ON STRINGS OF CONSECUTIVE INTEGERS
WITH A DISTINCT NUMBER OF PRIME FACTORS

JEAN-MARIE DE KONINCK, JOHN B. FRIEDLANDER, AND FLORIAN LUCA

Abstract. Let \( \omega(n) \) be the number of distinct prime factors of \( n \). For any positive integer \( k \) let \( n = n_k \) be the smallest positive integer such that \( \omega(n+1), \ldots, \omega(n+k) \) are mutually distinct. In this paper, we give upper and lower bounds for \( n_k \). We study the same quantity when \( \omega(n) \) is replaced by \( \Omega(n) \), the total number of prime factors of \( n \) counted with repetitions.

Let \( \omega(n) \) and \( \Omega(n) \) denote respectively the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \) counted with repetitions. For any positive integer \( k \) let \( m = m_k \) be the smallest positive integer \( m \) such that \( \Omega(m+1), \ldots, \Omega(m+k) \) are mutually distinct. Using a computer, we easily obtain that \( n_2 = 4 \), \( n_3 = 27 \), \( n_4 = 416 \), \( n_5 = 14321 \), \( n_6 = 461889 \), \( n_7 = 46908263 \) and \( n_8 = 7362724274 \), and also that \( m_2 = 2 \), \( m_3 = 5 \), \( m_4 = 14 \), \( m_5 = 59 \), \( m_6 = 725 \), \( m_7 = 6317 \), \( m_8 = 189374 \), \( m_9 = 755967 \) and \( m_{10} = 683441870 \). In this paper, we give upper and lower bounds for \( n_k \) and \( m_k \).

We start by improving these trivial estimates as follows.

Theorem 1. The inequality
\[
n_k \geq \exp((2 + o(1))k \log k)
\]
holds as \( k \to \infty \). Furthermore, the inequality
\[
m_k \geq \exp((1/2 + o(1))k \log k)
\]
holds as \( k \to \infty \). 

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The problem of finding lower and upper bounds for \( n_k \) and \( m_k \) was raised in the recent book \([1]\) by the first author. We remark that, after writing this paper, we noticed that the first of these bounds is essentially equivalent to one due to Erdős \([2]\). We were somewhat surprised that we could not find any other work on these problems.

**Proof.** We start with the first inequality. Assume that \( \omega(n + 1), \ldots, \omega(n + k) \) are mutually distinct. Let \( \varepsilon \in (0, 1) \) be arbitrarily small but fixed. Put \( s = \lfloor k^{1-\varepsilon} \rfloor \).

Let \( i_1, \ldots, i_s \) be \( s \) distinct integers in \( \{1, \ldots, k\} \) such that \( \omega(n + i_j) \geq k - j \) for \( j = 1, \ldots, s \). Let \( A_{i_j} \) be the set of prime factors of \( n + i_j \). Note that if \( j \neq \ell \) and \( p \in A_{i_j} \cap A_{i_\ell} \), then \( p \mid (n + i_j) - (n + i_\ell) = (i_j - i_\ell) \) and \( 1 \leq |i_j - i_\ell| \leq k - 1 \). Since \( \omega(m) \ll \log m / \log \log m \) holds for all positive integers, we get that

\[
\# \left( A_{i_j} \cap A_{i_\ell} \right) \leq c_1 \frac{\log k}{\log \log k}
\]

holds for all \( j \neq \ell \) with some absolute constant \( c_1 \). By the Principle of Inclusion and Exclusion,

\[
\# \left( \bigcup_{j=1}^{s} A_{i_j} \right) \geq \sum_{j=1}^{s} \# A_{i_j} - \sum_{1 \leq j \leq \ell \leq s} \# \left( A_{i_j} \cap A_{i_\ell} \right)
\]

\[
\geq ks - s\frac{(s+1)}{2} - c_1 \frac{s}{\log \log k} \geq (1 - \varepsilon)k^{2-\varepsilon}
\]

provided that \( k > k_\varepsilon \). Thus, using the Prime Number Theorem once more, we have

\[
(n + k)^s \geq \prod_{j=1}^{s} (n + i_j) \geq \prod_{1 \leq i \leq (1-\varepsilon)k^{2-\varepsilon}} p_i
\]

\[
\geq \exp((2 - \varepsilon + o(1))ks \log k)
\]

as \( k \to \infty \). This leads to \( n \geq \exp((2 - \varepsilon + o(1))k \log k) \) as \( k \to \infty \), which implies the desired conclusion since \( \varepsilon \in (0, 1) \) was arbitrary.

