ON INJECTIVITY OF QUASIREGULAR MAPPINGS

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ABSTRACT. We give sufficient conditions for a planar quasiregular mapping to be injective in terms of the range of the differential matrix.

1. INTRODUCTION

A holomorphic function \( f : \Omega \to \mathbb{C} \) of one complex variable is a local homeomorphism if and only if \( f' \neq 0 \) in \( \Omega \). If, in addition, \( \Omega \) is convex and \( \text{Re}\, f' \geq 0 \), then \( f \) is either constant or injective in \( \Omega \) [2]. In this paper we establish the analogues of these well-known facts for quasiregular mappings \( f : \Omega \to \mathbb{C} \). By definition \( f \in W^{1,2}_{\text{loc}}(\Omega) \) is quasiregular if there exists a constant \( k < 1 \) such that \( |f_{\bar{z}}| \leq k|f_z| \) a.e. in \( \Omega \). Such a mapping can be called \( K \)-quasiregular with \( K = (1 + k)/(1 - k) \), or \( K \)-quasiconformal if it is also injective.

Theorem 1.1. Let \( \Omega \subset \mathbb{C} \) be a domain. If \( f : \Omega \to \mathbb{C} \) is a nonconstant quasiregular mapping and \( \text{Re}\, f_z \geq 0 \) almost everywhere, then \( f \) is a local homeomorphism.

The proof is based on the celebrated theorem of Poincaré and Bendixson [5] and its extension by Brouwer [6], about local structure of integral curves of a continuous planar vector field near its critical point. Example 5.1 will show that the assumption \( \text{Re}\, f_z \geq 0 \) cannot be replaced with \( |\arg f_z| \leq \pi/2 + \epsilon \), for any \( \epsilon > 0 \). To ensure that \( f \) is injective in a convex domain \( \Omega \subset \mathbb{C} \) we must restrict the range of \( f_z \) even further.

Theorem 1.2. Let \( \Omega \subset \mathbb{C} \) be a convex domain. If \( f : \Omega \to \mathbb{C} \) is a nonconstant quasiregular mapping and \( \text{Re}\, f_z = 0 \) almost everywhere, then \( f \) is a homeomorphism.

This theorem admits the following reformulation (see section 3): if \( \psi \) is a differentiable real-valued function on a convex domain \( \Omega \subset \mathbb{R}^2 \) and the gradient mapping \( \nabla \psi : \Omega \to \mathbb{R}^2 \) is quasiregular, then \( \nabla \psi \) is either injective or constant. This is no longer true in dimensions \( n \geq 3 \), as is demonstrated by Example 5.3. Also, the assumption \( \text{Re}\, f_z = 0 \) in Theorem 1.2 cannot be replaced with \( |\arg f_z| < \epsilon \), for any \( \epsilon > 0 \), by Example 5.2. However, the situation is different when \( \Omega = \mathbb{C} \). This can

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be expected since by Picard’s theorem an entire function whose derivative omits two values is linear and therefore is either constant or injective. For quasiregular mappings we have the following.

**Theorem 1.3.** If \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a non-constant quasiregular mapping and \( \Re f_z \geq 0 \) almost everywhere, then \( f \) is a homeomorphism.

The sharpness of the assumption is demonstrated by Example 5.1. As a corollary of Theorems 1.1 and 1.3 we obtain a converse to the following theorem [4, Theorem 6.3.1]:

**Theorem 1.4.** If \( f \in W^{1,2}_{\text{loc}}(\mathbb{C}) \) is a homeomorphic solution to the reduced Beltrami equation

(1.1) \[ f_z = \lambda(z) \Re f_z, \quad |\lambda(z)| \leq k < 1, \]

then \( \Re f_z \) does not change sign.

**Corollary 1.5.** If \( f \in W^{1,2}_{\text{loc}}(\mathbb{C}) \) is a solution of (1.1) such that \( \Re f_z \) does not change sign, then \( f \) is a homeomorphism.

