REDUCTION THEOREMS FOR NOETHER’S PROBLEM

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Abstract. Let $K$ be any field, and $G$ be a finite group. Let $G$ act on the rational function field $K(x(g) : g \in G)$ by $K$-automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether’s problem asks whether $K(G)$ is rational (= purely transcendental) over $K$. We will give several reduction theorems for solving Noether’s problem. For example, let $\tilde{G} = G \times H$ be a direct product of finite groups. Theorem. Assume that $K(H)$ is rational over $K$. Then $K(\tilde{G})$ is rational over $K(G)$. In particular, if $K(G)$ is rational (resp. retract rational) over $K$, so is $K(\tilde{G})$ over $K$.

1. Introduction and statements

Let $K$ be any field, and $G$ be a finite group. Let $G$ act on the rational function field $K(x(g) : g \in G)$ by $K$-automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether’s problem asks whether $K(G)$ is rational (= purely transcendental) over $K$. For a survey of Noether’s problem, see Swan’s paper [10].

The purpose of this article is to prove several reduction theorems when we try to solve Noether’s problem for some group.

Our first result generalizes [6, Proposition 7] (the case $p = 2$).

Theorem 1.1. Let $K$ be a field with $\text{char } K = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \to \mathbb{Z}/p\mathbb{Z} \to \tilde{G} \to G \to 1$, where $G$ is a finite group. Then $K(\tilde{G})$ is rational over $K(G)$.

The meaning of the above conclusion is that there is a $K$-embedding of $K(G)$ into $K(\tilde{G})$, i.e. an injective $K$-linear homomorphism of fields from $K(G)$ into $K(\tilde{G})$, so that $K(\tilde{G})$ is rational over $K(G)$.

Note that, for any field $K$, if $G$ and $\tilde{G}$ are finite groups so that $K(\tilde{G})$ is rational over $K(G)$, then $K(\tilde{G})$ is rational (resp stably rational, retract rational) over $K$ provided that so is $K(G)$. (Recall that “rational” $\Rightarrow$ “stably rational” $\Rightarrow$ “retract rational”. Here, a field $L$ is stably rational over $K$ if there exists a field which is rational over both $L$ and $K$. For the definition of retract rationality, see [9, Definition 3.2].)

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Successive applications of the above theorem yield the following.

**Corollary 1.2.** Let \( K \) be a field with \( \text{char } K = p > 0 \) and \( \tilde{G} \) be a group extension defined by \( 1 \to H \to \tilde{G} \to G \to 1 \), where \( H \) and \( G \) are finite groups. If either

(i) \( H \) is a cyclic \( p \)-group or

(ii) \( H \) is an abelian \( p \)-group lying in the center of \( \tilde{G} \) or

(iii) \( H \) is a \( p \)-group and \( \tilde{G} \cong H \times G \),

then \( K(\tilde{G}) \) is rational over \( K(G) \).

**Proof.** In each of these cases, there is a subgroup \( C \cong \mathbb{Z}/p\mathbb{Z} \) of \( H \) such that \( C \) is normal in \( \tilde{G} \) and the extension \( 1 \to H/C \to \tilde{G}/C \to G \to 1 \) belongs again to the same type (i), (ii) or (iii). \( \square \)

Case (iii) of this result may be thought of as a generalization of Kuniyoshi’s Theorem: If \( K \) is a field with \( \text{char } K = p > 0 \) and \( G \) is a finite \( p \)-group, then \( K(G) \) is rational over \( K \) [5]. Also, by Kuniyoshi’s Theorem, one can view case (iii) of Corollary 1.2 as a consequence of our next result.

**Theorem 1.3.** Let \( K \) be any field, and let \( H \) and \( G \) be finite groups. If \( K(H) \) is rational (resp. stably rational, retract rational) over \( K \), so is \( K(H \times G) \) over \( K(G) \).

In particular, if both \( K(H) \) and \( K(G) \) are rational (resp. stably rational, retract rational) over \( K \), so is \( K(H \times G) \) over \( K \).

**Remark 1.4.** Saltman already proved in [8] Theorem 1.5] that if \( K(H) \) and \( K(G) \) are retract rational over \( K \), then so is \( K(H \times G) \) over \( K \).