We now deal with the second inequality. Let \( m = m_k \). For any given prime number \( p \) and positive integer \( n \) we let \( \nu_p(n) \) be the exact exponent with which \( p \) appears in the prime factorization of \( n \). For each \( p \leq k \) let \( i_p \in \{1, \ldots, k\} \) be such that

\[
\nu_p(m + i) = \max_{1 \leq i \leq k} \nu_p(m + i).
\]

If more than one value for \( i_p \in \{1, \ldots, k\} \) exists for which equality (1) is satisfied, we simply pick one of them. Clearly, the set \( \mathcal{I} \) of indices \( i_p \) so chosen satisfies

\[
\# \mathcal{I} \leq \pi(k).
\]

An elementary argument (see, for example, Lemma 2 in \([3]\)) shows that if we write

\[
m + i = a_ib_i,
\]

where the largest prime factor of \( a_i \) is \( \leq k \) and the smallest prime factor of \( b_i \) exceeds \( k \), then

\[
\prod_{1 \leq i \leq k \atop i \in \mathcal{I}} a_i \leq k^k.
\]
In particular,

\[(3) \quad \sum_{1 \leq i \leq k \atop i \notin I} \Omega(a_i) = \Omega \left( \prod_{1 \leq i \leq k \atop i \notin I} a_i \right) < \frac{k \log k}{\log 2} < 2k \log k.\]

Let

\[\mathcal{J} = \{i \notin I : \Omega(a_i) > k^{1/2}\}.\]

Then inequality 3 shows that

\[(4) \quad \#\mathcal{J} < 2k^{1/2} \log k.\]

Finally, let

\[\mathcal{K} = \{i \notin I \cup \mathcal{J} : \Omega(m + i) \leq k^{2/3}\}.\]

Since the numbers \(\Omega(m + j)\) are distinct for \(j = 1, \ldots, k\), it follows that

\[(5) \quad \#\mathcal{K} \leq k^{2/3}.\]

Let \(\mathcal{S} = \{1, \ldots, k\} - (I \cup \mathcal{J} \cup \mathcal{K})\) and put \(s = \#\mathcal{S}\). Let \(\varepsilon > 0\) be fixed. Estimates 2, 11 and 15 show that

\[s \geq k - \pi(k) - 2k^{1/2} \log k - k^{2/3} > (1 - \varepsilon)k,\]

provided that \(k > k_{\varepsilon}\). Note that if \(i \in \mathcal{S}\), then

\[\Omega(a_i) \leq k^{1/2} = \left(k^{2/3}\right)^{3/4} \leq \Omega(m + i)^{3/4},\]

so that

\[\Omega(b_i) = \Omega(m + i) - \Omega(a_i) \geq \Omega(m + i) - \Omega(m + i)^{3/4} \geq (1 - \varepsilon)\Omega(m + i)\]

for all \(i \in \mathcal{S}\), assuming that \(k > k_{\varepsilon}\). Thus, since the \(\Omega(m + i)\) are distinct,

\[(m + k)^s \geq \prod_{i \in \mathcal{S}} b_i > k^{\sum_{i \in \mathcal{S}} \Omega(b_i)} \geq \left(k^{\sum_{i \in \mathcal{S}} \Omega(m + i)}\right)^{(1 - \varepsilon)} > \left(k^{\sum_{j=1}^s \Omega(m + i)}\right)^{(1 - \varepsilon)} > \exp((1/2 - \varepsilon)s^2 \log k).\]

Hence,

\[m_k \geq \exp((1/2 - \varepsilon)s \log k) > \exp((1/2 - 2\varepsilon)k \log k).\]

Since \(\varepsilon > 0\) is arbitrary, we get the desired conclusion.

We next turn our attention to upper bounds for \(n_k\) and \(m_k\). We have the following result.

**Theorem 2.** The inequalities

\[n_k \leq \exp\left(\frac{6}{\log 2 + o(1)}k^2(\log k)^2\right)\]

and

\[m_k \leq \exp\left(\frac{4}{\log 2 + o(1)}k^2(\log k)^2\right)\]

hold as \(k \to \infty\).
Proof: We assume that \( k \geq 2 \). Again, we deal first with \( n_k \). We let \( A \) be a positive integer depending on \( k \), to be determined later. We let \( q_1 < q_2 < \cdots < q_m < \cdots \) be all the consecutive prime numbers exceeding \( k \). For \( j = 1, \ldots, k \), we put \( T_j = j(j-1)/2 \) and

\[
M_j = \prod_{\ell=T_j,A+1}^{T_{j+1}} q_\ell.
\]

