ON THE LUSTERNIK-SCHNIRELMANN CATEGORY
OF SPACES
WITH 2-DIMENSIONAL FUNDAMENTAL GROUP

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(Communicated by Daniel Ruberman)

Abstract. The following inequality
\[ \text{cat}_{LS} X \leq \text{cat}_{LS} Y + \left\lceil \frac{\text{hd}(X) - r}{r + 1} \right\rceil \]
holds for every locally trivial fibration \( f : X \to Y \) between ANE spaces which
admits a section and has the \( r \)-connected fiber, where \( \text{hd}(X) \) is the homotopical
dimension of \( X \). We apply this inequality to prove that
\[ \text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil \]
for every complex \( X \) with \( \text{cd}(\pi_1(X)) \leq 2 \), where \( \text{cd}(\pi_1(X)) \) denotes the coho-
omological dimension of the fundamental group of \( X \).

1. Introduction

In [DKR] we proved that if the Lusternik-Schnirelmann category of a closed \( n \)-
manifold, \( n \geq 3 \), equals 2, then the fundamental group of \( M \) is free. In the opposite
direction we proved that if the fundamental group of an \( n \)-manifold is free, then
\( \text{cat}_{LS} M \leq n - 2 \). J. Strom proved that \( \text{cat}_{LS} X \leq \frac{2}{3} n \) for every \( n \)-complex, \( n > 4 \),
with free fundamental group [St]. Yu. Rudyak conjectured that the coefficient
2/3 in Strom’s result could be improved to 1/2. Precisely, he conjectured that the function \( f \) defined as
\( f(n) = \max \{ \text{cat}_{LS} M^n \} \) is asymptotically \( \frac{2}{3} n \), where the
maximum is taken over all closed \( n \)-manifolds with free fundamental group.

In this paper we prove Rudyak’s conjecture. Our method gives the same estimate
for \( n \)-complexes. Moreover, we give the same asymptotic upper bound for \( \text{cat}_{LS} \) of
\( n \)-complexes with the fundamental group of cohomological dimension \( \leq 2 \). In view
of this, the following generalization of Rudyak’s conjecture seems to be natural.

Conjecture 1.1. For every \( k \) the function \( f_k \) defined as
\[ f_k(n) = \max \{ \text{cat}_{LS} M^n \mid \text{cd}(\pi_1(M^n)) \leq k \} \]
is asymptotically \( \frac{1}{2} n \).

The smallest \( k \) when it is unknown is 3.
This paper is organized as follows. Section 2 is an introduction to the Lusternik-Schnirelmann category based on an analogy with dimension theory. Section 3 contains a fibration theorem for cat_{LS}. In Section 4 this fibration theorem is applied for the proof of Rudyak’s conjecture.

2. Kolmogorov-Ostrand’s Approach to the Lusternik-Schnirelmann Category

A subset $A \subset X$ of a topological space $X$ is called $X$-contractible if it can be contracted to a point in $X$. A cover $\mathcal{U}$ of a topological space $X$ by $X$-contractible sets is called $X$-contractible. By definition, $\text{cat}_{LS} X \leq n$ if there is an $X$-contractible open cover $\mathcal{U} = \{U_0, \ldots, U_n\}$ of $X$ that consists of $n + 1$ sets.

