THE LAITINEN CONJECTURE
FOR FINITE SOLVABLE OLIVER GROUPS

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ABSTRACT. For smooth actions of G on spheres with exactly two fixed points, the Laitinen Conjecture proposed an answer to the Smith question about the G-modules determined on the tangent spaces at the two fixed points. Morimoto obtained the first counterexample to the Laitinen Conjecture for $G = \mathrm{Aut}(A_6)$. By answering the Smith question for some finite solvable Oliver groups G, we obtain new counterexamples to the Laitinen Conjecture, presented for the first time in the case where G is solvable.

0. THE LAITINEN CONJECTURE

In 1960, P. A. Smith [17] posed the following question. If a finite group G acts smoothly on a sphere with exactly two fixed points a and b, is it true that the real G-modules determined on the tangent spaces at a and b are *always* isomorphic?

Smith Equivalence. For a finite group G, two real G-modules U and V are called *Smith equivalent* if as real G-modules, $U \cong T_a(S)$ and $V \cong T_b(S)$ for a smooth action of G on a homotopy sphere S with exactly two fixed points a and b.

The Smith equivalence of real G-modules has been studied by many authors, and the Smith question has been answered in many cases (see [12] and [13] for long lists of related references, and see [4], [6], [9], [14] and [20] for very recent results).

In 1996, E. Laitinen suggested an answer to the Smith question in the case where G is an *Oliver group* (that is, G has a smooth fixed point free action on a disk [11] or, equivalently, G has a smooth one fixed point action on a sphere [7]) and G acts on the homotopy sphere S in such a way that the *Laitinen Condition* is satisfied: the fixed point set S^g is connected for every $g \in G$ of order 2^k for $k \geq 3$.

The answer (stated as a conjecture) is expressed by using the number r_G of real conjugacy classes in G of the elements $g \in G$ not of prime power order.

Laitinen Conjecture. *For any finite Oliver group G with $r_G \geq 2$, there exist two Smith equivalent real G-modules which are not isomorphic, and the action of G on the homotopy sphere in question satisfies the Laitinen Condition.*

For a finite group G with $r_G = 0$ or 1, any two Smith equivalent real G-modules are isomorphic provided the Laitinen Condition holds; see [8, Lemmas 1.4 and 2.1]. Therefore, in the Laitinen Conjecture, the condition that $r_G \geq 2$ is necessary.

By [8, Theorem A], for a finite perfect group G, there exist two non-isomorphic Smith equivalent real G-modules, and the Laitinen Condition is satisfied if and only if $r_G \geq 2$. In particular, the Laitinen Conjecture holds when G is perfect.

Morimoto [9] has obtained the first counterexample to the Laitinen Conjecture by proving that for $G = \mathrm{Aut}(A_6)$, any two Smith equivalent real G-modules are isomorphic. Here $r_G = 2$ (cf. [13]) and G is a non-solvable (and thus Oliver) group. We obtain for the first time counterexamples to the Laitinen Conjecture for finite solvable Oliver groups G. To reach this goal, we proceed as follows.

In Section 1, we restate the Smith isomorphism question and describe the Smith diagram of inclusions. In Section 2, we study some conditions imposed on two real G-modules which will be useful in proving that two Smith equivalent real G-modules are isomorphic. In Section 3, we prove that two Smith equivalent real G-modules are related to each other in a specific way depending on G and the subgroups of G of index 2, and we present a refinement of the Smith diagram. In Section 4, we recall how to obtain two Smith equivalent real G-modules which are not isomorphic, and we emphasize the role played by the subgroup G^{nil} of G. In Section 5, we prove six propositions for six different finite solvable Oliver groups G to answer the Smith question — negatively in one case and affirmatively in five other cases. As a result, we obtain counterexamples to the Laitinen Conjecture in four cases of G.

We refer the reader to [2], [3], and [5] for the basic information on transformation groups that we use in this paper.

1. The Smith diagram

In this section, G always denotes an arbitrary finite group. Let $RO(G)$ be the real representation ring of G. As a set, $RO(G)$ consists of the differences $U - V$ of real G-modules U and V, where $U - V = U' - V'$ if and only if $U \oplus V' \cong U' \oplus V$. In particular, $U - V = 0$ if and only if $U \cong V$.

Let $Sm(G)$ be the subset of $RO(G)$ consisting of the differences $U - V$ of Smith equivalent real G-modules U and V. Then the Smith isomorphism question can be restated as follows: *Is it true that $Sm(G) = 0$?*

Definition 1.1. For a set $\mathcal{N}$ of natural numbers, two real G-modules U and V are called $\mathcal{N}$-*matched* if $\chi_U(g) = \chi_V(g)$ for each element $g \in G$ of order $|g| \in \mathcal{N}$, where χ_U and χ_V are the characters of U and V, respectively.

For a set $\mathcal{N}$ of natural numbers, let $Sm(G)_{\mathcal{N}}$ consist of the differences $U - V$ of Smith equivalent $\mathcal{N}$-matched real G-modules U and V. Henceforth, we set

$$\mathcal{P} = \{1\} \cup \{p^k \colon p \text{ prime}, \ k \geq 1\}, \ \mathcal{R} = \{2^k \colon k \geq 3\}, \text{ and } \mathcal{S} = \mathcal{P} \setminus \mathcal{R}.$$

Now, we recall arguments showing that $Sm(G) = Sm(G)_{\mathcal{S}}$.

Theorem 1.2. *Any two Smith equivalent real G-modules are $\mathcal{S}$-matched, and thus*

$$Sm(G) = Sm(G)_{\mathcal{S}}.$$

In particular, if G has no element of order 8, then $Sm(G)_{\mathcal{P}} = Sm(G)$.