Let us emphasize that the notion of quasiregularity is invariant under an affine change of variables. Accordingly, the assumption \( \Re f_z \geq 0 \) in Theorems 1.1 and 1.3 can be replaced by somewhat geometrically pleasing conditions on the differential matrix \( Df \). Let \( \mathbb{R}^{2 \times 2} \) denote the 4-dimensional linear space of \( 2 \times 2 \) matrices equipped with the inner product \( \langle X, Y \rangle = \text{Tr} (XY^*) \), for \( X, Y \in \mathbb{R}^{2 \times 2} \). Each nonzero matrix \( N \in \mathbb{R}^{2 \times 2} \) gives rise to a 3-dimensional subspace perpendicular to \( N \),

\[ \mathbb{H}_N = \{ X : \langle X, N \rangle = 0 \} \subset \mathbb{R}^{2 \times 2}. \]

There are three types of such subspaces:

- \( \mathbb{H}_N \) is said to be a **positive subspace** if \( \det N > 0 \).
- \( \mathbb{H}_N \) is said to be a **negative subspace** if \( \det N < 0 \).
- \( \mathbb{H}_N \) is said to be **singular** if \( \det N = 0 \).

A 3-dimensional subspace \( \mathbb{H}_N \) splits \( \mathbb{R}^{2 \times 2} \) into two half-spaces. If \( \det N > 0 \) we call them **positive half-spaces** of \( \mathbb{R}^{2 \times 2} \). If \( \det N < 0 \) we call them **negative half-spaces**. If \( \det N = 0 \) the corresponding half-spaces are called **singular half-spaces**. Now, upon an affine change of variable, the assumption \( \Re f_z \geq 0 \) tells us that the essential range of \( Df \) lies in a positive half-space. Precisely, this means that there is a constant matrix \( N \in \mathbb{R}^{2 \times 2} \) of positive determinant such that

(1.2) \[ \langle Df(z), N \rangle \geq 0 \quad \text{a.e. in } \Omega. \]

As a matter of fact this amounts to saying that the homotopy between \( f \) and the \( \mathbb{R} \)-linear map \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), with \( DL = N \),

\[ f^t = (1 - t)f + tL, \quad 0 \leq t \leq 1, \]

keeps the distortion function of \( f^t \) decreasing as \( t \) increases from 0 to 1. For example, if \( L = \text{id} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), then condition (1.2) reads as \( \Re f_z \geq 0 \), so \( f^t(z) = (1 - t)f(z) + tz \) and

\[ \left| \frac{f_z^t}{f_z} \right|^2 = \frac{|f_z|^2}{\left( \Re f_z + \frac{t}{1-t} \right)^2 + (\Im f_z)^2} \searrow 0 \quad \text{as } t \nearrow 1. \]
The limit map \( f^1(z) = z \) is a homeomorphism. Recall the Hurwitz-type theorems for quasiregular mappings; see [9] II 5.3 and [11] Lemma 3.

**Theorem 1.6.** If \( f_j : \Omega \to \mathbb{C} \) is a sequence of [locally] \( K \)-quasiconformal mappings which converges uniformly on compact sets to \( f : \Omega \to \mathbb{C} \), then \( f \) is either constant or [locally] \( K \)-quasiconformal.

Now, heuristically, by virtue of Theorem 1.6, it should come as no surprise that Condition \( (1.2) \) yields local injectivity of \( f \). However, our proof still requires the Poincaré-Bendixson analysis of the integral curves of the vector fields \( f^i \), \( 0 < t \leq 1 \).

In view of these observations Theorem 1.1 is a statement on differential inclusions; let

\[
\mathcal{U}(K, N) = \left\{ X \in \mathbb{R}^{2 \times 2} : |X|^2 \leq K \det X, \quad \langle X, N \rangle \geq 0 \right\}
\]

for some \( K \geq 1 \) and \( N \in \mathbb{R}^{2 \times 2} \) with positive determinant.

Every nonconstant solution to the differential inclusion

\[
Df(z) \in \mathcal{U}(K, N) \quad \text{a.e. in } \Omega, \quad f \in W^{1,2}_{\text{loc}}(\Omega)
\]

is a local homeomorphism.

We refer the reader to [8] for a survey on differential inclusions.