Put \( M = \prod_{j=1}^{k} M_j \) and let \( N \) be the smallest positive integer such that \( M_j \) divides \( N + j \) for each \( j \) with \( 1 \leq j \leq k \). Such an integer \( N \) exists by the Chinese Remainder Theorem. Note that \( N + k < M \). Indeed, if not, then \( N = M - i \) for some \( i \in \{1, \ldots, k\} \), and by taking some \( j \neq i \in \{1, \ldots, k\} \) (which exists because \( k \geq 2 \)), we would get that \( M_j \mid N + j = M + (j - i) \); therefore \( M_j \mid j - i \), which is impossible. Let \( n = MA + N \) be a positive integer with \( \lambda \in [M, 2M] \). Note that

\[
n + j = M\lambda + (N + j) = M_j ((M/M_j)\lambda + (N + j)/M_j), \quad j = 1, \ldots, k.
\]

By setting \( A_j = (N + j)/M_j \) and \( B_j = M/M_j \), it follows that

\[
jA = T_{j+1}A - T_jA = \omega(M_j) \leq \omega(n + j) \leq jA + \omega(B_j\lambda + A_j),
\]

so that if \( \lambda \) is such that

\[
\omega(B_j\lambda + A_j) < A, \quad \text{for all } j = 1, \ldots, k - 1,
\]

then

\[
jA \leq \omega(n + j) < jA + A \leq \omega(n + j + 1) \quad \text{for all } j = 1, \ldots, k - 1.
\]

Hence, we certainly have that \( \omega(n + 1), \ldots, \omega(n + k) \) are pairwise distinct.

It now remains to estimate \( A \) and \( M \) such that we can guarantee the existence of a positive integer \( \lambda \in [M, 2M] \) with the property that all of the inequalities (6) hold.

We claim that \( A_j \) and \( B_j \) are coprime. Indeed, to see this, note first that

\[
B_j = M/M_j = \prod_{1 \leq \ell \leq k, \ell \neq j} M_\ell.
\]

If there exists a prime \( p \mid (A_j, B_j) \), then we get that \( p \mid M_\ell \) for some \( \ell \neq j \). Since \( M_\ell \mid N + \ell \), we get that \( p \mid N + \ell \). But obviously \( p \mid A_j \mid N + j \); therefore \( p \mid (N + \ell) - (N + j) = (\ell - j) \), and \( 1 \leq |\ell - j| < k \). Thus \( p < k \), which is impossible because all prime factors of \( M \) exceed \( k \), proving the claim.

Now note that since \( N + k \leq M \), we have

\[
B_j\lambda + A_j \leq \frac{1}{M_j} (M\lambda + N + k) < \frac{2M\lambda}{M_j} \leq \frac{4M^2}{M_j} < M^2
\]

for all \( \lambda \in [M, 2M] \) and \( j = 1, \ldots, k \), when \( k \geq 3 \), because in this case all primes dividing \( M \) exceed 4 and \( N + k < M \). Thus, writing \( \tau(m) \) for the number of divisors of \( m \), we obtain

\[
\tau(B_j\lambda + A_j) \leq 2 \sum_{d \mid B_j\lambda + A_j, d \leq M} 1.
\]
Summing the above inequality over all \( \lambda \in [M,2M] \) and changing the order of summation, we find that

\[
\sum_{\lambda \in [M,2M]} \tau(B_j \lambda + A_j) \leq 2 \sum_{\lambda \in [M,2M]} \sum_{d \leq M} \sum_{d \mid B_j \lambda + A_j} 1 \leq 2 \sum_{d \leq M} \sum_{\lambda \in [M,2M]} \sum_{B_j \lambda + A_j \equiv 0 \pmod{d}} 1 \leq 2 \sum_{d \leq M} \left( \left\lfloor \frac{M}{d} \right\rfloor + 1 \right) \leq 4M \sum_{d \leq M} \frac{1}{d} \leq 4M(\log M + 1).
\]

(7)

In the above chain of inequalities, we used the fact that, since \( A_j \) and \( B_j \) are coprime, the congruence \( B_j \lambda + A_j \equiv 0 \pmod{d} \) has at most \( \left\lfloor M/d \right\rfloor + 1 \) solutions \( \lambda \in [M,2M] \). This is true assuming that \( d \) and \( B_j \) are coprime. When \( d \) and \( B_j \) are not coprime, then this congruence has no integer solution \( \lambda \). Thus, if \( \lambda \) is such that \( \omega(B_j \lambda + A_j) \geq A \), then \( \tau(B_j \lambda + A_j) \geq 2^A \) and inequality (7) shows that

\[
\#\{\lambda \in [M,2M] : \omega(B_j \lambda + A_j) \geq A\} \leq \frac{4M(\log M + 1)}{2^A}.
\]