Proof. Let U and V be two real G-modules such that $U \cong T_a(S)$ and $V \cong T_b(S)$ for a smooth action of G on a homotopy sphere S with $S^G = \{a, b\}$. We shall prove that $\chi_U(g) = \chi_V(g)$ for any element $g \in G$ of order $|g| \in \mathcal{S}$, i.e. $|g| = 1, 2, 4$, or p^k for an odd prime p and an integer $k \geq 1$.

Clearly, $\chi_U(e) = \dim S = \chi_V(e)$ for the neutral element $e \in G$. For $g \in G$ with $|g| = p^k$ for a prime p and an integer $k \geq 1$, one gets $\dim U^g = \dim S^g = \dim V^g$ by Smith theory and the Slice Theorem.

If $\dim S^g > 0$, then S^g is connected, so U and V are isomorphic when restricted to the cyclic subgroup of G generated by g, and thus $\chi_U(g) = \chi_V(g)$. If $\dim S^g = 0$, then $\chi_U(g) = \chi_V(g)$ in each of the following cases:

   (1) p is odd and $k = 1$: by the results of Atiyah–Bott [1],
   (2) p is odd and $k \geq 2$: by the results of Sanchez [15],
   (3) $p = 2$ and $k = 1$ or 2: by character theory arguments.

Clearly, if G has no element of order 8, then $Sm(G)_{\mathcal{P}} = Sm(G)_{\mathcal{S}} = Sm(G)$. $\qquad\square$

For a set $\mathcal{N}$ of natural numbers, let $RO(G)_{\mathcal{N}}$ consist of the differences $U - V$ of $\mathcal{N}$-matched real G-modules U and V. Clearly, $RO(G)_{\mathcal{N}}$ is a subgroup of $RO(G)$.

Consider the homomorphism

$$RO(G)_{\mathcal{N}} \to \mathbb{Z}, \ U - V \mapsto \dim U^G - \dim V^G, \ \text{and its kernel}$$

$$RO(G)_{\mathcal{N}}^G = \{U - V \in RO(G)_{\mathcal{N}} : \dim U^G = \dim V^G\}.$$

By the Slice Theorem, the following lemma holds.

Lemma 1.3. *The difference of two Smith equivalent $\mathcal{N}$-matched real G-modules is in the kernel $RO(G)_{\mathcal{N}}^G$, i.e. $Sm(G)_{\mathcal{N}} \subseteq RO(G)_{\mathcal{N}}^G$.*

As free abelian subgroups, $RO(G)_{\mathcal{P}}^G \leq RO(G)_{\mathcal{S}}^G \leq RO(G)$. Thus, Theorem 1.2 and Lemma 1.3 yield the following *Smith diagram* consisting of inclusions:

$$
\begin{array}{ccc}
Sm(G)_{\mathcal{P}} & \longrightarrow & RO(G)_{\mathcal{P}}^G \\
\downarrow & & \downarrow \\
Sm(G) & \longrightarrow & RO(G)_{\mathcal{S}}^G.
\end{array}
$$

We say that two elements g and h of G are *real conjugate in G* (written $g \sim^{\pm} h$) if g is conjugate to h or h^{-1}. Then $\sim^{\pm}$ is an equivalence relation on G, and the resulting class $(g)^{\pm}$ is called the *real conjugate class* in G of g.

Recall that as a group, $RO(G)$ is a free abelian group whose rank is equal to the number of real conjugacy classes $(g)^{\pm}$ in G of elements $g \in G$.

Definition 1.4. For a set $\mathcal{N}$ of natural numbers, define $r_G^{\mathcal{N}}$ to be the number of real conjugacy classes $(g)^{\pm}$ in G of elements $g \in G$ such that $|g| \notin \mathcal{N}$.

Clearly, $r_G^{\mathcal{P}}$ (resp. $r_G^{\mathcal{S}}$) is the number of real conjugacy classes of elements of G whose order is divisible by two distinct primes (resp. by two distinct primes or 8). Henceforth, following [8], we shall write $r_G = r_G^{\mathcal{P}}$.

Lemma 1.5. *For a set $\mathcal{N}$ of natural numbers, the following holds.*
   (1) *The free abelian group $RO(G)_{\mathcal{N}}^G = 0$ if and only if $r_G^{\mathcal{N}} = 0$ or 1.*
   (2) *For $r_G^{\mathcal{N}} \geq 2$, the rank of $RO(G)_{\mathcal{N}}^G$ equals $r_G^{\mathcal{N}} - 1$.*

Proof. As a group, $RO(G)$ is a free abelian group whose rank is the dimension of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$. According to [16, Corollary 1, p. 96], $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$ can be regarded as the space of all real-valued functions χ on G that are constant on each real conjugacy class $(g)^{\pm}$, for $g \in G$. Every χ is of the form $\chi = \sum_{(g)^{\pm}} r_{(g)^{\pm}} f_{(g)^{\pm}}$, where $r_{(g)^{\pm}} \in \mathbb{R}$ and $f_{(g)^{\pm}}$ takes the value 1 on $(g)^{\pm}$ and 0 otherwise.