We recall [CLOT] that a sequence $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_{n+1} = X$ is called categorical of length $n+1$ if each difference $O_{i+1} \setminus O_i$ is contained in an $X$-contractible open set. It was proven in [CLOT] that $\text{cat}_{LS} X \leq n$ if and only if $X$ admits a categorical sequence of length $n+1$.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space $X$. Formally, it is a function $U : A \to 2^X \setminus \{\emptyset\}$ from the index set to the set of nonempty subsets of $X$. Thus, it is allowed to have $U_\alpha = U_\beta$ for $\alpha \neq \beta$. The sets $U_\alpha$ in the family $\mathcal{U}$ will be called elements of $\mathcal{U}$. The multiplicity of $\mathcal{U}$ (or the order) at a point $x \in X$, denoted $\text{Ord}_x \mathcal{U}$, is the number of elements of $\mathcal{U}$ that contain $x$. The multiplicity of $\mathcal{U}$ is defined as $\text{Ord} \mathcal{U} = \sup_{x \in X} \text{Ord}_x \mathcal{U}$. A family $\mathcal{U}$ is a cover of $X$ if $\text{Ord}_x \mathcal{U} \neq 0$ for all $x$. A cover $\mathcal{U}$ is a refinement of another cover $\mathcal{C}$ ($\mathcal{U}$ refines $\mathcal{C}$) if for every $U \in \mathcal{U}$ there exists $C \in \mathcal{C}$ such that $U \subset C$. We recall that the covering dimension of a topological space $X$ does not exceed $n$, $\dim X \leq n$, if for every open cover $\mathcal{C}$ of $X$ there is an open refinement $\mathcal{U}$ with $\text{Ord} \mathcal{U} \leq n+1$.

We recall that a family $\mathcal{F}$ of subsets of a topological space $X$ is called locally finite if for every $x \in X$ there is a neighborhood $U$ of $x$ which has a nonempty intersection at most with finitely many sets from $\mathcal{F}$. The following proposition makes the LS-category analogous to the covering dimension.

**Proposition 2.1.** For a paracompact topological space $X$, $\text{cat}_{LS} X \leq n$ if and only if $X$ admits an $X$-contractible locally finite open cover $\mathcal{V}$ with $\text{Ord} \mathcal{V} \leq n+1$.

**Proof.** If $\text{cat}_{LS} X \leq n$, then by the definition, $X$ admits an open contractible cover that consists of $n+1$ sets and therefore its multiplicity is at most $n+1$.

Let $\mathcal{V}$ be a contractible cover of $X$ of multiplicity $\leq n+1$. We construct a categorical sequence $O_0 \subset O_1 \subset \cdots \subset O_{n+1}$ of length $n+1$. We define $O_1 = \{x \in X \mid \text{Ord}_x \mathcal{V} = n+1\}$. Note that

$$O_1 = \bigcup_{\{V_0, \ldots, V_n\} \subset \mathcal{V}} V_0 \cap \cdots \cap V_n.$$

Note that this is a disjoint union and every nonempty summand is $X$-contractible. Thus $O_1$ is $X$-contractible. Next, we define $O_2 = \{x \in X \mid \text{Ord}_x \mathcal{V} \geq n\}$. Then

$$O_2 \setminus O_1 = \bigcup_{\{V_0, \ldots, V_{n-1}\} \subset \mathcal{V}} (V_0 \cap \cdots \cap V_{n-1} \setminus O_1)$$

is a disjoint union of closed in $O_2$ subsets. Since $\mathcal{V}$ is locally finite, the family of nonempty summands

$$\{V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \mid V_0, \ldots, V_{n-1} \in \mathcal{V}, V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \neq \emptyset\}$$

is...
is locally finite. We recall that every disjoint locally finite family of closed subsets is discrete. Hence there are open (in $O_2$ and hence in $X$) disjoint neighborhoods $W_{V_0, \ldots, V_{n-1}}$ of these summands $V_0 \cap \cdots \cap V_{n-1} \setminus O_1$. By taking $W_{V_0, \ldots, V_{n-1}} \cap V_0$ we may assume that the neighborhood of the summand $V_0 \cap \cdots \cap V_{n-1} \setminus O_1$ is contained in $V_0$. Thus, we may assume that all neighborhoods $W_{V_0, \ldots, V_{n-1}}$ are $X$-contractible. Define $O_3 = \{ x \in X \mid \text{Ord}_x V \geq n-1 \}$ as the union of $(n-1)$-fold intersections and so on. In general, $O_k = \{ x \in X \mid \text{Ord}_x V \geq n-k+2 \}$. Similarly,