Summing the above inequality over \( j = 1, \ldots, k-1 \), we get that

\[
\sum_{j=1}^{k-1} \#\{\lambda \in [M,2M] : \omega(B_j \lambda + A_j) \geq A\} \leq \frac{4(k-1)M(\log M + 1)}{2^A}.
\]

(8)

Hence, assuming that

\[
M > \frac{4(k-1)M(\log M + 1)}{2^A},
\]

we see that there exists a number \( \lambda \in [M,2M] \) such that all inequalities (6) are satisfied, and therefore

\[
n < n + 1 = M\lambda + N + 1 < 2M^2 + M < 3M^2.
\]

(9)

It remains to estimate the size of the minimal integer \( A \) depending on \( k \) such that inequality (9) holds. Clearly, \( M \) has \( Ak(k+1)/2 \) prime factors, which are all the consecutive primes starting with the first one exceeding \( k \). Thus, by the Prime Number Theorem,

\[
M = \exp((1/2 + o(1))k^2A(\log k^2A))
\]

as \( k \to \infty \) uniformly in \( A \geq 1 \). Thus, inequality (9) is fulfilled when

\[
A \log 2 > \log(4(k-1)) + \log(\log M + 1) = (3 + o(1)) \log k + O(\log \log k + \log A).
\]

This shows that given \( \varepsilon > 0 \), we may choose \( A = \lfloor (3/\log 2 + \varepsilon) \log k \rfloor \), and then inequality (9) is fulfilled once \( k > k_\varepsilon \). With this choice of \( A \), we have that

\[
M < \exp((3/\log 2 + 2\varepsilon)k^2(\log k)^2)
\]

provided that \( k \) is sufficiently large, and now inequality (9) shows that

\[
n < \exp((6/\log 2 + 5\varepsilon)k^2(\log k)^2)
\]

if \( k \) is sufficiently large with respect to \( \varepsilon \), which implies the desired estimate as \( k \to \infty \), since \( \varepsilon \in (0,1) \) may be chosen arbitrarily small.

We now turn our attention to the upper bound for \( m_k \). We follow the same line of attack, based on the Chinese Remainder Theorem, although the details are somewhat different.
We assume again that \( k \geq 2 \); we take \( M_0 = (k!)^2 \), \( M_j = q_j^{i_A} \) for \( j = 1, \ldots, k \), and let \( N \) be the smallest positive integer \( m \) in the arithmetic progression

\[
m + j \equiv 0 \pmod{M_j}, \quad j = 0, \ldots, k.
\]

Here,

\[
M = \prod_{j=0}^{k} M_j = (k!)^2 \prod_{j=1}^{k} q_j^{i_A} = \exp((1/2 + o(1))k^2A\log k)
\]

as \( k \to \infty \). Let \( m = M\lambda + N \) again be such that \( \lambda \in [M, 2M] \). Then

\[
m + i = iM_i \left( \frac{M}{iM_i} + \frac{N + i}{iM_i} \right), \quad \text{for all } i = 1, \ldots, k,
\]

so that if we set \( A_i = (N + i)/(iM_i) \) and \( B_i = M/(iM_i) \), we have

\[
\Omega(m + i) = \Omega(i) + \Omega(M_i) + \Omega(B_i\lambda + A_i).
\]

Now since \( i \leq k \), it follows that \( \Omega(i) \leq (\log k)/\log 2 \). Furthermore, \( \Omega(M_i) = iA \). Thus, if

\[
\Omega(B_i\lambda + A_i) < A - (\log k)/\log 2, \quad \text{for all } i = 1, \ldots, k - 1,
\]

then

\[
\Omega(m + i) < A(i + 1) = \Omega(M_{i+1}) < \Omega(m + i + 1), \quad \text{for all } i = 1, \ldots, k - 1,
\]

which certainly shows that \( \Omega(m + 1), \ldots, \Omega(m + k) \) are pairwise distinct.

Next, let \( i \in \{1, \ldots, k\} \). As in the analysis of the \( n_k \) case, one shows that \( A_i \) and \( B_i \) are coprime and that \( B_i\lambda + A_i < M\lambda + N + k < 2M^2 + M < 3M^2 \). Furthermore, since \( M_0/i \) is a divisor of \( B_i \) for all \( i = 1, \ldots, k \) and \( M_0/i = (k!)^2/i \) is divisible by all primes \( p \leq k \), it follows that the smallest prime factor of \( B_i\lambda + A_i \) exceeds \( k \).