The restriction to the space $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{N}}$ corresponds to imposing on χ the condition that $\chi(g) = r_{(g)\pm} = 0$ for every element $g \in G$ with $|g| \in \mathcal{N}$. Therefore, the functions $f_{(g)\pm}$ with $|g| \notin \mathcal{N}$ form a basis of $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{N}}$, and thus

$$\operatorname{rank} RO(G)_{\mathcal{N}} = \dim \left(\mathbb{R} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{N}} \right) = r_G^{\mathcal{N}}.$$

As $\chi \mapsto \chi^G$ is an epimorphism of $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{N}}$ onto $\mathbb{Z}$ when $RO(G)_{\mathcal{N}} \neq 0$,

$$\operatorname{rank} RO(G)_{\mathcal{N}}^G = \dim \left(\mathbb{R} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{N}}^G \right) = r_G^{\mathcal{N}} - 1,$$

which completes the proof (cf. [8, the proof of Lemma 2.1, p. 303]).   □

2. $\mathcal{P}$-matched Smith equivalent real G-modules

Lemma 2.1. *Let G be a finite group and let U and V be two Smith equivalent real G-modules. Then the following three conditions are equivalent.*

   (1) *U and V are $\mathcal{P}$-matched, where $\mathcal{P} = \{p^k \colon p \text{ prime}, \ k \geq 1\} \cup \{1\}$.*
   (2) *U and V are $\mathcal{R}$-matched, where $\mathcal{R} = \{2^k \colon k \geq 3\}$.*
   (3) *If $\dim U^g = \dim V^g = 0$ for $g \in G$ with $|g| \in \mathcal{R}$, then $\chi_U(g) = \chi_V(g)$.*

Proof. As the G-modules U and V are Smith equivalent, U and V are $\mathcal{S}$-matched by Theorem 1.2. Hence, as $\mathcal{P} = \mathcal{R} \cup \mathcal{S}$, the conditions (1) and (2) are equivalent. Clearly, (2) implies (3) by Definition 1.1. Finally, (3) implies (2) since if $|g| \in \mathcal{R}$, then $\dim U^g = \dim S^g = \dim V^g$ and thus $\chi_U(g) = \chi_V(g)$ by (3) if $\dim S^g = 0$ or by the fact that S^g is connected if $\dim S^g > 0$ (cf. the proof of Theorem 1.2).   □

Corollary 2.2. *Let G be a finite group. Then $Sm(G) = Sm(G)_{\mathcal{P}}$ if and only if (3) in Lemma 2.1 holds for any two Smith equivalent real G-modules U and V.*

Definition 2.3. For a finite group G, define i_G as the number of isomorphism classes of the irreducible real G-modules W such that $\dim W^g = 0$ for some element $g \in G$ with $|g| \in \mathcal{R}$.

   Let G be a finite group. Then for any element $g \in G$ of order 2 or 4, it follows that $gH \subset (g)^{\pm}$ for $H = \langle g^2 \rangle$, the cyclic subgroup of G generated by g^2.

Definition 2.4. Let G be a finite group. We say that G satisfies the 2-*condition* if $gH \subset (g)^{\pm}$ for every element $g \in G$ with $|g| \in \mathcal{R}$, where $H = \langle g^2 \rangle$.

Theorem 2.5. *Let G be a finite group with $i_G = 0$ or 1 or such that G satisfies the 2-condition. Then any two Smith equivalent real G-modules are $\mathcal{P}$-matched, and thus $Sm(G) = Sm(G)_{\mathcal{P}}$.*

Proof. Let U and V be two real G-modules such that $U \cong T_a(S)$ and $V \cong T_b(S)$ for a smooth action of G on a homotopy sphere S with $S^G = \{a, b\}$. We shall prove that U and V are $\mathcal{P}$-matched, which (by Lemma 2.1) is equivalent to proving that if $\dim U^g = \dim V^g = 0$ for $g \in G$ with $|g| \in \mathcal{R}$, then $\chi_U(g) = \chi_V(g)$.

   Assume $i_G = 0$. Let $|g| \in \mathcal{R}$. The Slice Theorem implies that $\dim S^g \geq \dim W^g$ for every irreducible summand W of U or V. As $i_G = 0$, $\dim W^g > 0$ and therefore $\dim U^g = \dim V^g > 0$. So, there is nothing to prove (cf. [8, Lemma 2.6]).

   Assume $i_G = 1$. Let $\dim U^g = \dim V^g = 0$ with $|g| \in \mathcal{R}$. As $i_G = 1$, the irreducible summands of U and V are all isomorphic to the unique irreducible real G-module, say W, such that $\dim W^g = 0$. Therefore $U \cong mW \cong V$ for an integer $m \geq 1$, and thus $\chi_U(g) = \chi_V(g)$.

Assume G satisfies the 2-condition. Set $\chi = \chi_U - \chi_V$. We claim that $\chi(g) = 0$ for any $g \in G$ of order 2^k for $k \geq 0$. We prove this claim by induction on k.

If $k = 0$, then $g = e$ and thus $\chi(g) = \dim U - \dim V = 0$. Fix $k \geq 1$, and assume $\chi(h) = 0$ for all $h \in G$ of order 2^ℓ with $\ell < k$. Let $g \in G$ be of order 2^k, and let $C = \langle g \rangle$ and $H = \langle g^2 \rangle$. As G satisfies the 2-condition, $gH \subset (g)^{\pm}$ and thus $\chi(gh) = \chi(g)$ for all $h \in H$. Therefore $\sum_{h \in H} \chi(gh) = |H|\chi(g)$. By the assumption, $\chi(h) = 0$ for all $h \in H$. Hence, by the fixed point set character formula,

$$\dim U^g - \dim V^g = \frac{1}{|C|} \sum_{x \in C} \chi(x) = \frac{1}{|C|} \sum_{h \in H} \big(\chi(gh) + \chi(h)\big) = \frac{|H|}{|C|}\chi(g).$$

As $\dim U^g = \dim V^g$, it follows that $\chi(g) = 0$, and thus $\chi_U(g) = \chi_V(g)$. □

3. A REFINEMENT OF THE SMITH DIAGRAM

Let G be a finite group and let H be a normal subgroup of G. If X is a G-space, then X^H has a natural action of G/H such that $(X^H)^{G/H} = X^G$.