## 2. Proof of Theorem 1.1

Let \( f \) and \( \Omega \) be as in the statement of Theorem 1.1. By virtue of Theorem 1.6 it suffices to prove that \( f^\lambda(z) := f(z) + \lambda z \) is a local homeomorphism for \( \lambda > 0 \). To simplify notation, we write \( f \) instead of \( f^\lambda \), keeping in mind that \( \Re f_z \geq \lambda > 0 \) a.e. in \( \Omega \). The local index of \( f \) at \( z_0 \in \Omega \) is an integer defined by the rule

\[
n_f(z_0) = \frac{1}{2\pi} \frac{\Delta}{0 \leq \theta < 2\pi} \arg \left[ f(z_0 + re^{i\theta}) - f(z_0) \right],
\]

where the increment of the argument of \( f \) does not depend on the choice of radius \( r \), provided \( r \) is sufficiently small. It is a general topological fact that \( f \) is locally injective if and only if \( n_f(z_0) = 1 \) for every \( z_0 \in \Omega \). Since nonconstant quasiregular mappings are orientation-preserving [12] Theorem I.4.5], we have \( n_f(z_0) \geq 1 \). It remains to show that \( n_f(z_0) \leq 1 \) for every \( z_0 \in \Omega \). It involves no loss of generality in assuming that \( z_0 = 0 \), \( f(z_0) = 0 \), \( \Omega \) is the open unit disk, and \( f(z) \neq 0 \) for \( 0 < |z| \leq 1 \). Let \( \Omega_\alpha \) denote the punctured unit disk, \( \Omega_\alpha = \{ z : 0 < |z| < 1 \} \). We shall consider the integral curves of \( f \) in \( \Omega_\alpha \), that is, solutions of the differential equation

\[
\frac{dz}{dt} = f(z) : 0 < |z(t)| < 1 \quad \text{for } a < t < b.
\]

By virtue of Peano’s Existence Theorem, through every point \( z_0 \in \Omega_\alpha \) there passes an integral curve, though uniqueness is not guaranteed. However, \( z = z(t) \) can be extended (as a solution) over a maximal interval of existence, say \( a_- < t < b_+ \), \(-\infty < a_- < b_+ < \infty \). Moreover \( z(t) \) tends to \( \partial \Omega_\alpha = \{ 0 \} \cup S^1 \) as \( t \to a_- \) and \( t \to b_+ \). The extension of \( z(t) \) need not be unique and the maximal interval of existence depends on the extension. Clearly \( z = z(t) \) is of class \( C^{1,\alpha}(a_-, b_+) \), \( 0 < \alpha < 1 \). Since \( f(0) = 0 \), it is possible in general that there is an injective solution \( z = z(t) \in \Omega_\alpha \) for \( 0 \leq t \leq \delta \) such that \( \lim_{t \to \delta} z(t) = z(0) \). Then \( \gamma = \{ z(t) : 0 \leq t < \delta \} \) is a rectifiable Jordan curve in \( \Omega_\alpha \). However, under our assumption, such curves are not present. Indeed, suppose such a \( \gamma \) exists. To reach a contradiction, we let
U denote the bounded component of \( \mathbb{C} \setminus \gamma \); it is a simply connected region in \( \Omega \). We integrate \( f_z \) over \( \Omega \) by using Stokes’ theorem:

\[
(2.2) \quad \iint_{\Omega} f_z \, dx \, dy = \frac{1}{2i} \iint_{\Omega} f \, d\bar{z} = \frac{1}{2i} \int_{\gamma} f \, d\bar{z} = \frac{1}{2i} \int_{0}^{\delta} |f(z(t))|^2 \, dt.
\]

This shows that

\[
(2.3) \quad \iint_{\Omega} (\text{Re} \, f_z) \, dx \, dy = 0,
\]

which is impossible since \( \text{Re} \, f_z > 0 \) almost everywhere.