$$O_{k+1} \setminus O_k = \bigcup_{\{V_0, \ldots, V_{n-k}\} \subset \mathcal{V}} (V_0 \cap \cdots \cap V_{n-k} \setminus O_k)$$

is a disjoint union of closed in $O_{k+1}$ subsets. Since the family of nonempty summands in this union is locally finite, there are open in $O_{k+1}$, and hence in $X$, disjoint neighborhoods of these summands $V_0 \cap \cdots \cap V_{n-k} \setminus O_k$ such that each neighborhood lies in some $X$-contractible set $V \in \mathcal{V}$.

Then $O_{n+1}$ is the union of elements of $\mathcal{V}$ (1-fold intersections) and hence $O_{n+1} = X$. The categorical sequence conditions are satisfied.

A family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$ if every subfamily of $k$ elements forms a cover of $X$.

**Example.** Let

$$U = \bigcup_{i \in \mathbb{Z}} (m_i, m(i+1) - 1)$$

be the union of disjoint intervals in $\mathbb{R}$ of length $m - 1$ with the distance 1 between any two consecutive intervals. Let $U = \{ T_r U \mid r = 0, \ldots, m-1 \}$ be the family of translates $T_r U = \{ x + r \mid x \in U \}$ of $U$. Clearly, $\mathcal{U}$ is a 3-cover of $\mathbb{R}$ that consists of $m$ subsets.

If we take the intervals of length $m - k$ and the distance $k$,

$$U = \bigcup_{i \in \mathbb{Z}} (m_i, m(i+1) - k),$$

then $\mathcal{U} = \{ T_r U \mid r = 0, \ldots, m-1 \}$ is a $(k+2)$-cover that consists of $m$ subsets. The proof can be derived from the following:

**Proposition 2.2.** A family $\mathcal{U}$ that consists of $m$ subsets of $X$ is an $(n+1)$-cover of $X$ if and only if $\text{Ord}_x \mathcal{U} \geq m-n$ for all $x \in X$.

**Proof.** If $\text{Ord}_x \mathcal{U} < m-n$ for some $x \in X$, then $n+1 = m - (m-n) + 1$ elements of $\mathcal{U}$ do not cover $x$.

If $n+1$ elements of $\mathcal{U}$ do not cover some $x$, then $\text{Ord}_x \mathcal{U} \leq m-(n+1) < m-n$. □

Inspired by the work of Kolmogorov on Hilbert’s 13th problem, Ostrand gave the following characterization of the covering dimension [Os].

**Theorem 2.3 (Ostrand).** A metric space $X$ is of dimension $\leq n$ if and only if for each open cover $\mathcal{C}$ of $X$ and each integer $m \geq n+1$, there exist $m$ disjoint families of open sets $\mathcal{U}_1, \ldots, \mathcal{U}_m$ such that their union $\bigcup \mathcal{U}_i$ is an $(n+1)$-cover of $X$ and it refines $\mathcal{C}$.

Let $\mathcal{U}$ be a family of subsets in $X$ and let $A \subset X$. We denote by $\mathcal{U}|_A = \{ U \cap A \mid U \in \mathcal{U} \}$ the restriction of $\mathcal{U}$ to $A$. 

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Definition 2.4. Let $f : X \to Y$ be a map. An open cover $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ of $X$ is called uniformly $f$-contractible if for every $y \in Y$ there is a neighborhood $V$ such that the restriction $\mathcal{U}_{f^{-1}(V)}$ of $\mathcal{U}$ to the preimage $f^{-1}(V)$ consists of $X$-contractible sets.

We will use uniformly $f$-contractible covers to give in the next section an alternative extension of the Lusternik-Schnirelmann category to mappings. The standard extension $\text{cat}_{LS}(f)$ satisfies the equalities $\text{cat}_{LS}(1_X) = \text{cat}_{LS} X$ and $\text{cat}_{LS}(c) = 0$, where $1_X$ and $c : X \to *$ are the identity map and the constant map respectively. Our extension $\text{cat}_{LS}^*$ satisfies the opposite: $\text{cat}_{LS}^*(c) = \text{cat}_{LS} X$. Also it satisfies $\text{cat}_{LS}^*(1_X) = 0$ for locally contractible spaces (see §3).