For a set $\mathcal{N}$ of natural numbers, consider the homomorphism

$$RO(G)_{\mathcal{N}} \to RO(G/H), \ U - V \mapsto U^H - V^H, \ \text{and its kernel}$$

$$RO(G, H)_{\mathcal{N}} = \{U - V \in RO(G)_{\mathcal{N}} : U^H \cong V^H \text{ as } G/H\text{-modules}\}.$$

Clearly, $RO(G, G)_{\mathcal{N}} = RO(G)_{\mathcal{N}}^G$ and $RO(G, H)_{\mathcal{N}} \leq RO(G)_{\mathcal{N}}^G$ for any normal subgroup H of G. We note that in [13] $RO(G, H)_{\mathcal{P}}$ is denoted by $IO(G, H)$.

Lemma 3.1. *Let G be a finite group and let $\mathcal{N}$ be a set of natural numbers. Then*

$$Sm(G)_{\mathcal{N}} \subseteq \bigcap_L RO(G, L)_{\mathcal{N}},$$

where L runs over all subgroups of G with index 1 or 2.

Proof. By Lemma 1.3, $Sm(G)_{\mathcal{N}} \subseteq RO(G)_{\mathcal{N}}^G = RO(G, G)_{\mathcal{N}}$. Let L be a subgroup of G of index 2. Then L is normal in G. Let $U - V \in Sm(G)_{\mathcal{N}}$ be represented by two Smith equivalent $\mathcal{N}$-matched real G-modules U and V. Then U^L and V^L are both direct sums of 1-dimensional real G/L-modules and, as real G/L-modules, $U^L \cong V^L$ by [9, Proposition 2.2]. Therefore $U - V \in RO(G, L)_{\mathcal{N}}$. □

Lemma 3.2. *Let G be a finite group and let $\mathcal{N}$ be a set of natural numbers. Then*

$$\bigcap_L RO(G, L)_{\mathcal{N}} = RO(G, \bigcap_L L)_{\mathcal{N}},$$

where L runs over all subgroups of G with index 1 or 2.

Proof. Set $H = \bigcap_L L$, where L runs over all subgroups of G with index 1 or 2. For any subgroup L of G with index 1 or 2, $g^2 \in L$ for all $g \in G$, and therefore $g^2 \in H$ for all $g \in G$. Hence the elements of G/H are of order 1 or 2, and thus the quotient group G/H is an elementary abelian 2-group.

Let $x_1, \ldots x_n$ be a set of generators of G/H. For each $i = 1, \ldots, n$, let H_i be the inverse image of $\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$ under the quotient map $G \to G/H$. As $RO(G, H)_{\mathcal{N}} = \{U - V \in RO(G)_{\mathcal{N}} : U^H \cong V^H \text{ as } G/H\text{-modules}\}$ and

$$RO(G/H)_{\mathcal{N}} = \bigoplus_{i=1}^{n} RO(G/H_i)_{\mathcal{N}},$$

the equality $RO(G, H)_{\mathcal{N}} = \bigcap_L RO(G, L)_{\mathcal{N}}$ holds. □

Set $G^{1,2} = \bigcap_L L$, where L runs over all subgroups of G with index 1 or 2. Then, by Lemmas 3.1 and 3.2, the following corollary holds.

Corollary 3.3. *Let G be a finite group and let $\mathcal{N}$ be a set of natural numbers. Then $Sm(G)_{\mathcal{N}} \subseteq RO(G, G^{1,2})_{\mathcal{N}}$.*

Corollary 3.3 and Theorem 1.2 yield immediately the following refinement of the Smith diagram from Section 1:

$$
\begin{array}{ccccc}
Sm(G)_{\mathcal{P}} & \longrightarrow & RO(G, G^{1,2})_{\mathcal{P}} & \longrightarrow & RO(G)_{\mathcal{P}}^{G} \\
\downarrow & & \downarrow & & \downarrow \\
Sm(G) & \longrightarrow & RO(G, G^{1,2})_{\mathcal{S}} & \longrightarrow & RO(G)_{\mathcal{S}}^{G}
\end{array}
$$

where all maps are inclusions.

4. Construction of Smith equivalent G-modules

Let G be a finite group. For a prime p, let $O^p(G)$ be the smallest group among normal subgroups of G with index being a power of p. Let $\mathcal{L}^p(G)$ consist of the subgroups of G which contain $O^p(G)$. Set $\mathcal{L}(G) = \bigcup \mathcal{L}^p(G)$, where the union is over all primes p. Let $\mathcal{P}(G)$ be the family of prime power order subgroups of G, including the trivial subgroup of G. A finite group G is called a *gap group* if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and if there exists a real G-module V which is $\mathcal{L}(G)$-*free*, i.e. $\dim V^L = 0$ for each $L \in \mathcal{L}(G)$ and V satisfies the *gap condition* asserting that $\dim V^P > 2 \dim V^H$ for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$.

For a set $\mathcal{N}$ of natural numbers, let $RO(G)_{\mathcal{N}}^{\mathcal{L}}$ consist of the differences $U - V$ in $RO(G)$ of $\mathcal{L}(G)$-free and $\mathcal{N}$-matched real G-modules U and V.

Now, we restate the Realization Theorem obtained in [13, p. 850].