**Corollary 1.5.** Let \( \tilde{G} = H \times G \) be a direct product of finite groups, and let \( K \) be a field. Assume that: (i) \( H \) is an abelian group with exponent \( e \), i.e. \( e = \max\{\text{ord}(h) : h \in H\} \); and (ii) the field \( K \) contains the \( e \)-th roots of unity. Then \( K(\tilde{G}) \) is rational over \( K(G) \).

In particular, \( K((\mathbb{Z}/2\mathbb{Z}) \times G) \) is rational over \( K(G) \) for every field \( K \) and every finite group \( G \).

**Proof.** First, by Corollary 1.2 (iii), we can assume that the characteristic of \( K \) does not divide \( e \). Then, in this case, the result follows from Theorem 1.3 and Fischer’s Theorem: \( K(H) \) is rational over \( K \) for every finite abelian group \( H \) of exponent \( e \) and every field \( K \) containing a primitive \( e \)-th root of unity [10] Theorem 6.1]. \( \square \)

**Remark 1.6.** We don’t know whether Theorem 1.3 (or even Corollary 1.5) is valid for \( \tilde{G} \), which is a semi-direct product but not a direct product. In fact, we don’t know whether there exist distinct prime numbers \( p \) and \( q \) such that \( \tilde{G} = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z} \) is a non-abelian semi-direct product and \( \mathbb{C}(\tilde{G}) \) is not rational over \( \mathbb{C} \). Note that \( \mathbb{C}(\tilde{G}) \) is retract rational over \( \mathbb{C} \) [8] Theorem 3.5].

However, consider the non-abelian group \( \tilde{G} = \mathbb{Z}/17\mathbb{Z} \rtimes \mathbb{Z}/16\mathbb{Z} \), where \( \mathbb{Z}/16\mathbb{Z} \) acts faithfully on \( \mathbb{Z}/17\mathbb{Z} \). By [8] Theorems 3.1 and 5.11], \( \mathbb{Q}(\tilde{G}) \) is not even retract rational over \( \mathbb{Q} \), while it is known that \( \mathbb{C}(\tilde{G}) \) is rational over \( \mathbb{C} \) [2].

For wreath products, we will prove the following.

**Theorem 1.7.** Let \( K \) be any field, and let \( H \wr G \) be the wreath product of finite groups \( H \) and \( G \). If \( K(H) \) is rational (resp. stably rational) over \( K \), so is \( K(H \wr G) \) over \( K(G) \).
In particular, if both $K(H)$ and $K(G)$ are rational (resp. stably rational) over $K$, so is $K(H \wr G)$ over $K$.

**Remark 1.8.** Assuming that $K$ is an infinite field, Saltman showed in [8, Theorem 3.3] that $K(H \wr G)$ is retract rational over $K$ provided that so are $K(H)$ and $K(G)$.

Obvious combinations of the above theorems produce new rationality results as, for example, the following one.

**Corollary 1.9.** Let $K$ be a field, $p$ a prime number, and $G$ a $p$-Sylow subgroup of some finite symmetric group. If $K(\mathbb{Z}/p\mathbb{Z})$ is rational over $K$, so is $K(G)$ over $K$.

In particular, if $K$ is a field containing the $p$-th roots of unity, then $K(G)$ is rational over $K$.

**Proof.** It is well known (e.g. [7, p. 177]) that $G$ is a direct product of iterated wreath products $\mathbb{Z}/p\mathbb{Z} \cdots \mathbb{Z}/p\mathbb{Z}$. Apply Theorem 1.3 and Theorem 1.7. \hfill \square

We also have an application to Noether's problem for dihedral groups.

**Corollary 1.10.** Let $K$ be any field, $n$ be an odd integer, and $D_n$ be the dihedral group of order $2n$.

(a) $K(D_{2n})$ is rational over $K(D_n)$. In particular, if $K(D_n)$ is rational (resp. retract rational) over $K$, so is $K(D_{2n})$.

(b) If $K(\mathbb{Z}/n\mathbb{Z})$ is rational over $K$, then both $K(D_n)$ and $K(D_{2n})$ are stably rational over $K$.

**Proof.** (a) If $n$ is odd, then $D_{2n} = \langle \sigma, \tau : \sigma^{2n} = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle$ is a direct product of the groups $\langle \sigma^2, \tau \rangle \cong D_n$ and $\langle \sigma^n \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Apply Corollary 1.5.