Theorem 2.5. Let $\mathcal{U} = \{U_0, \ldots, U_n\}$ be an open cover of a normal topological space $X$. Then for any $m = n, n+1, \ldots, \infty$ there is an open $(n+1)$-cover $\mathcal{U}_m = \{U_0, \ldots, U_m\}$ such that for $k > n$, $U_k = \bigcup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$.

In particular, if $\mathcal{U}$ is $X$-contractible, the cover $\mathcal{U}_m$ is $X$-contractible. If $\mathcal{U}$ is uniformly $f$-contractible for some $f : X \to Z$, the cover $\mathcal{U}_m$ is uniformly $f$-contractible.

Proof. We construct the family $\mathcal{U}_m$ by induction on $m$. For $m = n$ we take $\mathcal{U}_m = \mathcal{U}$.

Let $\mathcal{U}_{m-1} = \{U_0, \ldots, U_{m-1}\}$ be the corresponding family for $m > n$. By Proposition 2.2 Ord$_n \mathcal{U} \geq m-n$. Consider $Y = \{x \in X \mid \text{Ord}_x \mathcal{U} = m-n\}$. Clearly, it is a closed subset of $X$. If $Y = \emptyset$, then by Proposition 2.2 $\mathcal{U}$ is an $n$-cover and we can add $U_m = U_0$ to obtain a desired $(n+1)$-cover. Assume that $Y \neq \emptyset$. We show that for every $i \leq n$, the set $Y \cap U_i$ is closed in $X$. Let $x$ be a limit point of $Y \cap U_i$ that does not belong to $U_i$. Let $U_{i_1}, \ldots, U_{i_{m-n}}$ be the elements of the cover $\mathcal{U}$ that contain $x \in Y$. The limit point condition implies that $(U_{i_1} \cap \cdots \cap U_{i_{m-n}}) \cap (Y \cap U_i) \neq \emptyset$. Therefore $\text{Ord}_y \mathcal{U} = m-n+1$ for all $y \in Y \cap U_i \cap U_{i_0} \cap \cdots \cap U_{i_{m-n}}$, a contradiction.

We define recursively $F_0 = Y \cap U_0$ and $F_{i+1} = Y \cap U_{i+1} \setminus \bigcup_{k=0}^n (U_k)$, where $\{F_i\}_{i=0}^n$ is a disjoint finite family of closed subsets with $\bigcup_{i=0}^n F_i = Y$. Since $X$ is normal, we can fix disjoint open neighborhoods $V_i$ of $F_i$ with $V_i \subset U_i$. Define $U_m = \bigcup_{i=m}^n V_i$. In view of Proposition 2.2 $U_0, \ldots, U_{m-1}, U_m$ is an $(n+1)$-cover.

Clearly, if all $U_i$ are $X$-contractible, $i \leq n$, then $U_m$ is $X$-contractible. If all $U_i$ are uniformly $f$-contractible, for some $f : X \to Z$, then $U_m$ is uniformly $f$-contractible.

Corollary 2.6. For a normal topological space $X$, $\text{cat}_{LS} X \leq n$ if and only if for any $m > n$, $X$ admits an open $(n+1)$-cover by $m$ $X$-contractible sets.

This corollary is a $\text{cat}_{LS}$-analog of Ostrand’s theorem. It also can be found in CLOT with further reference to Cm.

3. Fibration theorems for $\text{cat}_{LS}$

Definition 3.1. The $*$-category $\text{cat}_{LS}^*$ of a map $f : X \to Y$ is the minimal $n$, if it exists, such that there is a uniformly $f$-contractible open cover $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ of $X$.

