ON STRINGS OF CONSECUTIVE INTEGERS
WITH A DISTINCT NUMBER OF PRIME FACTORS

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ABSTRACT. Let \( \omega(n) \) be the number of distinct prime factors of \( n \). For any positive integer \( k \) let \( n = n_k \) be the smallest positive integer such that \( \omega(n+1), \ldots, \omega(n+k) \) are mutually distinct. In this paper, we give upper and lower bounds for \( n_k \). We study the same quantity when \( \omega(n) \) is replaced by \( \Omega(n) \), the total number of prime factors of \( n \) counted with repetitions.

Let \( \omega(n) \) and \( \Omega(n) \) denote respectively the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \) counted with repetitions. For any positive integer \( k \) let \( n = n_k \) be the smallest positive integer such that \( \Omega(n+1), \ldots, \Omega(n+k) \) are mutually distinct. Using a computer, we easily obtain that
\[
\begin{align*}
n_2 &= 4, \\
n_3 &= 27, \\
n_4 &= 416, \\
n_5 &= 461889, \\
n_6 &= 14321, \\
n_7 &= 46908263 \\
n_8 &= 7362724274, \\
n_9 &= 461889, \\
n_{10} &= 7362724274, \\
n_{11} &= 461889, \\
n_{12} &= 7362724274, \\
n_{13} &= 461889, \\
n_{14} &= 7362724274, \\
n_{15} &= 461889, \\
n_{16} &= 7362724274, \\
n_{17} &= 461889, \\
n_{18} &= 7362724274, \\
n_{19} &= 461889, \\
n_{20} &= 7362724274.
\end{align*}
\]

We give upper and lower bounds for \( n_k \) and \( m_k \). Let \( p_i \) be the \( i \)-th prime number. Let \( n = n_k \). Since the set \( \{\omega(n+j) : j = 1, \ldots, k\} \) consists of \( k \) nonnegative integers, it follows that one of \( n+j \) for \( j = 1, \ldots, k \) must have at least \( k \) distinct prime factors. Thus,
\[
n + k \geq \prod_{i=1}^{k} p_i = \exp((1 + o(1))p_k) = \exp((1 + o(1))k \log k)
\]
as \( k \to \infty \) by the Prime Number Theorem; therefore
\[
n_k \geq \exp((1 + o(1))k \log k) \quad \text{as} \quad k \to \infty.
\]

Similarly, letting \( m = m_k \), we get that \( \Omega(m+i) \geq k \) for some \( i \in \{1, \ldots, k\} \). Thus, \( m + k \geq 2^k \), giving \( m_k \geq \exp((\log 2 + o(1))k) \) as \( k \to \infty \).

We start by improving these trivial estimates as follows.

**Theorem 1.** The inequality
\[
n_k \geq \exp((2 + o(1))k \log k)
\]
holds as \( k \to \infty \). Furthermore, the inequality
\[
m_k \geq \exp((1/2 + o(1))k \log k)
\]
holds as \( k \to \infty \).
The problem of finding lower and upper bounds for \( n_k \) and \( m_k \) was raised in the recent book \([1]\) by the first author. We remark that, after writing this paper, we noticed that the first of these bounds is essentially equivalent to one due to Erdős \([2]\). We were somewhat surprised that we could not find any other work on these problems.

**Proof.** We start with the first inequality. Assume that \( \omega(n+1), \ldots, \omega(n+k) \) are mutually distinct. Let \( \varepsilon \in (0, 1) \) be arbitrarily small but fixed. Put \( s = \lfloor k^{1-\varepsilon} \rfloor \).

Let \( i_1, \ldots, i_s \) be \( s \) distinct integers in \( \{1, \ldots, k\} \) such that \( \omega(n+i_j) \geq k-j \) for \( j = 1, \ldots, s \). Let \( \mathcal{A}_{i_j} \) be the set of prime factors of \( n+i_j \). Note that if \( j \neq \ell \) and \( p \in \mathcal{A}_{i_j} \cap \mathcal{A}_{i_\ell} \), then \( p \mid (n+i_j) - (n+i_\ell) = (i_j - i_\ell) \) and \( 1 \leq |i_j - i_\ell| \leq k-1 \). Since \( \omega(m) \ll \log m / \log \log m \) holds for all positive integers, we get that

\[
\#(\mathcal{A}_{i_j} \cap \mathcal{A}_{i_\ell}) < c_1 \frac{\log k}{\log \log k}
\]

holds for all \( j \neq \ell \) with some absolute constant \( c_1 \). By the Principle of Inclusion and Exclusion,