(b) It is easy to check that, for odd $n$, the map $\mathbb{Z}/n\mathbb{Z} \times D_n \rightarrow (\mathbb{Z}/n\mathbb{Z}) \wr (\mathbb{Z}/2\mathbb{Z})$ given by $(a, \sigma^b \tau^c) \mapsto ((a + b, a - b), \epsilon)$ is well-defined and is an isomorphism (see Section 4 for the definition of the wreath product). Hence, the stable rationality of $K(D_n)$ over $K$ follows from Theorem 1.3 and Theorem 1.7. Then, by (a), also $K(D_{2n})$ is stably rational over $K$. \hfill \square

**Remark 1.11.** Note that part (a) is implicit in [4].

If $n$ is an odd integer and $K(\mathbb{Z}/n\mathbb{Z})$ is rational over $K$, the first-named author is able to show that $K(D_n)$ is indeed rational over $K$, by using other methods.

We will prove Theorem 1.1, Theorem 1.3, and Theorem 1.7 in Section 2, Section 5, and Section 4, respectively.

**Standing notation.** If $G$ is a finite group, we will write $V = \bigoplus_{g \in G} K \cdot x(g)$ as the regular representation space of $G$, where $G$ acts on $V$ by $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Recall the definition $K(G) := K(x(g) : g \in G)^G$ given at the beginning of this section.

2. **Proof of Theorem 1.1**

Before proving Theorem 1.1, we recall two basic facts.

**Theorem 2.1** [Hajja and Kang [3, Theorem 1]]. Let $G$ be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field of $n$ variables over a field $L$. Suppose that

(i) for any $\sigma \in G$, $\sigma(L) \subset L$;

(ii) the restriction of the action of $G$ to $L$ is faithful;
(iii) for any $\sigma \in G$, 

$$
\begin{pmatrix}
\sigma(x_1) \\
\sigma(x_2) \\
\vdots \\
\sigma(x_n)
\end{pmatrix} = A(\sigma) \cdot 
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} + B(\sigma),
$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over $L$.

Then there exist $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ so that $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ with $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

**Theorem 2.2** (Ahmad, Hajja and Kang [1, Theorem 3.1]). Let $L$ be any field, $L(x)$ the rational function field of one variable over $L$, and $G$ a group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G$ or $L^G(f)$ for some polynomial $f \in L[x]$. In fact, if the integer $m := \min\{\deg g(x) : g(x) \in L[x]^G, g(x) \notin L\}$ does exist, then $L(x)^G = L^G(f(x))$ for any $f(x) \in L[x]^G$ satisfying $\deg f = m$.

**Proof of Theorem** In this section, $K$ is a field with $\text{char } K = p > 0$ and $1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$. Let $c$ be a generator of the normal subgroup $\mathbb{Z}/p\mathbb{Z}$ and $\pi : \tilde{G} \rightarrow G$ be the given epimorphism.

Step 1. Let $u : G \rightarrow \tilde{G}$ be a section of $\pi$.

As before let $\tilde{V} = \bigoplus_{g \in \tilde{G}} K \cdot x(\tilde{g})$ and $V = \bigoplus_{g \in G} K \cdot x(g)$ be the regular representation spaces of $\tilde{G}$ and $G$ respectively.

Step 2. For each $g \in G$, define

$$
y(g) = \sum_{0 \leq i \leq p-1} x(c^i u(g)) \in \tilde{V},
$$

$$
z(g) = \sum_{0 \leq i \leq p-1} ix(c^i u(g)) \in \tilde{V},
$$

$$
z = \sum_{g \in G} z(g) \in \tilde{V},
$$

$$
W = \bigoplus_{g \in G} K \cdot y(g).
$$

Note that $c \cdot y(g) = y(g)$. As $G$-spaces, $W$ and $V$ are $G$-equivariant. Hence $K(W)^\tilde{G} \simeq K(G)$.

Step 3. We will examine the action of $\tilde{G}$ on $z(g)$ and $z$.

It is clear that $c \cdot z(g) = z(g) - y(g)$.

For any $h, g \in G$, suppose that $u(h) \cdot u(g) = c^m \cdot u(hg)$ and $u(h) \cdot c \cdot u(h)^{-1} = c^n$.