Theorem 4.1. *For a finite Oliver gap group G, every element of $RO(G)_{\mathcal{P}}^{\mathcal{L}}$ is the difference of two real G-modules U and V, where $U \cong T_a(S)$ and $V \cong T_b(S)$ for a smooth action of G on some sphere S such that $S^G = \{a, b\}$ and S^g is connected for every element $g \in G$ of prime power order. In particular, $RO(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq Sm(G)_{\mathcal{P}}$.*

By arguing as in the proof of [13, Subgroup Lemma, p. 858], we see that for any finite group G and any set $\mathcal{N}$ of natural numbers,

$$
RO\left(G, \bigcap_p O^p(G)\right)_{\mathcal{N}} \leq RO(G)_{\mathcal{N}}^{\mathcal{L}} \leq \bigcap_p RO(G, O^p(G))_{\mathcal{N}} \leq RO(G)_{\mathcal{N}}^{G}.
$$

Set $G^{\mathrm{nil}} = \bigcap_p O^p(G)$. The group G^{nil} is the smallest group among all normal subgroups H of G such that the quotient group G/H is nilpotent.

Corollary 4.2. *Let G be a finite Oliver gap group. Then*

$$
RO(G, G^{\mathrm{nil}})_{\mathcal{P}} \subseteq RO(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq Sm(G)_{\mathcal{P}}.
$$

For any normal subgroup H of G, let $\overline{r}_{G/H}$ be the number of real conjugacy classes in G/H of cosets gH which contain elements of G not of prime power order. Then $r_G \geq \overline{r}_{G/H} \geq r_{G/H}$. We note that in [13], $r_G = a_G$ and $\overline{r}_{G/H} = b_{G/H}$.

The following lemma goes back to [13, Second Rank Lemma, p. 856].

Lemma 4.3. *For a finite group G and $H \trianglelefteq G$, the following two conclusions hold.*
  (1) *$RO(G, H)_{\mathcal{P}} = 0$ if and only if $r_G = \overline{r}_{G/H}$.*
  (2) *For $r_G > \overline{r}_{G/H}$, the rank of $RO(G, H)_{\mathcal{P}}$ equals $r_G - \overline{r}_{G/H}$.*

5. Examples of Smith equivalent G-modules

By [11], the smallest finite Oliver group is the alternating group A_5 of order 60, and the smallest finite solvable Oliver groups are of order 72, including $\mathbb{Z}_3 \times S_4$ and $S_3 \times A_4$. By [13], $Sm(A_5) = 0$ (see [13, Theorems C1–C3, pp. 851–852] for computation of $Sm(G)$ for specific finite groups G, including $G = A_n$ or S_n).

In the GAP libraries [21], SmallGroup(ord, n) denotes the finite n-th group of order ord. We compute $Sm(\text{SmallGroup}(ord, n))$ for some values of ord and n.

First, we prove that the Laitinen Conjecture holds for $G = \text{SmallGroup}(72, 42)$; next, we show that $Sm(G) = 0$ for $G = \text{SmallGroup}(72, 43)$.

Proposition 5.1. *For $G = \text{SmallGroup}(72, 42)$, the following holds.*

(1) *G is a finite solvable Oliver group, and G is a gap group.*
(2) *$r_G = 3$ and $Sm(G)_{\mathcal{P}} = Sm(G) \cong \mathbb{Z}$.*

Moreover, each element of $Sm(G)$ is the difference of two real G-modules U and V with $U \cong T_a(S)$ and $V \cong T_b(S)$ for a smooth action on some sphere S such that $S^G = \{a, b\}$ and S^g is connected for every $g \in G$ of prime power order.

Proof. The group $G = \text{SmallGroup}(72, 42)$ is isomorphic to $\mathbb{Z}_3 \times S_4$ and can be regarded as a subgroup of S_7 generated by the elements $(1, 2, 3)$, $(4, 5)$, $(4, 5, 6, 7)$. In particular, G is a finite solvable Oliver group, with $O^2(G) \cong \mathbb{Z}_3 \times A_4$ and $O^3(G) \cong S_4$. Moreover, $G^{\text{nil}} \cong A_4$, $G/G^{\text{nil}} \cong \mathbb{Z}_6$, and we have $r_G = 3$, corresponding to the elements

$$x = (1, 2, 3)(4, 5), \; y = (1, 2, 3)(4, 5)(6, 7), \; \text{and } z = (1, 2, 3)(4, 5, 6, 7)$$

of order 6, 6, and 12, respectively. As $xG^{\text{nil}} \neq yG^{\text{nil}}$ and $xG^{\text{nil}} = zG^{\text{nil}}$, we obtain that $\overline{r}_{G/G^{\text{nil}}} = 2$. As $G^{1,2} = O^2(G)$ we have $xG^{1,2} \neq yG^{1,2}$ and $xG^{1,2} = zG^{1,2}$. Hence $\overline{r}_{G/G^{1,2}} = 2$. By applying [19, Theorem B] for $C = \langle z \rangle$, we deduce that G is a gap group, and thus by Corollaries 3.3 and 4.2 and Lemma 4.3,

$$\mathbb{Z} \cong RO(G, G^{\text{nil}})_{\mathcal{P}} \subseteq RO(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq Sm(G)_{\mathcal{P}} \subseteq RO(G, G^{1,2})_{\mathcal{P}} \cong \mathbb{Z}.$$

The inclusions occurring here are actually equalities, because $RO(G, G^{\text{nil}})_{\mathcal{P}}$ is the kernel of a homomorphism determined on $RO(G)_{\mathcal{P}}$ containing $RO(G, G^{1,2})_{\mathcal{P}}$. As G has no element of order 8, $Sm(G) = Sm(G)_{\mathcal{P}} \cong \mathbb{Z}$ by Theorem 1.2. Moreover, the claim about the elements of $Sm(G)$ follows from Theorem 4.1. $\qquad\square$