Next we shall rule out the integral curves \( \gamma = \{z(t) \mid a_- < t < b_+\} \) such that

\[
\lim_{t \to a_-} z(t) = \lim_{t \to b_+} z(t) = 0.
\]

Call such curves **elliptic loops** in \( \Omega \). According to the celebrated Poincaré-Bendixon-Brouwer Theory \([6]\) such curves are present in every elliptic sector of \( \Omega \). We shall not give a definition of an elliptic sector here as the need will not arise. The interested reader is referred to \([5, 6, 7]\) for the definition and thorough discussion of sectors. The proof of nonexistence of elliptic loops is much the same as above. Adding the point 0 to \( \gamma \) we obtain a Jordan curve, closure of \( \gamma \) in \( \Omega_\circ \). Let \( U \) denote the bounded component of \( \mathbb{C} \setminus \gamma \). To avoid delicate questions of rectifiability of \( \gamma \) we remove from \( U \) a small disk \( D_\epsilon = \{z \mid |z| = \epsilon\} \subset U \). Then we integrate as before:

\[
(2.4) \quad \iint_{U \setminus D_\epsilon} f_z \, dx \, dy = \frac{1}{2i} \iint_{\partial(U \setminus D_\epsilon)} f \, d\bar{z} = \frac{1}{2i} \int_{\gamma_1} f \, d\bar{z} + \frac{1}{2i} \int_{\gamma_2} f \, d\bar{z},
\]

where \( \gamma_1 = \partial U \setminus D_\epsilon \) and \( \gamma_2 = U \cap \partial D_\epsilon \). As before, the real part of the first integral term vanishes. The second term can be made as small as we wish. Indeed, we have

\[
(2.5) \quad \left| \frac{1}{2i} \int_{\gamma_2} f \, d\bar{z} \right| \leq \frac{1}{2} \int_{|z| = \epsilon} |f| |dz| = \pi \epsilon \|f\|_{\infty}.
\]

Passing to the limit as \( \epsilon \to 0 \) we find that

\[
(2.6) \quad \int_{U} (\text{Re} \, f_z) \, dx \, dy = 0,
\]

which gives the desired contradiction.

Therefore, there are no elliptic sectors in \( \Omega \). We now come to the fundamental theorem of Brouwer \([6]\) Theorem 5], which asserts that the index of \( f \) at the point 0 is given by

\[
(2.7) \quad n_f(0) = 1 + \frac{n_e - n_h}{2},
\]

where \( n_e \) stands for the number of elliptic sectors and \( n_h \geq 0 \) stands for the number of hyperbolic sectors in \( \Omega \). We just proved that \( n_e = 0 \). Since \( n_h \geq 0 \) and \( n_f(0) \geq 1 \), this is only possible if \( n_h = 0 \) and \( n_f(0) = 1 \), as claimed.

3. **Proof of Theorem 1.2**

Let \( f = u + iv \). In this notation the condition \( \text{Re} \, f_z = 0 \) reads as \( u_x + v_y = 0 \). Therefore there exists a real-valued function \( \psi \) such that

\[
\psi_x = -v \quad \text{and} \quad \psi_y = u
\]

or, using complex notation, \( \nabla \psi = \psi_x + i\psi_y = if \). For \( (a, b) \in \Omega \) we define

\[
\psi^{a, b}(x, y) = \psi(x, y) - [\psi(a, b) + (x - a)\psi_x(a, b) + (y - b)\psi_y(a, b)].
\]
Due to the local injectivity of $f$, $(a, b)$ is an isolated critical point of $\psi_{a,b}$. Since the topological index of $\nabla \psi_{a,b}(a, b)$ is equal to 1, by [1] Lemma 3.1 there is a neighborhood $U$ of $(a, b)$ such that either

(i) $\psi_{a,b} > 0$ in $U \setminus (a, b)$ or
(ii) $\psi_{a,b} < 0$ in $U \setminus (a, b)$.

We claim that only one of the above alternatives occurs for all $(a, b) \in \Omega$. Suppose to the contrary that (i) occurs at $(a_1, b_1)$ and (ii) occurs at $(a_2, b_2)$. Consider the function $\phi(t) = \psi(a_1 + t(a_2 - a_1), b_1 + t(b_2 - b_1))$ which is defined on some open interval containing $[0, 1]$, because $\Omega$ is convex. Since any tangent line to the graph of $\phi$ stays (locally) on one side of the graph, the Mean Value Theorem implies that $\phi'$ does not have any points of local extremum. Therefore, $\phi'$ is monotone, and $\phi$ is either convex or concave. However, this contradicts our assumptions about $(a_1, b_1)$ and $(a_2, b_2)$.