Note that $m$ is an integer depending on $g$ and $h$, and $n$ is invertible in $K$. When the element $h$ is fixed, we may write $m = m(g)$ to emphasize the dependence of $m$ on $g$.

We find that $u(h) \cdot z(g) = \sum_{0 \leq i \leq p-1} ix(u(h)c^i u(g)) = \sum_{0 \leq i \leq p-1} ix(c^m u(h)u(g)) = \sum_{0 \leq i \leq p-1} ix(c^{m+n} u(hg)) = c^m \cdot (1/n) \sum_{0 \leq i \leq p-1} ix(c^i u(hg)) = (1/n)z(hg) - (m/n)y(hg)$.

It follows that $u(h) \cdot z = (1/n)z - \sum_{g \in G} (m(g)/n)y(hg)$, where $m(g)$ denotes the integer $m$ depending on $g$. 

Step 4. Define $\widetilde{W} = W \oplus K \cdot z$. Then $\widetilde{W}$ is a faithful $\widetilde{G}$-subspace of $\widetilde{V}$. By Theorem 2.1, $K(\tilde{G})$ is rational over $K(W)^\tilde{G}$.

Consider the pair $\widetilde{W}$ and $W$ and apply Theorem 2.2. We find that $K(\widetilde{W})^\tilde{G}$ is rational over $K(W)^\tilde{G}$. Since $K(W)^\tilde{G} = K(W)^G \simeq K(G)$, we are done.

3. Proof of Theorem 1.3

Without loss of generality, we may assume that neither $H$ nor $G$ is the trivial group.

Step 1. Write $\tilde{G} = H \times G$.

Let $U = \bigoplus_{h \in H} K \cdot x(h)$ and $V = \bigoplus_{g \in G} K \cdot x(g)$ be the regular representation spaces of $H$ and $G$ respectively.

For any element $\tilde{g} \in \tilde{G}$, any $u \otimes v \in U \otimes_K V$, define $\tilde{g} \cdot (u \otimes v) = (h \cdot u) \otimes (g \cdot v)$ if $\tilde{g} = hg$, where $h \in H$ and $g \in G$. It is easy to see that $U \otimes_K V$ is isomorphic to the regular representation space of $G$.

Step 2. Define

$$u_0 = \sum_{h \in H} x(h) \in U, \quad v_0 = \sum_{g \in G} x(g) \in V,$$

$$\widetilde{U} = \sum_{u \in U} K \cdot u \otimes v_0 \subset U \otimes_K V, \quad \widetilde{V} = \sum_{v \in V} K \cdot u_0 \otimes v \subset U \otimes_K V.$$ 

It is easy to see that $\widetilde{U} \oplus \widetilde{V}$ is a faithful $\tilde{G}$-subspace of $U \otimes_K V$. Moreover, when restricted to the action of $H$, the space $\widetilde{U}$ is $H$-equivariant isomorphic to the space $U$, and similarly for $\widetilde{V}$ and $V$ as $G$-spaces.

Step 3. By Theorem 2.1, $K(\tilde{G}) = K(U \otimes_K V)^\tilde{G}$ is rational over $K(\widetilde{U} \oplus \widetilde{V})^\tilde{G}$.

On the other hand, $K(\widetilde{U} \oplus \widetilde{V})^\tilde{G} = (K(\widetilde{U} \oplus \widetilde{V})^H)^\tilde{G}$, which is $K$-isomorphic to $K(H) \cdot K(\tilde{G})$. We conclude that $K(\tilde{G})$ is rational over $K(H) \cdot K(\tilde{G})$. (Note that the composite $K(H) \cdot K(\tilde{G})$ is a free composite; i.e. the transcendence degree of it is the sum of those of $K(H)$ and $K(\tilde{G})$.)

Step 4. If $K(H)$ is rational (resp. stably rational) over $K$, it is easy to see that so is $K(H) \cdot K(G)$ over $K$. Thus $K(\tilde{G})$ is rational (resp. stably rational) over $K(G)$.

As to the retract rationality, from the definition of retract rationality [9], Definition 3.2], it is not difficult to show that (i) if $K(H)$ is retract rational over $K$, then $K(H) \cdot K(G)$ is retract rational over $K(G)$; and (ii) if both $K(H)$ and $K(G)$ are retract rational, then $K(H) \cdot K(G)$ is retract rational over $K$. Hence the result.