Note that $\text{cat}_{LS}^* c = \text{cat}_{LS} X$ for a constant map $c : X \to pt$. More generally, $\text{cat}_{LS}^* \pi = \text{cat}_{LS} X$ for the projection $\pi : X \times Y \to Y$.

Theorem 3.2. The inequality $\text{cat}_{LS} X \leq \dim Y + \text{cat}_{LS}^* f$ holds true for any continuous map of a normal space.
Proof. The requirements to the spaces in the theorem are that the Ostrand theorem holds true for $Y$; i.e. they are fairly general (say, $Y$ is normal).

Let $\dim Y = n$ and $\cat_{LS}^*(f) = m$. Let $\mathcal{U} = \{U_0, \ldots, U_m\}$ be a uniformly $f$-contractible cover of $X$. For $y \in Y$ denote by $V_y$ a neighborhood of $y$ from the definition of the uniform $f$-contractibility. In view of Theorem 2.4 there is a refinement $\mathcal{V} = V_0 \cup \cdots \cup V_{n+m}$ of the cover $\{V_y \mid y \in Y\}$ of $Y$ such that each family $V_i$ is disjoint and $\mathcal{V}$ is an $(n+1)$-cover. Let $V_i = \bigcup V_i$.

We apply Theorem 2.3 to extend the family $\mathcal{U}$ to a uniformly $f$-contractible $(m+1)$-cover $\{U_0, \ldots, U_{n+m}\}$. Consider the family $\mathcal{W} = \{f^{-1}(V_i) \cap U_i \} _{0 \leq i \leq n+m}$. Note that it is X-contractible. Thus, in order to get the inequality $\cat_{LS}^*(X) \leq n + m$ it suffices to show that $\mathcal{W}$ is a cover of $X$. Since $\mathcal{V}$ is an $(n+1)$-cover, by Proposition 2.2 every $y \in Y$ is covered by $m + 1$ elements of $\mathcal{V}$, $V_0, \ldots, V_m$. Since $\{U_0, \ldots, U_{n+m}\}$ is an $(m+1)$-cover, the family $U_0, \ldots, U_m$ covers $X$. Therefore the family $f^{-1}(V_0) \cap U_0, \ldots, f^{-1}(V_m) \cap U_m$ covers the fiber $f^{-1}(y)$. Since $y \in Y$ is arbitrary, $\mathcal{W}$ covers all $X$.

Corollary 3.3 (Corollary 9.35 [CLOT], [OW]). Let $p : X \to Y$ be a closed map of ANE. If each fiber $p^{-1}(y)$ is contractible in $X$, then $\cat_{LS}^*(X) \leq \dim Y$.

Proof. In this case $\cat_{LS}^*(p) = 0$. Indeed, since $X$ is an ANE, a contraction of $p^{-1}(y)$ to a point can be extended to a neighborhood $U$. Since the map $p$ is closed there is a neighborhood $V$ of $y$ such that $p^{-1}(V) \subset U$.

We recall that the homotopical dimension of a space $X$, $\text{hd}(X)$, is the minimal dimension of a CW-complex homotopy equivalent to $X$ [CLOT].

Proposition 3.4. Let $p : E \to X$ be a fibration with $(n-1)$-connected fiber where $n = \text{hd}(X)$. Then $p$ admits a section.

Proof. Let $h : Y \to X$ be a homotopy equivalence with the homotopy inverse $g : X \to Y$, where $Y$ is a CW-complex of dimension $n$. Since the fiber of $p$ is $(n-1)$-connected, the map $h$ admits a lift $h' : Y \to E$. Let $H$ be a homotopy connecting $h \circ g$ with $1_X$. By the homotopy lifting property there is a lift $H' : X \times I \to E$ of $H$ with $H|_{X \times \{0\}} = h' \circ g$. Then the restriction $H|_{X \times \{1\}}$ is a section.