Suppose for the sake of definiteness that only (i) occurs in the entire domain $\Omega$. It follows that $\psi$ is strictly convex in $\Omega$. Being the gradient mapping of a strictly convex function, the map $if(z)$ is injective [13 Corollary 26.3.1], and so is $f$. □

4. PROOF OF THEOREM 1.3

Once we know that $f$ is locally quasiconformal, by Theorem 1.3 its global injectivity is a consequence of integral estimates near $\infty$. The following elementary, though interesting, fact yields Theorem 1.3.

**Proposition 4.1.** If $f : \mathbb{C} \to \mathbb{C}$ is locally $K$-quasiconformal and $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \geq \lambda^2$, almost everywhere for some $\lambda > 0$, then $f$ is injective. Precisely we have

$$|f(z_1) - f(z_2)| \geq \frac{\lambda}{\sqrt{k}}|z_1 - z_2|.$$  \hspace{1cm} (4.1)

**Proof.** We may assume that $f(0) = 0$ and $f(1) = 1$, by rescaling if necessary. Stoilow factorization provides us with a normalized $K$-quasiconformal map $\chi : \mathbb{C} \to \mathbb{C}$, $\chi(0) = 0$, $\chi(1) = 1$, such that $H(\omega) = f(\chi(\omega))$ is an entire function. We aim to show that $H(\omega) \equiv \omega$. Since $f$ is locally injective so is $H$. In particular, $H'(\omega) \neq 0$. By the chain rule we have the following lower bound of the derivative:

$$|H'(\omega)|^2 = J_f(z) J_\chi(\omega) \geq \lambda^2 J_\chi(\omega).$$

Choose and fix a sufficiently small positive number $\epsilon$, for instance $0 < \epsilon < \frac{1}{\sqrt{\lambda}}$ will suffice. Then we have

$$\frac{1}{|H'(\omega)|^{2\epsilon}} \leq \frac{1}{\lambda^{2\epsilon} J_\chi(\omega)}.$$ 

Consider the entire function

$$F(\omega) = [H'(\omega)]^{-\epsilon} = \sum_{m=0}^{\infty} a_m \omega^m.$$ 

Integration over the disk $B = \{ \omega \mid |\omega| \leq R \}$ yields

$$\sum_{m=0}^{\infty} \frac{|a_m|^2}{m + 1} R^{2m} = \int_B |F(\omega)|^2 \, d\omega \leq \frac{1}{\lambda^{2\epsilon}} \int_B \frac{d\omega}{J_\chi(\omega)}.$$ 

The integral average on the right-hand side exhibits a power growth with respect to $R$. Although the precise value of the power is immaterial for the forthcoming
arguments, we demonstrate here the use of Astala’s area distortion theorem, [3], to obtain a sharp power.

**Lemma 4.2.** If $\chi : \mathbb{C} \to \mathbb{C}$ is $K$-quasiconformal and $B \subset \mathbb{C}$ is a disk, then for every $0 < \epsilon < \frac{1}{K-1}$ we have

$$
\int_B \frac{d\omega}{J_\chi^z(\omega)} \leq C_K \left( \frac{|B|}{|\chi(B)|} \right)^\epsilon.
$$

In particular, if $\chi(0) = 0$ and $\chi(1) = 1$, then for $R \geq 1$,

$$
\frac{1}{\pi R^2} \int_{|\omega| \leq R} \frac{d\omega}{J_\chi^z(\omega)} \leq C_K(\epsilon) R^2 \epsilon (1-1/K).
$$

**Proof.** See [4, Theorem 13.2.7].

We just arrived at the inequality

$$
\sum_{m=0}^{\infty} \frac{|a_m|^2}{m+1} R^{2m} \leq \frac{C_K(\epsilon)}{\lambda^{2\epsilon}} R^{2 \epsilon (1-1/K)},
$$

where $2 \epsilon (1-1/K) < 2/K < 2$. Therefore, $a_m = 0$ for $m \geq 1$. This means that $F(\omega)$ is constant, and so is $H'(\omega)$. Hence $H(\omega) = \omega$ because of the normalization $H(0) = 0$ and $H(1) = 1$. In conclusion, $f(z)$ is the inverse of $\chi(\omega)$, and we have

$$
|D\chi(\omega)|^2 \leq K J_\chi(\omega) = \frac{K}{J_f(z)} \leq \frac{K}{\lambda^2},
$$

