graph \( \Gamma \). Then there exists a decoration \( \hat{\Gamma} \) of \( \Gamma \) and a V-spin structure \( \hat{c} \) on the associated plumbed 4-V-manifold \( P(\hat{\Gamma}) \) such that

\[
w(\Sigma(\Gamma), P(\hat{\Gamma}), \hat{c}) = -\bar{\mu}(\Sigma(\Gamma)).
\]

To apply Theorem 1.9 efficiently, we must find a “good” spin 4-V-manifold \( X \) to evaluate the \( w \)-invariant \( w(\Sigma, X, c) \). For Seifert homology 3-spheres \( \Sigma = \Sigma(a_1, \ldots, a_n) \), we can take the canonical V-manifold \( X \) to be the total space of the \( D^2 \)-bundle over the V-sphere \( S^2 \) associated with the Seifert fibration \( \Sigma \to S^2 \) which can be regarded as an \( S^1 \)-V-bundle over a V-sphere \( S^2 \). If one of the \( a_i \)'s is even, then \( X \) admits a unique V-spin structure \( c \). In this case, we can take a spin resolution \( P(\Gamma) \) of \( X \) with an even weighted star-shaped graph \( \Gamma \) which satisfies \( \Sigma(\Gamma) \cong \Sigma \). Then a decoration \( \hat{\Gamma} \) of \( \Gamma \) is obtained by drawing circles enclosing linear arms emanating from the central vertex, and the plumbed V-manifold \( P(\hat{\Gamma}) \) is diffeomorphic to \( X \) with induced V-spin structure \( \hat{c} \) isomorphic to \( c \). Then by Theorem 3.1, we have \( w(\Sigma, P(\hat{\Gamma}), \hat{c}) = -\bar{\mu}(\Sigma(\Gamma)) \). When all \( a_i \)'s are odd, we must take other choices of \( X \), such as plumbed V-manifolds \( P(\hat{\Gamma}) \) for some decorated plumbing graph \( \hat{\Gamma} \) or “4-dimensional Seifert fibrations” [8].

In the following, we list several results by Y. Matsumoto [11] of estimates on \( w \)-invariants to determine the bounding genera.

Proposition 3.2 ([11], §4, Proposition 4.4]). Let \( p, q, m \) be positive integers with \( \gcd(p, q) = 1 \). Then

1. \( ||\Sigma(p, q, pqm \pm 1)|| \leq 1 \) for \( m \) even;
2. if \( m \) is odd and \( \text{Arf}(K(p, q)) = 0 \), then

\[
||\Sigma(p, q, pqm \pm 1)|| \leq (p - 1)(q - 1)/2.
\]

Remark 3.3. The Arf invariant of the \( (p, q) \)-torus knot \( K(p, q) \) is as follows [11], §4, Remark):

\[
\text{Arf}(K(p, q)) = \text{Arf}(K(q, p)) = \begin{cases} 
\frac{1 - p^2}{8} \pmod{2}, & p: \text{odd}, q: \text{even}, \\
0, & p, q: \text{odd}.
\end{cases}
\]

Example 3.4. \( ||\Sigma(2, 3, 11)|| \leq 1, ||\Sigma(2, 7, 13)|| \leq 3.\)

As an application of Theorem 1.9 in this case, we have the following:

Proposition 3.5. Let \( p, q \) be coprime positive integers and \( m \) be a positive odd integer.

1. If \( w(p, q, pq \pm 1) > 0 \), then

\[
||\Sigma(p, q, pqm \pm 1)|| \geq w(p, q, pq \pm 1) + 1;
\]

and

2. if \( w(p, q, pq \pm 1) < 0 \), then

\[
||\Sigma(p, q, pqm \pm 1)|| \geq -w(p, q, pq \pm 1).
\]