Write

\[
B_i\lambda + A_i = U_iV_i,
\]

where all prime factors of \( U_i \) are \( \leq M^{1/2} \) and all prime factors of \( V_i \) are \( > M^{1/2} \). Clearly, \( \Omega(V_i) \leq 4 \) because \( M > 9 \). We will now bound from above the number of \( \lambda \) such that \( U_i \) is not squarefree for some \( i = 1, \ldots, k \). There exists a prime \( p \in [k, M^{1/2}] \) such that \( B_i\lambda + A_i \equiv 0 \pmod{p^2} \). For a fixed prime \( p \), the number of integers \( \lambda \in [M, 2M] \) for which the above congruence holds is at most \( [M/p^2] + 1 \leq 2M/p^2 \). Thus,

\[
\#\{\lambda \in [M, 2M] : p^2 \mid B_i\lambda + A_i \text{ for some } p \in [k, M^{1/2}]\} \leq 2M \sum_{p > k} \frac{1}{p^2} \ll \frac{M}{k \log k}
\]

uniformly in \( i \in \{1, \ldots, k\} \). Summing this over all \( i \in \{1, \ldots, k\} \), we get that

\[
\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : U_i \text{ is not squarefree}\} \ll \frac{M}{\log k}.
\]

In particular, if \( k \) is large, then

\[
\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \Omega(B_i\lambda + A_i) > \omega(B_i\lambda + A_i) + 4\} \ll \frac{M}{2}.
\]
Let \( \lambda \) be some number in \([M, 2M]\) such that \( \Omega(B_i \lambda + A_i) \leq \omega(B_i \lambda + A_i) + 4 \). As we have seen, there are at least \( M/2 \) such values for \( \lambda \). If there is such a positive integer \( \lambda \) with the additional property that

\[
\omega(B_i \lambda + A_i) < A - \frac{\log k}{\log 2} - 4, \quad \text{for all } i = 1, \ldots, k, \tag{12}
\]

it follows that inequalities \((11)\) are satisfied. So, let us look at the number of \( \lambda \in [M, 2M] \) such that at least one of the inequalities \((12)\) fails. The argument used in the proof of the upper bound for \( n_k \) (based on the fact that \( \tau(m) \geq 2^{\omega(m)} \)) shows that

\[
\sum_{i=1}^{k} \# \{ \lambda \in [M, 2M] : \omega(B_i \lambda + A_i) \geq A - (\log k)/\log 2 - 4 \} \leq \frac{4(k - 1)M(\log M)}{2^{A - (\log k)/(\log 2) - 4}}.
\]

Thus, if

\[
\frac{4(k - 1)M(\log M)}{2^{A - (\log k)/(\log 2) - 4}} < \frac{M}{2}, \tag{13}
\]

then the number of \( \lambda \in [M, 2M] \) such that at least one of the inequalities \((12)\) fails is \( < M/2 \). Since we have \( \geq M/2 \) values of \( \lambda \) to choose from, it follows that one can indeed choose such a value of \( \lambda \) for which all inequalities in \((11)\) hold. Clearly, with such a value of \( \lambda \), we have that \( m_k \leq m = M \lambda + N < 3M^2 \). Inequality \((19)\) is equivalent, via estimate \((10)\), to

\[
A \log 2 - (\log k) - 4 \log 2 \geq \log(8(k - 1)) + 2 \log k + O(\log \log k + \log A),
\]

which holds if we first fix \( \varepsilon > 0 \), then take \( k > k_\varepsilon \), and finally choose \( A = [(4/\log 2 + \varepsilon) \log k] \). With this choice of \( A \), we have

\[
M < \exp((2/\log 2 + 2\varepsilon)(\log k)^2)
\]

once \( k > k_\varepsilon \). Therefore,

\[
m_k < 3M^2 < \exp((4/\log 2 + 5\varepsilon)(\log k)^2)
\]

if \( k \) is large with respect to \( \varepsilon \), which implies the desired inequality since \( \varepsilon > 0 \) can be chosen arbitrarily small.

\( \square \)

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DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ LAVAL, QUÉBEC G1K 7P4, CANADA
E-mail address: jmdk@mat.ulaval.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 3G3, CANADA
E-mail address: frdlndr@math.toronto.edu

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, C.P. 58089, MORELIA, MICHOACÁN, MÉXICO
E-mail address: fluca@matmor.unam.mx