$$
|\chi(\omega_1) - \chi(\omega_2)| \leq \frac{\sqrt{K}}{\lambda} |\omega_1 - \omega_2|,
$$

which is equivalent to (1.1). \qed

Returning to Theorem 1.3, we consider the quasiregular mappings

$$
f^\lambda(z) = f(z) + \lambda z, \quad \lambda \geq 0.
$$

Clearly,

$$
J_f^z(z) = J_f(z) + \lambda^2 + 2\lambda \text{Re} f_z \geq \lambda^2.
$$

Hence $f^\lambda$ is a $K$-quasiconformal mapping of $\mathbb{C}$ onto itself, for all $\lambda > 0$. Passing to the limit as $\lambda \to 0$, by Theorem 1.6 we conclude that $f$ is injective in the entire plane. \qed

5. Examples

**Example 5.1.** For every $\epsilon > 0$ there is a nonconstant quasiregular map $f : \mathbb{C} \to \mathbb{C}$ whose $z$-derivative lies in the sector

$$
\text{Re} f_z \geq -\epsilon |\text{Im} f_z| \quad \text{a.e. in } \mathbb{C},
$$

and yet $f$ fails to be injective.

**Proof.** We need only consider $0 < \epsilon \leq 2$. Let us introduce a parameter $\delta = \frac{\epsilon}{2 + \sqrt{4 - \epsilon^2}} \leq \frac{\epsilon}{2} \leq 1$ so that $\epsilon = \frac{\delta^2}{1+\delta^2}$. First we define a quasiconformal homeomorphism of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ onto the complex plane with...
a slit along the nonnegative $x$-axis:

$$f(z) = \begin{cases} \frac{2z^2}{|z|\sqrt{1 + \delta^2}}, & \text{if } \text{Re } z \geq -\delta \text{ Im } z, \\ (i-\epsilon)z - i\bar{z}, & \text{if } \text{Re } z \leq -\delta \text{ Im } z. \end{cases}$$

(5.2)

A straightforward computation shows that

$$(i-\epsilon)z - i\bar{z} = \frac{2z^2}{|z|\sqrt{1 + \delta^2}}$$
on the ray $z = (i-\delta)t$, $t > 0$.

Thus $f$ is continuous on $\mathbb{H}$. Moreover, its complex derivatives outside this ray are

$$f_z = \begin{cases} \frac{2z^2}{|z|\sqrt{1 + \delta^2}}, & \text{if } \text{Re } z > -\delta \text{ Im } z, \\ i-\epsilon, & \text{if } \text{Re } z < -\delta \text{ Im } z, \end{cases}$$

and

$$f_{\bar{z}} = \begin{cases} \frac{-2z^2}{|z|\sqrt{1 + \delta^2}}, & \text{if } \text{Re } z > -\delta \text{ Im } z, \\ -i, & \text{if } \text{Re } z < -\delta \text{ Im } z. \end{cases}$$

In any case we find that

$$|f_{\bar{z}}| \leq \frac{1}{\sqrt{1+\epsilon^2}} |f_z|.$$

(5.3)

Regarding the condition (5.1), we have

$$\text{Re } f_z \geq \begin{cases} -\delta |\text{Im } f_z|, & \text{if } \text{Re } z > -\delta \text{ Im } z, \\ -\epsilon |\text{Im } f_z|, & \text{if } \text{Re } z < -\delta \text{ Im } z. \end{cases}$$

In any case, $\text{Re } f_z \geq -\epsilon |\text{Im } f_z|$. Next we note that $f: \mathbb{H} \to \mathbb{C}$ extends continuously to the real line with values in the nonnegative real axis. Precisely, we have

$$f(x+i0) = \begin{cases} \frac{2x}{\sqrt{1+\delta^2}}, & \text{for } x \geq 0, \\ -\epsilon x, & \text{for } x \leq 0. \end{cases}$$

By reflection about the $x$-axis, we extend $f$ to the lower half-plane, that is, by setting $f(z) = \overline{f(\bar{z})}$ if $\text{Im } z \leq 0$. Inequality (5.3) holds almost everywhere in $\mathbb{C}$. But $f$ is no longer injective near any neighborhood of the origin, because

$$f(z) = f(-z) \quad \text{if } \text{Re } z \geq -\delta |\text{Im } z|.$$ 

It only remains to verify (5.1) in the lower half-plane. For $\text{Im } z < 0$ we have $f_z(z) = \overline{f_{\bar{z}}(\bar{z})}$. Hence

$$\text{Re } f_z(z) = \text{Re } f_z(\bar{z}) \geq -\epsilon |\text{Im } f_z(\bar{z})| = -\epsilon |\text{Im } f_{\bar{z}}(\bar{z})|,$$

as desired.