Proof. Let \( k \) be a non-negative integer such that \( m = 2k + 1 \). Let \( X \) be the disk \( V \)-bundle over \( S^2 \) associated with the Seifert fibration \( \Sigma(p, q, pqm \pm 1) \). Then \( b_2^+(X) = 0 \) and \( b_2^-(X) = 1 \). Since one of the \( p, q, pqm \pm 1 \) is even, \( X \) admits
a unique $V$-spin structure. By Theorem 3.3 and a formula of W. Neumann, $\bar{\mu}(\Sigma(p,q,r)) = \bar{\mu}(\Sigma(p,q,r + 2kpq))$ for any non-negative integer $k$, we have
\[
w(p,q,pqm \pm 1) = w(p,q,pq(2k + 1) \pm 1) = w(p,q,pq \pm 1 + 2kpq) = -\bar{\mu}(\Sigma(p,q,pq \pm 1 + 2kpq)) = -\bar{\mu}(\Sigma(p,q,pq \pm 1)) = w(p,q,pq \pm 1).
\]
Hence the assertion follows. \hfill \qed

The following is an application of Proposition 3.5.

Proposition 3.6. $|\Sigma(2,7,14m - 1)| = 3$ for any positive odd integer $m$.

Proof. The $w$-invariant of $\Sigma(2,7,13)$ is $w(2,7,13) = 2 > 0$ and hence by Proposition 3.5, we have
\[
|\Sigma(2,7,14m - 1)| \geq w(2,7,13) - 0 + 1 = 2 - 0 + 1 = 3.
\]
On the other hand, by Matsumoto’s estimate
\[
|\Sigma(2,7,14m - 1)| = |\Sigma(2,7,14 \cdot (2k - 1) - 1)| \leq \frac{(2 - 1)(7 - 1)}{2} = 3.
\]
Therefore $|\Sigma(2,7,14m - 1)| = 3$ for any odd $m$. \hfill \qed

The following two propositions are cases where we could not determine the bounding genera.

Proposition 3.7. $2 \leq |\Sigma(2,7,14m + 1)| \leq 3$ for any positive odd integer $m$.

Proof. The $w$-invariant of $\Sigma(2,7,15)$ is $w(2,7,15) = -2 < 0$ and hence by Proposition 3.5, we have $|\Sigma(2,7,14m + 1)| \geq -(2) - 1 + 1 = 2$. On the other hand, by Matsumoto’s estimate $|\Sigma(2,7,14m + 1)| \leq (2)(7 - 1)/2 = 3$. Hence the assertion follows. \hfill \qed

Proposition 3.8. $2 \leq |\Sigma(3,5,15m + 1)| \leq 4$, $3 \leq |\Sigma(3,5,15m - 1)| \leq 4$ for any positive odd integer $m$.

Proof. The $w$-invariant of $\Sigma(3,5,16)$ is calculated to be $w(3,5,16) = -2 < 0$ and hence by Proposition 3.5, $|\Sigma(3,5,15m + 1)| \geq -(2) - 1 + 1 = 2$ for $m$ odd. On the other hand, the $w$-invariant of $\Sigma(3,5,14)$ is calculated to be $w(3,5,14) = 2 > 0$ and hence by Proposition 3.5, $|\Sigma(3,5,15m - 1)| \geq 3$ for $m$ odd. Since Arf $(K(3,5)) = 0$, Proposition 3.8 can be applied to obtain $|\Sigma(3,5,15m + 1)| \leq (3 - 1)(5 - 1)/2 = 4$ for any $m$ odd. \hfill \qed

The above estimate is sharpened by Matsumoto for small $m$’s.

Proposition 3.9 ([11 §4, Proposition 4.5]). Suppose that Arf $(K(p,q)) = 0$. Let $m$ be an odd integer such that $0 < m \leq \lfloor p/2 \rfloor \lfloor q/2 \rfloor + 1$. Then $|\Sigma(p,q,pqm \pm 1)| \leq (p - 1)(q - 1)/2 - 1$.

This proposition enables us to determine the bounding genera in the following case.

Proposition 3.10. $|\Sigma(2,7,14m + 1)| = 2$ for $m = 1, 3$. 

Proof. By Proposition 3.7 we have $|\Sigma(2, 7, 14m + 1)| \geq 2$ for any odd $m$. On the other hand, by Matsumoto’s estimates (Proposition 3.9), we have $|\Sigma(2, 7, 14m \pm 1)| \leq (2 - 1)(7 - 1)/2 - 1 = 2$ for $m = 1, 3$ and therefore $|\Sigma(2, 7, 14m + 1)| = 2$ for $m = 1, 3$.