We introduce a fiberwise version of Ganea’s fibration. First we recall that the $k$-th Ganea’s fibration $p_k : E_k(Z, z_0) \to Z$ over a path connected space $Z$ with a fixed base point $z_0$ is the fiberwise join product of $k+1$ copies of Serre’s path fibrations $p_0 : PZ \to Z$. We recall that $PZ$ consists of paths $\phi$ in $Z$ with the initial point $z_0$ and $p_0$ takes $\phi$ to $\phi(1)$. Note that $p_0$ is a Hurewicz fibration and since the fiberwise join of Hurewicz fibrations is a Hurewicz fibration, so are all $p_k$ [SV]. Also we note that the fiber of $p_0$ is the loop space $\Omega Z$ and therefore, the fiber of $p_k$ is the join product $*^{k+1}\Omega Z$ of $k+1$ copies of $\Omega Z$ (see [CLOT] for more details).

Theorem 3.5 (Ganea, Svarc). For a path connected normal space $X$ with a non-degenerate base point, $\cat_{LS}(X) \leq k$ if and only if the Ganea fibration $p_k : E_k(Z, z_0) \to Z$ has a section.

The proof can be found in [CLOT], [SV].
Hurewicz fibration. Let ˜p_k : ˜E_k → Z × Z denote the fiberwise join of k + 1 copies of ˜p_0. We call ˜p_k the extended Ganea fibration. Note that for every z_0 ∈ Z, the preimage ˜p_k−1(Z × {z_0}) is homeomorphic to E_k(Z, z_0) and the restriction of ˜p_k to ˜p_k−1(Z × {z_0}) is the Ganea fibration p_k with the base point z_0.

Now let f : X → Y be a locally trivial bundle with a path connected fiber Z and let f admit a section s : Y → X. We define a space
\[ E_0 = \{ \phi ∈ C(I, X) \mid s(f(\phi(I))) = \{\phi(0)\}\} \]
to be the space of all paths φ in X with the initial point s(y) for some y ∈ Y such that the image of φ is contained in the fiber f−1(y). The topology in E_0 is inherited from C(I, X). We define a map ξ_0 : E_0 → X by the formula ξ_0(φ) = φ(1). Then ξ_k : E_k → X is defined as the fiberwise join of k + 1 copies of ξ_0. Formally, we define E_k inductively as a subspace of the join E_0 * E_{k−1}:
\[ E_k = \bigcup\{ \phi * \psi ∈ E_0 * E_{k−1} \mid ξ_0(\phi) = ξ_{k−1}(\psi)\}, \]
which is the union of all intervals [φ, ψ] = φ * ψ with the endpoints φ ∈ E_0 and ψ ∈ E_{k−1} such that ξ_0(φ) = ξ_{k−1}(ψ). There is a natural projection ξ_k : E_k → X that takes all points of each interval [φ, ψ] to φ(0).

Note that when f : X = Z × Y is a trivial bundle and a section s : Y → X is defined by a point z_0 ∈ Z, then E_k = E_k(Z, z_0) × Y and ξ_k = p_k × 1_Y where p_k : E_k → Z is the Ganea fibration.

**Lemma 3.6.** Let f : X → Y be a locally trivial bundle between paracompact spaces with a path connected fiber Z and with a section s : Y → X. Then
\begin{enumerate}
  \item For each k the map ξ_k : E_k → X is a Hurewicz fibration.
  \item The fiber of ξ_k is precisely the join of k + 1 copies of the space of paths from s(f(x)) to x which is homeomorphic to *k+1ΩZ.
  \item ξ_k has a section if and only if X has an open cover U = {U_0, ..., U_k} by sets, each of which admits a fiberwise deformation into s(Y).
\end{enumerate}