Proposition 5.2. *For $G = \text{SmallGroup}(72, 43)$, the following holds.*

(1) *G is a finite solvable Oliver group, and G is not a gap group.*
(2) *$r_G = 1$ and $Sm(G)_{\mathcal{P}} = Sm(G) = 0$.*

Proof. The group $G = \text{SmallGroup}(72, 43)$ is isomorphic to $(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ and can be regarded as a subgroup of S_7 generated by the elements $(1, 2, 3)$, $(4, 5, 6)$, $(5, 6, 7)$, $(1, 2)(4, 5)$. In particular, G is a finite solvable Oliver group with quotients D_6 and S_4, with $O^2(G) = \langle (1, 2, 3), (4, 5, 6), (5, 6, 7) \rangle \cong \mathbb{Z}_3 \times A_4$ and $O^3(G) = G$. Therefore, $G^{\text{nil}} = O^2(G)$ and $G/G^{\text{nil}} \cong \mathbb{Z}_2$.

As S_4 is not a gap group, it follows from [18, Corollary 4.6 and Proposition 6.4] that the product $D_6 \times S_4$ is not a gap group, and as G is a subgroup of $D_6 \times S_4$ with index 2, G is not a gap group by [10, Proposition 3.1].

Note that $r_G = 1$, corresponding to the element $(1, 2, 3)(4, 5)$ of G of order 6. Therefore $Sm(G)_{\mathcal{P}} = 0$ by Lemmas 1.3 and 1.5. As G has no element of order 8, $Sm(G) = Sm(G)_{\mathcal{P}} = 0$ by Theorem 1.2. $\qquad\square$

Now, for the first time, we give counterexamples to the Laitinen Conjecture for finite solvable Oliver groups G. In contrast to the conclusion (2) of Proposition 5.2, in four cases of G, we shall prove that $r_G \geq 2$ and $Sm(G)_\mathcal{P} = Sm(G) = 0$.

Proposition 5.3. *For $G = \mathrm{SmallGroup}(72, 44)$, the following holds.*

(1) *G is a finite solvable Oliver group, and G is a gap group.*
(2) *$r_G = 2$ and $Sm(G)_\mathcal{P} = Sm(G) = 0$.*

Proof. The group $G = \mathrm{SmallGroup}(72, 44)$ is isomorphic to $S_3 \times A_4$ and can be regarded a subgroup of S_7 generated by the elements $(1, 2)$, $(1, 2, 3)$, $(4, 5, 6)$, $(5, 6, 7)$. In particular, G is a finite solvable Oliver group, with

$$O^2(G) \cong \mathbb{Z}_3 \times A_4 \quad \text{and} \quad O^3(G) \cong S_3 \times \mathbb{Z}_2^2.$$

Hence, $G^{\mathrm{nil}} \cong \mathbb{Z}_3 \times \mathbb{Z}_2^2$ and $G/G^{\mathrm{nil}} \cong \mathbb{Z}_6$. Moreover, $r_G = 2$, corresponding to the following elements of order 6:

$$x = (1, 2)(4, 5, 6) \in O^2(G) \quad \text{and} \quad y = (1, 2, 3)(4, 5)(6, 7) \notin O^2(G).$$

Note that G is a gap group by [18, Lemma 5.1] and [10, Lemma 3.2].

As $G^{1,2} = O^2(G)$, we have $\overline{r}_{G/G^{1,2}} = 2$, corresponding to the cosets $xG^{1,2}$ and $yG^{1,2}$; thus $r_G = \overline{r}_{G/G^{1,2}}$, and hence $Sm(G)_\mathcal{P} = 0$ by Corollary 3.3 and Lemma 4.3. As G has no element of order 8, $Sm(G) = Sm(G)_\mathcal{P} = 0$ by Theorem 1.2. $\square$

Proposition 5.4. *For $G = \mathrm{SmallGroup}(288, 1025)$, the following holds.*

(1) *G is a finite solvable Oliver group, and G is a gap group.*
(2) *$r_G = 2$ and $Sm(G)_\mathcal{P} = Sm(G) = 0$.*

Proof. The group $G = \mathrm{SmallGroup}(288, 1025)$ is isomorphic to $(\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ and can be regarded as a subgroup of S_8 generated by the elements

$$(1, 2)(3, 4), (1, 3)(2, 4), (2, 3, 4), (5, 6)(7, 8),$$
$$(5, 7)(6, 8), (6, 7, 8), (1, 5)(2, 6)(3, 7)(4, 8).$$

In particular, G is a finite solvable Oliver group with

$$O^2(G) \cong (\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3)^2 \quad \text{and} \quad O^3(G) \cong (\mathbb{Z}_2^2 \times \mathbb{Z}_2^2) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_2;$$

thus $O^2(G)$ and $O^3(G)$ are subgroups of G with indexes 2 and 3, respectively. Moreover, $O^3(G)$ is generated by the following elements:

$$(1, 2)(3, 4), (1, 3)(2, 4), (5, 6)(7, 8), (5, 7)(6, 8),$$
$$(2, 3, 4)(6, 8, 7), (1, 5)(2, 6)(3, 7)(4, 8).$$

It follows that $G^{\mathrm{nil}} \cong (\mathbb{Z}_2^2 \times \mathbb{Z}_2^2) \rtimes \mathbb{Z}_3$ and $G/G^{\mathrm{nil}} \cong \mathbb{Z}_6$. Now, $r_G = 2$, corresponding to the following two elements of order 6:

$$x = (1, 2)(3, 4)(6, 7, 8) \in O^2(G) \text{ and}$$
$$y = (1, 5)(2, 6)(3, 7)(4, 8) * (2, 3, 4)(6, 7, 8)$$
$$= (1, 5)(2, 7, 4, 6, 3, 8) \notin O^2(G).$$

Note that $G \smallsetminus O^2(G)$ has the element y^3 of order 2, unique up to conjugacy, and the centralizer of y^3 in G is isomorphic to $A_4 \times \mathbb{Z}_2$. Therefore, G is a gap group by [19, Proposition 4.2 and Theorem C].