The following is a case where we could not determine the bounding genera even if we use the sharpened estimate.

**Proposition 3.11.** $2 \leq |\Sigma(3, 5, 15m + 1)| \leq 3$ and $|\Sigma(3, 5, 15m - 1)| = 3$ for $m = 1, 3$.

**Proof.** By Proposition 3.8 we have $|\Sigma(3, 5, 15m + 1)| \geq 2$ and $|\Sigma(3, 5, 15m - 1)| \geq 3$. On the other hand, for $m$ odd with $m \leq [3/2][5/2] + 1 = 3$, we have the inequality $|\Sigma(3, 5, 15m \pm 1)| \leq (3 - 1)(5 - 1)/2 - 1 = 3$ for $m = 1, 3$ by Matsumoto’s estimates (Proposition 3.9).

In the case where the $w$-invariant vanishes for Brieskorn homology 3-spheres $\Sigma(p, q, r)$ such as $\Sigma(2, 3, 12k - 1)$ for any integers $k$, we can apply the Fintushel-Stern invariant $R(p, q, r)$ [3]. The explicit formula of the invariant is given in terms of the trigonometric sums by using the Kawasaki $V$-index formula. W. Neumann and D. Zagier [14] derived the useful expression $R(\alpha_1, \ldots, \alpha_n) = 2b - 3$ by using the “$b$-invariant” of the Seifert fibration $\Sigma(\alpha_1, \ldots, \alpha_n)$ where $b$ satisfies $b + \sum_{i=1}^{n} \beta_i/\alpha_i = 1/\prod_{i=1}^{n} \alpha_i$ and $0 < \beta_i < \alpha_i$ with $\beta_i\alpha_i \equiv -1 \pmod{\alpha_i}$. By using this expression we have the following:

**Proposition 3.12.** Let $p, q, r$ be pairwise coprime positive integers. If $R(p, q, r) > 0$, then

$$|\Sigma(p, q, r + kpq)| \geq 1$$

for any non-negative integers $k$.

**Proof.** The $b$-invariants of $\Sigma(p, q, r)$ and $\Sigma(p, q, r + pq)$ coincide, and hence by the formula of W. Neumann and D. Zagier [14] we have $R(p, q, r) = R(p, q, r + pq)$. Therefore if $R(p, q, r) > 0$, then $R(p, q, r + kpq) > 0$, and by the theorem of R. Fintushel and R. Stern [3], $\Sigma(p, q, r + kpq)$ cannot be the boundary of an acyclic 4-manifold for any non-negative integer $k$.

As an application of Proposition 3.12 we have the following:

**Proposition 3.13.** $|\Sigma(2, 3, 12k - 1)| = 1$ for any non-negative integer $k$.

**Proof.** By Matsumoto’s estimate $|\Sigma(2, 3, 12k \pm 1)| \leq 1$ for any integer $k$. The $w$-invariant of $\Sigma(2, 3, 11)$ is $w(2, 3, 11) = 0$; hence we cannot apply Proposition 3.5. However, the Fintushel-Stern invariant $R(2, 3, 11) = 1 > 0$, and hence by Proposition 3.12 we have $|\Sigma(2, 3, 12k - 1)| \geq 1$ for any integer $k$, and the assertion follows. Note that in the case $\Sigma(2, 3, 12k + 1)$, the $w$-invariant of $\Sigma(2, 3, 13)$ is $w(2, 3, 13) = 0$ and the Fintushel-Stern invariant is $R(2, 3, 13) = -1 < 0$, and hence we cannot apply Proposition 3.5 nor Proposition 3.12. In fact, it is known that $\Sigma(2, 3, 13)$ [11, 14] and $\Sigma(2, 3, 25)$ [2] bound contractible smooth manifolds.

Matsumoto also gave estimates for the so-called Casson series of Brieskorn homology spheres.

**Proposition 3.14** (Casson’s series [11, §5, Proposition 5.1]). Let $p, q, r$ be odd integers satisfying $qr + rp + pq = -1$. Then $\|\Sigma(|p|, |q|, |r|)\| \leq 1$. 