**Proof.** i. In view of Dold’s theorem [Do] it suffices to show that ξ_k is a Hurewicz fibration over f−1(U) for all U ∈ U for some locally finite cover of X. We consider a cover U such that f admits a trivialization over U for all U ∈ U, i.e., fiberwise homeomorphisms h_U : f−1(U) → U × Z. Then the section s defines a map σ_U = π_2 ∘ h_U ∘ s : U → Z where π_2 : U × Z → Z is the projection to the second factor. If the map σ_U were constant, the fibration ξ_k over f−1(U) ≃ U × Z would be a Hurewicz fibration being homeomorphic to the product 1_U × p_k. In the general case the fibration ξ_k over f−1(U) is obtained as the pull-back of the extended Ganea fibration ˜p_k : ˜E_k → Z × Z under the map (σ_U × 1_Z) ∘ h_U : f−1(U) → Z × Z. Hence it is a Hurewicz fibration.

   ii. We note that the map ξ_k over the fiber (f−1(x), s(x)) coincides with the Ganea fibration p_k for Z. Therefore, the fiber of ξ_k coincides with the fiber of p_k; i.e., it is *k+1ΩZ.

   iii. Note that when Y = pt, iii turns into the Ganea-Švarc theorem. Thus, iii can be viewed as a fiberwise version of the Ganea-Švarc theorem.

Suppose ξ_k has a section σ : X → E_k. For each x ∈ X the element σ(x) of *k+1ΩF can be presented as the (k + 1)-tuple
\[ \sigma(x) = ((φ_0, t_0), ..., (φ_k, t_k)) \mid \sum t_i = 1, t_i ≥ 0. \]
We use the notation σ(x)_i = t_i. Clearly, σ(x)_i is a continuous function.
A section $\sigma : X \to E_k$ defines a cover $U = \{U_0, \ldots, U_k\}$ of $X$ as follows:

$$U_i = \{x \in X \mid \sigma(x)_i > 0\}.$$ 

By the construction of $U_i$ for $i \leq n$ for every $x \in U_i$ there is a canonical path connecting $x$ with $sf(x)$. We use these paths to contract a fiberwise deformation of $U_i$ into $s(Y)$.

Moreover, Borel’s construction

Theorem 4.1. Let $\text{cat} \leq r-1$ be a locally trivial fibration $f : X \to Y$ with an $r$-connected fiber $F$ admits a section. Then

$$\text{cat}_{LS}^* f \leq \left\lceil \frac{\text{hd}(X) - r}{r + 1} \right\rceil.$$ 

Moreover,

$$\text{cat}_{LS} X \leq \text{cat}_{LS} Y + \left\lceil \frac{\text{hd}(X) - r}{r + 1} \right\rceil.$$ 

Proof. Let $\text{cat}_{LS} Y = m$ and $\text{hd}(X) = n$.

Let $s : Y \to X$ be a section. By Lemma 8.4 if $\xi_k$ is a Hurewicz fibration with the fiber the join product $X^{k+1}F$ of $k + 1$ copies of the loop space $\Omega F$. Thus, it is $(k + (k + 1)r - 1)$-connected. By Proposition 8.4 there is a section $\sigma : X \to E_k$ whenever $k(r + 1) + r \geq n$. The smallest such $k$ is equal to $\left\lceil \frac{n+r}{r+1} \right\rceil$.

By Lemma 8.6 if a section $\sigma : X \to E_k$ defines a cover $U = \{U_0, \ldots, U_k\}$ by the sets fiberwise contractible to $s(Y)$. Let $U_{m+k} = \{U_0, \ldots, U_n+k\}$ be an extension of $U$ to a $(k+1)$-cover of $X$ from Theorem 2.6.

Let $V = \{V_0, \ldots, V_{m+k}\}$ be an open $Y$-contractible $(m+1)$-cover of $Y$. We show that the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in $X$ for all $i$. By Theorem 2.5 $U_i$ is fiberwise contractible into $s(Y)$ for $i \leq m + k$. Hence we can contract $f^{-1}(V_i) \cap U_i$ to $s(V_i)$ in $X$. Then we apply a contraction of $s(V_i)$ to a point in $s(Y)$.