\[
\# \left( \bigcup_{j=1}^s \mathcal{A}_{i_j} \right) \geq \sum_{j=1}^s \# \mathcal{A}_{i_j} - \sum_{1 \leq j < \ell \leq s} \# (\mathcal{A}_{i_j} \cap \mathcal{A}_{i_\ell}) \geq ks - s(s+1)/2 - c_1 s \frac{\log k}{\log \log k} > (1 - \varepsilon) k^{2-\varepsilon}
\]

provided that \( k > k_\varepsilon \). Thus, using the Prime Number Theorem once more, we have

\[
(n+k)^s \geq \prod_{j=1}^s (n+i_j) \geq \prod_{1 \leq i < (1-\varepsilon)k^{2-\varepsilon}} p_i \geq \exp((2 - \varepsilon + o(1))k s \log k)
\]

as \( k \to \infty \). This leads to \( n \geq \exp((2 - \varepsilon + o(1))k \log k) \) as \( k \to \infty \), which implies the desired conclusion since \( \varepsilon \in (0, 1) \) was arbitrary.

We now deal with the second inequality. Let \( m = m_k \). For any given prime number \( p \) and positive integer \( n \) we let \( \nu_p(n) \) be the exact exponent with which \( p \) appears in the prime factorization of \( n \). For each \( p \leq k \) let \( i_p \in \{1, \ldots, k\} \) be such that

\[
(1) \quad \nu_p(m+i_p) = \max_{1 \leq i \leq k} \nu_p(m+i).
\]

If more than one value for \( i_p \in \{1, \ldots, k\} \) exists for which equality (1) is satisfied, we simply pick one of them. Clearly, the set \( \mathcal{I} \) of indices \( i_p \) so chosen satisfies

\[
(2) \quad \# \mathcal{I} \leq \pi(k).
\]

An elementary argument (see, for example, Lemma 2 in \([3]\)) shows that if we write

\[
m + i = a_1 b_1,
\]

where the largest prime factor of \( a_i \) is \( \leq k \) and the smallest prime factor of \( b_i \) exceeds \( k \), then

\[
\prod_{1 \leq i \leq k} a_i \leq k^k.
\]
In particular,
\begin{equation}
\sum_{i \in \mathcal{I}} \Omega(a_i) = \Omega \left( \prod_{i \in \mathcal{I}} a_i \right) < \frac{k \log k}{\log 2} < 2k \log k.
\end{equation}

Let
\[ \mathcal{J} = \{ i \notin \mathcal{I} : \Omega(a_i) > k^{1/2} \}. \]

Then inequality (3) shows that
\begin{equation}
\# \mathcal{J} < 2k^{1/2} \log k.
\end{equation}

Finally, let
\[ \mathcal{K} = \{ i \notin \mathcal{I} \cup \mathcal{J} : \Omega(m+i) \leq k^{2/3} \}. \]

Since the numbers \( \Omega(m+j) \) are distinct for \( j = 1, \ldots, k \), it follows that
\begin{equation}
\# \mathcal{K} < k^{2/3}.
\end{equation}

Let \( \mathcal{S} = \{1, \ldots, k\} - (\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}) \) and put \( s = \# \mathcal{S} \). Let \( \varepsilon > 0 \) be fixed. Estimates (2), (4) and (5) show that
\[ s \geq k - \pi(k) - 2k^{1/2} \log k - k^{2/3} > (1 - \varepsilon)k, \]

provided that \( k > k_\varepsilon \). Note that if \( i \in \mathcal{S} \), then
\[ \Omega(a_i) \leq k^{1/2} = (k^{2/3})^{3/4} \leq \Omega(m+i)^{3/4}, \]

so that
\[ \Omega(b_i) = \Omega(m+i) - \Omega(a_i) \geq \Omega(m+i) - \Omega(m+i)^{3/4} \geq (1 - \varepsilon)\Omega(m+i) \]

for all \( i \in \mathcal{S} \), assuming that \( k > k_\varepsilon \). Thus, since the \( \Omega(m+i) \) are distinct,
\begin{align*}
(m+k)^s & \geq \prod_{i \in \mathcal{S}} b_i > k^{\sum_{i \in \mathcal{S}} \Omega(b_i)} \left( k^{\sum_{i \in \mathcal{S}} \Omega(m+i)} \right)^{(1-\varepsilon)} \\
& > \left( k^{\sum_{i=1}^s} \Omega(m+i) \right)^{(1-\varepsilon)} > \exp((1/2 - \varepsilon)s^2 \log k).
\end{align*}

Hence,
\[ m_k \geq \exp((1/2 - \varepsilon)s \log k) > \exp((1/2 - 2\varepsilon)k \log k). \]

Since \( \varepsilon > 0 \) is arbitrary, we get the desired conclusion. \( \square \)

We next turn our attention to upper bounds for \( n_k \) and \( m_k \). We have the following result.