4. Proof of Theorem 1.7

Step 1. Write $\tilde{G} = H \wr G$.

Recall the definition of the wreath product $H \wr G$.

Define $N = \bigoplus_{g \in G} H_g$, where each $H_g$ is a copy of $H$. When we write an element $x = (\cdots, x_g, \cdots) \in N$, it is understood that $x_g$ is the component of $x$ in $H_g$.

We will define a left action of $G$ on $N$ as follows. If $\sigma \in G$ and $x = (\cdots, x_g, \cdots) \in N$, define $\sigma x = y$ where $y = (\cdots, y_g, \cdots) \in N$ with $y_g = x_{\sigma^{-1} g}$.
The wreath product \( H \wr G \) is the semi-direct product \( N \rtimes G \). More precisely, if \( x, y \in N \) and \( \sigma, \tau \in G \), then \((x, \sigma) \cdot (y, \tau) = (x \cdot (\tau y), \sigma \tau)\). Thus we have

\[
(\sigma x)(\tau y) = (\sigma \tau)(\tau^{-1} x \cdot y),
\]

where \( \sigma, \tau \in G \) and \( x, y \in N \).

We will fix our notation for the group \( \tilde{G} = H \wr G \), which will be used in subsequent discussions. The groups \( N \) and \( G \) may be identified (in the usual way) with subgroups of \( \tilde{G} \). As above, if \( x \in N \) and \( \sigma \in G \), then \((x, \sigma)\) or \( x\sigma\) denotes an element (and the same element) in \( \tilde{G} \). For any \( g \in G \), let \( H_g \) be the subgroup of \( N \) consisting of elements \( x = (\cdots, x_g, \cdots) \) satisfying the condition that \( x_g = 1 \) for any \( g' \in G \setminus \{g\} \); define a group isomorphism \( \phi_g : H \to H_g \) such that, for any \( h \in H \), if \( x = \phi_g(h) \) and \( x = (\cdots, x_g, \cdots) \in H_g \), then \( x_g = h \).

Define a subgroup \( M = \sum_{g \in G \setminus \{1\}} H_g \). Note that the coset decomposition of \( \tilde{G} \) with respect to \( M \) is given as \( \tilde{G} = \bigcup (\sigma \cdot \phi_1(h)) M \) where \( \sigma \) and \( h \) run over all elements in \( G \) and \( H \) respectively.

Step 2. Let \( V = \bigoplus_{g \in G} K \cdot u(g) \) and \( W = \bigoplus_{x \in N} K \cdot v(x) \) be the regular representation spaces of \( G \) and \( N \) respectively.

Define an action of \( \tilde{G} \) on \( V \otimes_K W \) by \((gx) \cdot (u(g') \otimes v(y)) = u(gg') \otimes v(g'^{-1} x \cdot y)\) (following Equation (11)), where \( g, g' \in G \) and \( x, y \in N \).

It follows that \( V \otimes_K W \) is isomorphic to the regular representation space of \( \tilde{G} \).

Step 3. For each \( g \in G \), let \( W_g = \bigoplus_{h \in H} K \cdot v(\phi_g(h)) \) be the regular representation space of \( H_g \). For any \( g \in G \setminus \{1\} \), define

\[
w_g = \sum_{h \in H} v(\phi_g(h)) \in W_g.
\]

As in the proof of Theorem 13, we may regard \( \bigotimes_{g \in G \setminus \{1\}} W_g \) as the regular representation space of \( M \), and regard \( \bigotimes_{g \in G} W_g \) as the regular representation space of \( N \), i.e. \( W \). Define

\[
w' = \bigotimes_{g \in G \setminus \{1\}} w_g \in \bigotimes_{g \in G \setminus \{1\}} W_g.
\]

Define \( w_0 = v(1) \otimes w' \in W \) and \( u_0 = u(1) \otimes w_0 \in V \otimes_K W \).

Note that \( x \cdot u_0 = u_0 \) for any \( x \in M \).