As $G^{1,2} = O^2(G)$, we have $\overline{r}_{G/G^{1,2}} = 2$, corresponding to the cosets $xG^{1,2}$ and $yG^{1,2}$; thus $r_G = \overline{r}_{G/G^{1,2}}$, and hence $Sm(G)_\mathcal{P} = 0$ by Corollary 3.3 and Lemma 4.3. As G has no element of order 8, $Sm(G) = Sm(G)_\mathcal{P} = 0$ by Theorem 1.2. $\square$

Proposition 5.5. *For* $G = \mathrm{SmallGroup}(432, 734)$, *the following holds.*
  (1) G *is a finite solvable Oliver group, and* G *is not a gap group.*
  (2) $r_G = 2$ *and* $Sm(G)_{\mathcal{P}} = Sm(G) = 0$.

Proof. The group $G = \mathrm{SmallGroup}(432, 734)$ is isomorphic to $\mathrm{Aff}(2, 3)$ and can be regarded as a subgroup of S_9 generated by the elements

$$(1, 2, 3)(4, 5, 6)(7, 8, 9), (1, 9, 5)(2, 7, 6)(3, 8, 4), (4, 5, 6)(7, 9, 8),$$
$$(2, 5, 3, 9)(4, 8, 7, 6), (2, 8, 3, 6)(4, 9, 7, 5), (4, 7)(5, 8)(6, 9).$$

In particular, G is a finite solvable Oliver group, $O^3(G) = G$, and $O^2(G) = G^{\mathrm{nil}}$ is isomorphic to the solvable Oliver subgroup of S_9 generated by the elements

$$(1, 2, 3)(4, 5, 6)(7, 8, 9), (1, 9, 5)(2, 7, 6)(3, 8, 4), (4, 5, 6)(7, 9, 8),$$
$$(2, 5, 3, 9)(4, 8, 7, 6), (2, 8, 3, 6)(4, 9, 7, 5).$$

As G^{nil} is of order 216, $G/G^{\mathrm{nil}} \cong \mathbb{Z}_2$. Now, $r_G = 2$, corresponding to the elements

$$x = (2, 5, 3, 9)(4, 8, 7, 6) * (4, 5, 6)(7, 9, 8) = (2, 6, 5, 3, 8, 9)(4, 7) \in G^{\mathrm{nil}} \text{ and}$$

$$y = (4, 7)(5, 8)(6, 9) * (1, 2, 3)(4, 5, 6)(7, 8, 9) = (1, 2, 3)(4, 8, 6, 7, 5, 9) \notin G^{\mathrm{nil}},$$

both of order 6. Note that G is not a gap group by [18, Corollary 4.6].

As $G^{1,2} = G^{\mathrm{nil}}$, we have $\overline{r}_{G/G^{1,2}} = 2$, corresponding to the cosets $xG^{1,2}$ and $yG^{1,2}$. Thus $r_G = \overline{r}_{G/G^{1,2}}$ and $Sm(G)_{\mathcal{P}} = 0$ by Corollary 3.3 and Lemma 4.3. The elements $g \in G$ of 2-power order are real conjugate in G, and thus G satisfies the 2-condition in Definition 2.4. Hence $Sm(G) = Sm(G)_{\mathcal{P}} = 0$ by Theorem 2.5. □

Proposition 5.6. *For* $G = \mathrm{SmallGroup}(576, 8654)$, *the following holds.*
  (1) G *is a finite solvable Oliver group of order* 576, *and* G *is not a gap group.*
  (2) $r_G = 3$ *and* $Sm(G)_{\mathcal{P}} = Sm(G) = 0$.

Proof. The group $G = \mathrm{SmallGroup}(576, 8654)$ is isomorphic to $(A_4 \times A_4) \rtimes \mathbb{Z}_2^2$ and can be regarded as a subgroup of S_8 generated by the elements

$$(1, 2, 3), (2, 3, 4), (5, 6, 7), (6, 7, 8), (1, 2)(5, 6), (1, 5)(2, 6)(3, 7)(4, 8).$$

In particular, G is a finite solvable Oliver group, $O^3(G) = G$, and $O^2(G) = G^{\mathrm{nil}} \cong A_4 \times A_4$ is generated by the elements $(1, 2, 3)$, $(2, 3, 4)$, $(5, 6, 7)$, and $(6, 7, 8)$. Moreover, $G/G^{\mathrm{nil}} = \{G^{\mathrm{nil}}, a\,G^{\mathrm{nil}}, b\,G^{\mathrm{nil}}, c\,G^{\mathrm{nil}}\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ where $a = (1, 2)(5, 6)$, $b = (1, 5)(2, 6)(3, 7)(4, 8)$, and $c = ab = (1, 6)(2, 5)(3, 7)(4, 8)$.