**Theorem 2.** The inequalities
\[ n_k \leq \exp((6/\log 2 + o(1))k^2(\log k)^2) \]

and
\[ m_k \leq \exp((4/\log 2 + o(1))k^2(\log k)^2) \]

hold as \( k \to \infty \).
Proof. We assume that $k \geq 2$. Again, we deal first with $n_k$. We let $A$ be a positive integer depending on $k$, to be determined later. We let $q_1 < q_2 < \cdots < q_m < \cdots$ be all the consecutive prime numbers exceeding $k$. For $j = 1, \ldots, k$, we put $T_j = j(j - 1)/2$ and

$$M_j = \prod_{\ell = T_j, A + 1}^{T_j + A} q_\ell.$$

Put $M = \prod_{j=1}^{k} M_j$ and let $N$ be the smallest positive integer such that $M_j$ divides $N + j$ for each $j$ with $1 \leq j \leq k$. Such an integer $N$ exists by the Chinese Remainder Theorem. Note that $N + k < M$. Indeed, if not, then $N = M - i$ for some $i \in \{1, \ldots, k\}$, and by taking some $j \neq i \in \{1, \ldots, k\}$ (which exists because $k \geq 2$), we would get that $M_j \mid N + j = M + (j - i)$; therefore $M_j \mid j - i$, which is impossible. Let $n = M\lambda + N$ be a positive integer with $\lambda \in [M, 2M]$. Note that

$$n + j = M\lambda + (N + j) = M_j ((M/M_j)\lambda + (N + j)/M_j), \quad j = 1, \ldots, k.$$

By setting $A_j = (N + j)/M_j$ and $B_j = M/M_j$, it follows that

$$jA = T_{j+1}A - T_jA = \omega(M_j) \leq \omega(n + j) \leq jA + \omega(B_j\lambda + A_j),$$

so that if $\lambda$ is such that

$$\omega(B_j\lambda + A_j) < A, \quad \text{for all } j = 1, \ldots, k - 1,$$

then

$$jA \leq \omega(n + j) < jA + A \leq \omega(n + j + 1) \quad \text{for all } j = 1, \ldots, k - 1.$$

Hence, we certainly have that $\omega(n + 1), \ldots, \omega(n + k)$ are pairwise distinct.

It now remains to estimate $A$ and $M$ such that we can guarantee the existence of a positive integer $\lambda \in [M, 2M]$ with the property that all of the inequalities hold.

We claim that $A_j$ and $B_j$ are coprime. Indeed, to see this, note first that

$$B_j = M/M_j = \prod_{1 \leq \ell \leq k \atop \ell \neq j} M_\ell.$$

If there exists a prime $p \mid (A_j, B_j)$, we then get that $p \mid M_\ell$ for some $\ell \neq j$. Since $M_\ell \mid N + \ell$, we get that $p \mid N + \ell$. But obviously $p \mid A_j \mid N + j$; therefore $p \mid (N + \ell) - (N + j) = (\ell - j)$, and $1 \leq |\ell - j| < k$. Thus $p < k$, which is impossible because all prime factors of $M$ exceed $k$, proving the claim.

Now note that since $N + k \leq M$, we have

$$B_j\lambda + A_j \leq \frac{1}{M_j} (M\lambda + N + k) < \frac{2M\lambda}{M_j} \leq \frac{4M^2}{M_j} < M^2$$

for all $\lambda \in [M, 2M]$ and $j = 1, \ldots, k$, when $k \geq 3$, because in this case all primes dividing $M$ exceed 4 and $N + k < M$. Thus, writing $\tau(m)$ for the number of divisors of $m$, we obtain

$$\tau(B_j\lambda + A_j) \leq 2 \sum_{d \mid B_j\lambda + A_j \atop d \leq M} 1.$$
Summing the above inequality over all \( \lambda \in [M, 2M] \) and changing the order of summation, we find that

\[
\sum_{\lambda \in [M, 2M]} \tau(B_j \lambda + A_j) \leq 2 \sum_{\lambda \in [M, 2M]} \sum_{d \leq M} \frac{1}{d} \sum_{B_j \lambda + A_j \equiv 0 \pmod{d}} \leq 2 \sum_{d \leq M} \left( \frac{M}{d} + 1 \right) \leq 4M \sum_{d \leq M} \frac{1}{d} \leq 4M(\log M + 1).
\]