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In fact, we have the following:

**Proposition 3.15.** Let $p, q, r$ be odd integers satisfying $qr + rp + pq = -1$. Then $|\Sigma(p, q, r)| = 1$.

**Proof.** By the equality $qr + rp + pq = -1$, we see that the $b$-invariant of $\Sigma(p, q, r)$ is 2, and hence by the formula of W. Neumann and D. Zagier [14], we have $R(\Sigma(p, q, r)) = 2 \cdot 2 - 3 = 1 > 0$ and therefore $|\Sigma(p, q, r)| \geq 1$. Hence the assertion follows.

**Example 3.16.** $|\Sigma(2n + 1, 4n + 1, 4n + 3)| = 1$ for any positive integer $n$.

We have the following estimates of the bounding genera for Brieskorn homology 3-spheres in [11, §5, Proposition 5.5]. The upper bounds are given by Matsumoto.

**Proposition 3.17** (cf. [11, §5, Proposition 5.5]).

| $q$ | Brieskorn $\mathbb{Z}HS^4$ | $||\Sigma|| \leq$ | $w(\Sigma)$ | $R(\Sigma)$ | $|\Sigma|$ |
|-----|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| 3   | $\Sigma(2, 3, 12k \pm 1)$  | $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 3, 12k \pm 5)$ | $\pm 1$        | $0$             | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 5, 20k \pm 1)$ | $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 5, 20k \pm 3)$ | $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 5, 20k \pm 7)$ | $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 5, 20k \pm 9)$ | $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
| 5   | $\Sigma(2, 7, 28k \pm 1)$ | $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 7, 28k \pm 5)$ | $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 7, 28k \pm 9)$ | $\leq 1$        | $0$             | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 7, 28k \pm 11)$| $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 7, 28k \pm 13)$| $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
| 7   | $\Sigma(2, 9, 36k \pm 1)$ | $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 9, 36k \pm 5)$ | $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 9, 36k \pm 7)$ | $\leq 1$        | $0$             | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 9, 36k \pm 11)$| $\pm 1$        | $1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
|     | $\Sigma(2, 9, 36k \pm 13)$| $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |
| 9   | $\Sigma(2, 9, 36k \pm 17)$| $\leq 1$        | $0$             | $-1(+)\pm 1$, $1(-)$ | $\geq 0(+)\pm 1$, $1(-)$ |

**Remark 3.18** ([11, §5, Remark]). Matsumoto improved the estimates for several Brieskorn homology 3-spheres in the above lists. We give estimates below for them.

| $q$ | Brieskorn $\mathbb{Z}HS^4$ | $||\Sigma|| \leq$ | $w(\Sigma)$ | $R(\Sigma)$ | $|\Sigma|$ |
|-----|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| 3   | $\Sigma(2, 3, 12k + 1, k = 1, 2$ | $0$, $k = 1, 2$ | $0$             | $-1$ | $= 0$ |
|     | $\Sigma(2, 3, 12k + 1, k = 1, 2$ | $0$             | $0$             | $-1$ | $= 0$ |
| 5   | $\Sigma(2, 5, 7)$           | $0$             | $0$             | $-1$ | $= 0$ |
|     | $\Sigma(2, 5, 21)$          | $0$             | $0$             | $-1$ | $= 0$ |
| 7   | $\Sigma(2, 7, 28k - 13, k = 1, 2$ | $\leq 2, k = 1, 2$ | $2$             | $-1$ | $= 2$ |
|     | $\Sigma(2, 7, 13)$          | $\leq 3_{cr}$   | $2$             | $-1$ | $= 3$ |
|     | $\Sigma(2, 7, 15)$          | $\leq 2_{cr}$   | $-2$            | $-1$ | $= 2$ |
|     | $\Sigma(2, 7, 19)$          | $0$             | $0$             | $-1$ | $= 0$ |
| 9   | $\Sigma(2, 9, 11)$          | $\leq 3_{cr}$   | $-2$            | $-1$ | $= 2$ |
|     | $\Sigma(2, 9, 36k - 17, 1 \leq k \leq 3$ | $\leq 3, 1 \leq k \leq 3$ | $-2$ | $-1$ | $= 2$ |
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References


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