Let $K = \langle a \rangle O^2(G)$, which is a subgroup of G with index 2. Since K is a subgroup of $S_4 \times S_4$ with index 2, which is not a gap group by [18, Theorem 6.8], K itself is not a gap group by [10, Proposition 3.1], and therefore G is not a gap group again by [10, Proposition 3.1]. Now, $r_G = 3$, corresponding to the elements of order 6:

$$x = b * (1, 2, 3)(5, 6, 7) = (1, 6, 3, 5, 2, 7)(4, 8),$$
$$y = c * (2, 3, 4)(5, 7, 8) = (1, 6)(2, 7, 4, 5, 3, 8),$$
$$z = (1, 2)(3, 4)(6, 7, 8) \in G^{\mathrm{nil}}.$$

The cosets xG^{nil}, yG^{nil}, and $zG^{\mathrm{nil}} = G^{\mathrm{nil}}$ are distinct from each other.

There are three subgroups of G with index 2, namely $\langle a \rangle G^{\mathrm{nil}}$, $\langle b \rangle G^{\mathrm{nil}}$, and $\langle c \rangle G^{\mathrm{nil}}$. Therefore, $G^{1,2} = G^{\mathrm{nil}}$ and $\overline{r}_{G/G^{1,2}} = 3$, corresponding to the cosets $xG^{1,2}$, $yG^{1,2}$, and $zG^{1,2}$. Hence $r_G = \overline{r}_{G/G^{1,2}}$ and $Sm(G)_{\mathcal{P}} = 0$ by Corollary 3.3 and Lemma 4.3. As G has no element of order 8, $Sm(G) = Sm(G)_{\mathcal{P}} = 0$ by Theorem 1.2. □

References

[1] Atiyah, M.F., and Bott, R., *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. 88 (1968), 451–491. MR0232406 (38:731)

[2] Bredon, G.E., *Introduction to Compact Transformation Groups*, Pure and App. Math. 46, Academic Press, 1972. MR0413144 (54:1265)

[3] tom Dieck, T., *Transformation Groups*, de Gruyter Studies in Math. 8, Walter de Gruyter, 1987. MR889050 (89c:57048)

[4] Ju, X.M., *The Smith set of the group $S_5 \times C_2 \times \cdots \times C_2$*, accepted for publication in Osaka Journal of Mathematics.

[5] Kawakubo, K., *The Theory of Transformation Groups*, Oxford University Press, Oxford, 1991. MR1150492 (93g:57044)

[6] Koto, A., Morimoto, M., and Qi, Y., *The Smith sets of finite groups with normal Sylow 2-subgroups and small nilquotients*, J. Math. Kyoto Univ. 48 (2008), 219–227.

[7] Laitinen, E., and Morimoto, M., *Finite groups with smooth one fixed point actions on spheres*, Forum Math. 10 (1998), 479–520. MR1631012 (99k:57078)

[8] Laitinen, E., and Pawałowski, K., *Smith equivalence of representations for finite perfect groups*, Proc. Amer. Math. Soc. 127 (1999), 297–307. MR1468195 (99b:57070)

[9] Morimoto, M., *Smith equivalent $Aut(A_6)$-representations are isomorphic*, Proc. Amer. Math. Soc. 136 (2008), 3683–3688.

[10] Morimoto, M., Sumi, T., and Yanagihara, M., *Finite groups possessing gap modules*, in Geometry and Topology, Aarhus 1998 (ed. K. Grove, I.H. Madsen, E.K. Pedersen), Contemp. Math. 258, Amer. Math. Soc., Providence, RI, 2000, 329–342. MR1778115 (2001i:57050)

[11] Oliver, R., *Fixed point sets of group actions on finite acyclic complexes*, Comment. Math. Helv. 50 (1975), 155–177. MR0375361 (51:11556)

[12] Pawałowski, K., *Smith equivalence of group modules and the Laitinen conjecture. A survey*, in Geometry and Topology, Aarhus 1998 (ed. K. Grove, I.H. Madsen, E.K. Pedersen), Contemp. Math. 258, Amer. Math. Soc., Providence, RI, 2000, 343–350. MR1778116 (2001i:57051)

[13] Pawałowski, K., and Solomon, R., *Smith equivalence and finite Oliver groups with Laitinen number 0 or 1*, Algebr. Geom. Topol. 2 (2002), 843–895. MR1936973 (2003j:57057)

[14] Pawałowski, K., and Sumi, T., *Finite groups with Smith equivalent, nonisomorphic representations*, in Proc. of the 33$^{\rm rd}$ Symposium on Transformation Groups, Yokohama 2006 (ed. T. Kawakami), Wing Co. Ltd., Wakayama, Japan, 2007, 68–76. MR2313386 (2008d:57033)

[15] Sanchez, C.U., *Actions of groups of odd order on compact orientable manifolds*, Proc. Amer. Math. Soc. 54 (1976), 445–448. MR0407871 (53:11641)

[16] Serre, J-P., *Linear Representations of Finite Groups*, Grad. Texts in Math. 42, Springer-Verlag, 1977. MR0450380 (56:8675)

[17] Smith, P.A., *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. 66 (1960), 401–415. MR0125581 (23:A2880)

[18] Sumi, T., *Gap modules for direct product groups*, J. Math. Soc. Japan 53 (2001), 975–990. MR1852892 (2002j:57066)

[19] Sumi, T., *Gap modules for semidirect product groups*, Kyushu J. Math. 58 (2004), 33–58. MR2053718 (2005c:20017)

[20] Sumi, T., *Finite groups possessing Smith equivalent, nonisomorphic representations*, RIMS Kokyuroku no. 1569, Kyoto Univ. (2007), 170–179.

[21] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4, 2006 (http://www.gap-system.org).

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