(7)

In the above chain of inequalities, we used the fact that, since \( A_j \) and \( B_j \) are coprime, the congruence \( B_j \lambda + A_j \equiv 0 \pmod{d} \) has at most \([M/d] + 1\) solutions \( \lambda \in [M, 2M] \). This is true assuming that \( d \) and \( B_j \) are coprime. When \( d \) and \( B_j \) are not coprime, then this congruence has no integer solution \( \lambda \). Thus, if \( \lambda \) is such that \( \omega(B_j \lambda + A_j) \geq A \), then \( \tau(B_j \lambda + A_j) \geq 2^A \) and inequality (7) shows that

\[
\# \{ \lambda \in [M, 2M] : \omega(B_j \lambda + A_j) \geq A \} \leq \frac{4M(\log M + 1)}{2^A}.
\]

Summing the above inequality over \( j = 1, \ldots, k - 1 \), we get that

\[
\sum_{j=1}^{k-1} \# \{ \lambda \in [M, 2M] : \omega(B_j \lambda + A_j) \geq A \} \leq \frac{4(k-1)M(\log M + 1)}{2^A}.
\]

(8)

Hence, assuming that

\[
M > \frac{4(k-1)M(\log M + 1)}{2^A},
\]

we see that there exists a number \( \lambda \in [M, 2M] \) such that all inequalities (6) are satisfied, and therefore

\[
n < n + 1 = M\lambda + N + 1 < 2M^2 + M < 3M^2.
\]

(9)

It remains to estimate the size of the minimal integer \( A \) depending on \( k \) such that inequality (8) holds. Clearly, \( M \) has \( Ak(k+1)/2 \) prime factors, which are all the consecutive primes starting with the first one exceeding \( k \). Thus, by the Prime Number Theorem,

\[
M = \exp((1/2 + o(1))k^2 A(\log k^2 A))
\]
as \( k \to \infty \) uniformly in \( A \geq 1 \). Thus, inequality (8) is fulfilled when

\[
A \log 2 > \log(4(k-1)) + \log(\log M + 1) = (3 + o(1)) \log k + O(\log \log k + \log A).
\]

This shows that given \( \varepsilon > 0 \), we may choose \( A = \lfloor (3/\log 2 + \varepsilon) \log k \rfloor \), and then inequality (8) is fulfilled once \( k > k_\varepsilon \). With this choice of \( A \), we have that

\[
M < \exp((3/\log 2 + 2\varepsilon)k^2(\log k)^2)
\]

provided that \( k \) is sufficiently large, and now inequality (9) shows that

\[
n < \exp((6/\log 2 + 5\varepsilon)k^2(\log k)^2)
\]

if \( k \) is sufficiently large with respect to \( \varepsilon \), which implies the desired estimate as \( k \to \infty \), since \( \varepsilon \in (0, 1) \) may be chosen arbitrarily small.

We now turn our attention to the upper bound for \( m_k \). We follow the same line of attack, based on the Chinese Remainder Theorem, although the details are somewhat different.
We assume again that $k \geq 2$; we take $M_0 = (k!)^2$, $M_j = q_j^{A_j}$ for $j = 1, \ldots, k$, and let $N$ be the smallest positive integer $m$ in the arithmetic progression

$$m + j \equiv 0 \pmod{M_j}, \quad j = 0, \ldots, k.$$

Here,

$$M = \prod_{j=0}^{k} M_j = (k!)^2 \prod_{j=1}^{k} q_j^{A_j} = \exp((1/2 + o(1))k^2A \log k)$$

as $k \to \infty$. Let $m = M\lambda + N$ again be such that $\lambda \in [M, 2M]$. Then

$$m + i = iM_i \left( \frac{M}{iM_i} \lambda + \frac{N + i}{iM_i} \right), \quad \text{for all } i = 1, \ldots, k,$$

so that if we set $A_i = (N + i)/(iM_i)$ and $B_i = M/(iM_i)$, we have

$$\Omega(m + i) = \Omega(i) + \Omega(M_i) + \Omega(B_i\lambda + A_i).$$

Now since $i \leq k$, it follows that $\Omega(i) \leq (\log k)/\log 2$. Furthermore, $\Omega(M_i) = iA$. Thus, if

$$\Omega(B_i\lambda + A_i) < A - (\log k)/\log 2, \quad \text{for all } i = 1, \ldots, k - 1,$$

then

$$\Omega(m + i) < A(i + 1) = \Omega(M_{i+1}) < \Omega(m + i + 1), \quad \text{for all } i = 1, \ldots, k - 1,$$

which certainly shows that $\Omega(m + 1), \ldots, \Omega(m + k)$ are pairwise distinct.