Step 4. For any \( g \in G \), \( h \in H \), define

\[
u(g; h) = (g \cdot \phi_1(h)) \cdot u_0 = u(g) \otimes (v(\phi_1(h)) \otimes w') \in V \otimes_K W.
\]

Note that, for \( g, g' \in G \) and \( h, h' \in H \), we have \( g \cdot u(g'; h) = u(gg'; h) \cdot \phi_g(h) \cdot u(g'; h') = u(g'; h') \) if \( g \neq g' \).

For each \( g \in G \), define

\[
U_g = \bigoplus_{h \in H} K \cdot u(g; h) \subset V \otimes_K W,
\]

and define

\[
\hat{U} := \bigoplus_{g \in G} U_g \subset V \otimes_K W.
\]

It is not difficult to show that \( \hat{U} \) is a faithful \( \tilde{G} \)-subspace of \( V \otimes_K W \). Note that \( G \) permutes the spaces \( U_g \) \( (g \in G) \) regularly; \( H_g \) acts regularly on \( U_g \), while \( H_g \) acts trivially on \( U_{g'} \) if \( g \neq g' \).
Step 5. Apply Theorem 2.1 We find that $K(\tilde{G})$ is rational over $K(\tilde{U})^G$. It remains to show that $K(\tilde{G})$ is rational (resp. stably rational) over $K(G)$ provided that $K(H)$ is rational (resp. stably rational) over $K$.

We consider first the situation when $K(H)$ is rational over $K$. Since $G$ permutes the spaces $U_g$ ($g \in G$) regularly, we may choose a transcendence basis $\{z(g;i): 1 \leq i \leq d\}$ for $K(U_g)^{H_g}$ (where $d$ is the order of $H$); i.e. we may write $K(U_g)^{H_g} = K(z(g;i): 1 \leq i \leq d)$, such that $g \cdot z(g';i) = z(gg';i)$ for $1 \leq i \leq d$.

Thus $K(\tilde{U})^G = (K(\tilde{U})^N)^G = K(z(g;i): g \in G, 1 \leq i \leq d)^G$. Apply Theorem 2.1 It is easy to see that $K(z(g;i): g \in G, 1 \leq i \leq d)^G$ is rational over $K(z(g;1): g \in G)^G$, which is isomorphic to $K(G)$.

Step 6. Assume now that $K(H)$ is stably rational over $K$. More precisely, suppose that $K(H)\langle t_1, \ldots, t_m \rangle$ is rational over $K$.

Let $\{t(g;j)\}$ denote new indeterminates and define a $\tilde{G}$-space $\tilde{V}$ by

$$\tilde{V} := \bigoplus_{g \in G, 1 \leq j \leq m} K \cdot t(g;j),$$

where $g \cdot t(g';j) = t(gg';j)$ and $x \cdot t(g;j) = t(g;j)$ for any $g, g' \in G$, any $x \in N$, any $1 \leq j \leq m$.

Note that $\tilde{U} \oplus \tilde{V}$ is a faithful $\tilde{G}$-subspace of $(V \otimes_K W) \oplus \tilde{V}$. By Theorem 2.1 we find that $K((V \otimes_K W) \oplus \tilde{V})^\tilde{G}$ is rational over $K(V \otimes_K W)^\tilde{G} = K(\tilde{G})$. Again by Theorem 2.1 $K((V \otimes_K W) \oplus \tilde{V})^\tilde{G}$ is rational over $K(\tilde{U} \oplus \tilde{V})^\tilde{G}$.

Now $K(\tilde{U} \oplus \tilde{V})^N = \prod_{g \in G} K(U_g)^{H_g}(t(g;j): 1 \leq j \leq m)$, where each $K(U_g)^{H_g}$ is $K$-isomorphic to $K(H)$ with $g \cdot K(U_g)^{H_g} = K(U_{gg'})^H$ for any $g, g' \in G$. For each $g \in G$, the field $K(U_g)^{H_g}(t(g;j): 1 \leq j \leq m)$ is rational over $K$. As in Step 5, we may choose a transcendence basis $\{z(g;i): 1 \leq i \leq d + m\}$ for $K(U_g)^{H_g}(t(g;j): 1 \leq j \leq m)$ so that $G$ acts regularly on each set $\{z(g;i): g \in G\}$, for every $1 \leq i \leq d + m$. The remaining arguments are quite similar to Step 5 and are omitted.

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