**Example 5.2.** Let $\Omega = \{ z \in \mathbb{C}: \text{Re } z > 0 \}$ be the right half-plane. For every $M \geq 1$ there is a nonconstant quasiregular map $f: \Omega \to \mathbb{C}$ such that

$$\text{Re } f_z \geq M |\text{Im } f_z|$$

a.e. in $\Omega$ and $f$ is not injective.
Proof. For a given $M \geq 1$ we define
\[
\begin{cases}
(4M^2 + 4Mi)z + (4M^2 + 1)\bar{z}, & \text{if } 0 < \Re z \leq 2M \Im z, \\
(4M^2 - 4Mi)z + (4M^2 + 1)\bar{z}, & \text{if } 0 < \Re z \leq -2M \Im z, \\
(8M^2 - 1)z, & \text{if } \Re z \geq 2M |\Im z|.
\end{cases}
\]

The reader may wish to verify that $f$ is quasiregular and satisfies inequality (5.4). Moreover, $f$ fails to be injective, because $f(1 + 4Mi) = 1 - 8M^2 = f(1 - 4Mi)$. □

Example 5.3. For any $n \geq 3$ there exists a function $\psi \in C^1(\mathbb{R}^n)$ such that $\nabla \psi : \mathbb{R}^n \to \mathbb{R}^n$ is a nonconstant quasiregular mapping that is not a local homeomorphism.

Proof. Define
\[
\psi(x_1, \ldots, x_n) = \frac{(x_1^2 - x_2^2)^2 - 4x_1^2x_2^2}{x_1^2 + x_2^2} - \frac{x_3^2}{2} + \frac{1}{2} \sum_{k=4}^{n} x_k^2.
\]

Note that $\psi$ is homogeneous of degree 2. Since $\psi$ is smooth away from the origin, the homogeneity implies that the entries of the Hessian matrix $D^2\psi$ belong to $L^\infty(\mathbb{R}^n)$. We claim that $\det D^2\psi \geq 16$ a.e. Thanks to the block-diagonal form of $D^2\psi$ and $\prod_{k=3}^{n} \frac{\partial^2 \psi}{\partial x_k^2} = -1$ it suffices to show that $\det D^2u \leq -16$, where $u$ is the restriction of $\psi$ to the plane $\{x : x_k = 0 \forall k \geq 3\}$. Writing $u$ in complex notation,
\[
u(z) = \frac{\Re(z^4)}{|z|^2} = \frac{1}{2} \left( \frac{z^3}{z} + \frac{\bar{z}^3}{z} \right),
\]
we find that
\[
(5.5) \quad \nabla u = 2u\bar{z} = \frac{3\bar{z}^2}{z} - \frac{z^3}{|z|^2} = \frac{\bar{z}^3}{|z|^2} \left( 3 - \frac{z^4}{|z|^4} \right)
\]
and
\[
\det D^2u = 4(|u_{zz}|^2 - |u_{\bar{z}z}|^2) = -22 - 6\Re \frac{z^4}{|z|^4} \leq -16.
\]

This proves that $\nabla \psi$ is quasiregular and, moreover, belongs to the class BLD (bounded length distortion) introduced by Martio and Väisälä [10]. The last part of (5.5) shows that $\nabla u$ is homotopic to $z \mapsto z^3$ in $\mathbb{C} \setminus \{0\}$. Since $\nabla u$ has index $-3$ at 0, it follows that $\nabla u$ is not injective on any neighborhood of 0. Consequently $\nabla \psi$ is not injective on any neighborhood of any point $x \in \mathbb{R}^n$ with $x_1 = x_2 = 0$. In fact these points constitute the branch set of $\nabla \psi$, because $\nabla \psi$ is a local diffeomorphism elsewhere. □

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References

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