Now let $i \in \{1, \ldots, k\}$. As in the analysis of the $n_k$ case, one shows that $A_i$ and $B_i$ are coprime and that $B_i\lambda + A_i < M\lambda + N + k < 2M^2 + M < 3M^2$. Furthermore, since $M_0/i$ is a divisor of $B_i$ for all $i = 1, \ldots, k$ and $M_0/i = (k!)^2/i$ is divisible by all primes $p \leq k$, it follows that the smallest prime factor of $B_i\lambda + A_i$ exceeds $k$. Write

$$B_i\lambda + A_i = U_iV_i,$$

where all prime factors of $U_i$ are $\leq M^{1/2}$ and all prime factors of $V_i$ are $> M^{1/2}$. Clearly, $\Omega(V_i) \leq 4$ because $M > 9$. We will now bound from above the number of $\lambda$ such that $U_i$ is not squarefree for some $i = 1, \ldots, k$. There exists a prime $p \in [k, M^{1/2}]$ such that $B_i\lambda + A_i \equiv 0 \pmod{p^2}$. For a fixed prime $p$, the number of integers $\lambda \in [M, 2M]$ for which the above congruence holds is at most $[M/p^2] + 1 \leq 2M/p^2$. Thus,

$$\#\{\lambda \in [M, 2M] : p^2 \mid B_i\lambda + A_i \text{ for some } p \in [k, M^{1/2}]\} \leq 2M \sum_{p > k} \frac{1}{p^2} \leq \frac{M}{k \log k},$$

uniformly in $i \in \{1, \ldots, k\}$. Summing this over all $i \in \{1, \ldots, k\}$, we get that

$$\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : U_i \text{ is not squarefree}\} \ll \frac{M}{\log k}.$$

In particular, if $k$ is large, then

$$\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \Omega(B_i\lambda + A_i) > \omega(B_i\lambda + A_i) + 4\} \ll \frac{M}{2}.$$
Let \( \lambda \) be some number in \([M, 2M]\) such that \( \Omega(B_i\lambda + A_i) \leq \omega(B_i\lambda + A_i) + 4 \). As we have seen, there are at least \( M/2 \) such values for \( \lambda \). If there is such a positive integer \( \lambda \) with the additional property that

\[
\omega(B_i\lambda + A_i) < A - \frac{\log k}{\log 2} - 4, \quad \text{for all } i = 1, \ldots, k,
\]

it follows that inequalities (11) are satisfied. So, let us look at the number of \( \lambda \in [M, 2M] \) such that at least one of the inequalities (12) fails. The argument used in the proof of the upper bound for \( n_k \) (based on the fact that \( \tau(m) \geq 2^\omega(m) \)) shows that

\[
\sum_{i=1}^{k} \#\{ \lambda \in [M, 2M] : \omega(B_i\lambda + A_i) \geq A - (\log k)/\log 2 - 4 \} \leq \frac{4(k-1)M(\log M)}{2^A-(\log k)/(\log 2)-4}.
\]

Thus, if

\[
4(k-1)M(\log M) < M/2,
\]

then the number of \( \lambda \in [M, 2M] \) such that at least one of the inequalities (12) fails is \(< M/2 \). Since we have \( \geq M/2 \) values of \( \lambda \) to choose from, it follows that one can indeed choose such a value of \( \lambda \) for which all inequalities in (11) hold. Clearly, with such a value of \( \lambda \), we have that \( m_k \leq m = M\lambda + N < 3M^2 \). Inequality (13) is equivalent, via estimate (10), to

\[
A \log 2 - (\log k) - 4\log 2 > \log(8(k-1)) + 2\log k + O(\log \log k + \log A),
\]

which holds if we first fix \( \varepsilon > 0 \), then take \( k > k_\varepsilon \), and finally choose \( A = [(4/\log 2 + \varepsilon)\log k] \). With this choice of \( A \), we have

\[
M < \exp((2/\log 2 + 2\varepsilon)k^2(\log k)^2)
\]

once \( k > k_\varepsilon \). Therefore,

\[
m_k < 3M^2 < \exp((4/\log 2 + 5\varepsilon)k^2(\log k)^2)
\]

if \( k \) is large with respect to \( \varepsilon \), which implies the desired inequality since \( \varepsilon > 0 \) can be chosen arbitrarily small. \( \square \)

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