Similarly as in the proof of Theorem 2.2 we show that $\{W_i\}_{i=n+k}$ is a cover of $X$. Since $V$ is an $(m+1)$-cover, by Proposition 2.2 every $y \in Y$ is covered by at least $k+1$ elements $V_{i_0}, \ldots, V_{i_k}$ of $V$. By the construction $U_{i_0}, \ldots, U_{i_k}$ is a cover of $X$. Hence $W_{i_0}, \ldots, W_{i_k}$ covers $f^{-1}(y)$. 

4. The Lusternik-Schnirelmann category of complexes with low dimensional fundamental groups

Theorem 4.1. For every complex $X$ with $cd(\pi_1(X)) \leq 2$ the following inequality holds true:

$$\text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\text{hd}(X) - 1}{2} \right\rceil.$$ 

Proof. Let $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover of $X$. We consider Borel’s construction

$$\begin{array}{cccccc}
\tilde{X} & \leftarrow & \tilde{X} \times E\pi & \longrightarrow & E\pi \\
\downarrow & & \downarrow & & \\
X & \leftarrow_{g} & \tilde{X} \times_{\pi} E\pi & \longrightarrow & B\pi.
\end{array}$$
We claim that there is a section \( s : B\pi \to \tilde{X} \times\pi E\pi \) of \( f \). By the condition \( cd\pi \leq 2 \) we may assume that \( B\pi \) is a complex of dimension \( \leq 3 \). Note that \( f \) is a locally trivial bundle with the fiber \( \tilde{X} \). Since the fiber of \( f \) is simply connected, there is a lift of the 2-skeleton. The condition \( cd\pi \leq 2 \) implies \( H^3(B\pi, E) = 0 \) for every \( \pi \)-module. Thus, we have no obstruction for the lift of the 3-skeleton (see, for example, [Po], [Th] for the basics of obstruction theory with twisted coefficients).

We apply Theorem 3.7 to obtain the inequality

\[
\text{cat}_{LS} X \leq \text{cat}_{LS}(B\pi) + \left\lceil \frac{hd(\tilde{X} \times\pi E\pi) - 1}{2} \right\rceil.
\]

Since \( g \) is a fibration with the homotopy trivial fiber, the space \( \tilde{X} \times\pi E\pi \) is homotopy equivalent to \( X \). Thus, \( hd(\tilde{X} \times\pi E\pi) = hd(X) \). Note that the results of Eilenberg and Ganea [EG] in view of the Stallings-Swan theorem [Sta, Swan] imply that \( \text{cat}_{LS} B\pi = cd\pi \) for all groups \( \pi \). □

**Corollary 4.2.** For every complex \( X \) with free fundamental group,

\[
\text{cat}_{LS} X \leq 1 + \left\lceil \frac{\dim X - 1}{2} \right\rceil.
\]

Note that this estimate is sharp on \( X = S^1 \times \mathbb{C}P^n \).

**Corollary 4.3.** For every 3-dimensional complex \( X \) with free fundamental group, \( \text{cat}_{LS} X \leq 2 \).

This corollary can also be derived from the fact that in the case of a free fundamental group every 2-complex is homotopy equivalent to the wedge of circles and 2-spheres [KR].

It is unclear whether the estimate \( \text{cat}_{LS} X \leq 2 + \left\lceil \frac{\dim X - 1}{2} \right\rceil \) is sharp for complexes with \( cd(\pi_1(X)) = 2 \). It is sharp if the answer to the following question is affirmative.

**Question 4.4.** Does there exists a 4-complex \( K \) with free fundamental group and with \( \text{cat}_{LS}(K \times S^1) = 4 \)?

Indeed, for \( X = K \times S^1 \) we would have the equality \( 4 = 2 + \left\lceil \frac{5-1}{2} \right\rceil \). Note that \( cd(\pi_1(X)) = 2 \).

**Acknowledgement**

The author is thankful to the referee for valuable remarks.

**References**


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