PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 6, June 2009, Pages 2075–2083
S 0002-9939(09)09765-2
Article electronically published on January 15, 2009

REMARKS ON THE BLOW-UP OF SOLUTIONS TO A TOY MODEL FOR THE NAVIER-STOKES EQUATIONS

ISABELLE GALLAGHER AND MARIUS PAICU

(Communicated by Walter Craig)

ABSTRACT. In a 2001 paper, S. Montgomery-Smith provides a one-dimensional model for the three-dimensional, incompressible Navier-Stokes equations, for which he proves the blow-up of solutions associated with a class of large initial data, while the same global existence results as for the Navier-Stokes equations hold for small data. In this paper the model is adapted to the cases of two and three space dimensions, with the additional feature that the divergence-free condition is preserved. It is checked that a family of initial data constructed by Chemin and Gallagher, which is arbitrarily large yet generates a global solution to the Navier-Stokes equations in three space dimensions, actually causes blow-up for the toy model — meaning that the precise structure of the nonlinear term is crucial to understanding the dynamics of large solutions to the Navier-Stokes equations.

1. Introduction

Consider the Navier-Stokes equations in $\mathbb{R}^d$, for $d = 2$ or 3,

$$\text{(NS)} \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0, \end{cases}$$

where $u = (u^1, \dots, u^d)$ is the velocity of an incompressible, viscous, homogeneous fluid evolving in $\mathbb{R}^d$, and p is its pressure. Note that the divergence-free condition allows us to recover p from u through the formula

$$-\Delta p = \operatorname{div}\,(u \cdot \nabla u).$$

A formally equivalent formulation for (NS) can be obtained by applying the projector onto divergence-free vector fields, $\mathbb{P} \stackrel{\text{def}}{=} \mathrm{Id} - \nabla \Delta^{-1} \operatorname{div}$, to (NS), giving

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}(u \cdot \nabla u) = 0 \\ u_{|t=0} = u_0 = \mathbb{P} u_0. \end{cases}$$

This system has three important features:

(E) (the *energy* inequality): the $L^2(\mathbb{R}^d)$ norm of u is formally bounded for all times by that of the initial data;

(I) (the *incompressibility* condition): the solution satisfies for all times the constraint $\operatorname{div} u = 0$;

Received by the editors May 21, 2008.
2000 *Mathematics Subject Classification.* Primary 76D05.
Key words and phrases. Navier-Stokes equations, blow-up.

(S) (the *scaling* conservation): if u is a solution associated with the data u_0, then for any positive λ, the rescaled $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$ is a solution associated with $u_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x)$.

Of course the first two properties are related, as (I) is the ingredient enabling one to obtain (E), due to the special structure of the nonlinear term.

Taking (E) into account, one can prove the existence of global, possibly non-unique, finite energy solutions (see the fundamental work of J. Leray [11]). On the other hand the use of (S) and a fixed point argument enables one to prove the existence of a unique, global solution if the initial data is small in scale-invariant spaces (we will call any Banach space X satisfying $\|\lambda f(\lambda \cdot)\|_X = \|f\|_X$ for all $\lambda > 0$ a "scale-invariant space"): for instance the homogenenous Sobolev space $\dot H^{\frac d2 -1}$, Besov spaces $\dot B^{-1+\frac dp}_{p,\infty}$ for $p<\infty$ or the space BMO^{-1}. We recall that

$$\|f\|_{\dot B^s_{p,q}} \stackrel{\text{def}}{=} \left\| t^{-\frac s2} \|e^{t\Delta} f\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(\mathbb{R}^+; \frac{dt}{t})}$$

and

$$\|f\|_{BMO^{-1}} \stackrel{\text{def}}{=} \sup_{t>0} \left(t^{\frac12} \|e^{t\Delta} f\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^d \\ R>0}} R^{-\frac d2} \Big(\int_{P(x,R)} |e^{t\Delta} f(t,y)|^2 dy \Big)^{\frac12} \right),$$

where $P(x,R) = [0,R^2] \times B(x,R)$ and $B(x,R)$ denotes the ball in $\mathbb{R}^d$ of center x and radius R. We refer respectively to [6], [1] and [10] for proofs of the wellposedness of (NS) for small data in these spaces. When $d=2$, the smallness condition may be removed: this has been known since the work of J. Leray ([12]) in the energy space L^2 (which is scale-invariant in two space dimensions) and was proved in [7] and [8] for larger spaces, provided they are completions of the Schwartz class for the corresponding norm (Besov or BMO^{-1} norms).

It is well known and rather easy to see that the largest scale-invariant Banach space embedded in the space of tempered distributions is $\dot B^{-1}_{\infty,\infty}$. In three or more space dimensions, it is not known whether global solutions exist for smooth data that are arbitrarily large in $\dot B^{-1}_{\infty,\infty}$. We will not review here all the progress made in that direction in the past years, but merely recall a few of the main recent achievements concerning the possibility of blow-up of large solutions. Recently, D. Li and Ya. Sinai were able in [13] to prove the blow-up in finite time of solutions to the Navier-Stokes equations for complex initial data. We note that, like the system that we construct in the present paper, the complex Navier-Stokes system does not satisfy any energy inequality. Earlier, some numerical evidence was suggested to support the idea of finite time blow-up of (NS) (see for instance [15] or [9]). On the other hand, in [2] a class of large initial data was constructed, giving rise to a global, unique solution; this family will be presented below. Another type of example was provided in [3]. It should be noted that in both those examples, the special structure of the equation is crucial for obtaining the global wellposedness. In [14], S. Montgomery-Smith suggested a model for (NS) with the same scale invariance and for which the same global wellposedness results hold for small data. The interesting feature of the model is that it is possible (see [14]) to prove the blow-up in finite time of some solutions. The model is the following:

$$(\text{TNS}_1) \qquad \partial_t u - \Delta u = \sqrt{-\Delta}\,(u^2) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}.$$

The main ingredient in the proof of the existence of blow-up solutions consists of noticing that if the initial data has a positive Fourier transform, then that positivity is preserved for the solution at all further times. One can then use the Duhamel formulation of the solution and deduce a lower bound for the Fourier transform that blows up in finite time. We will not write more details here as we will be reproducing that computation in Section 2.

In this paper we adapt the construction of [14] to higher space dimensions. In order to have a proper model in higher dimensions it is important to preserve as many features of the Navier-Stokes equations as possible. Here we will seek to preserve the scaling property (S) as well as the divergence-free condition (I) (as we will see, condition (E) cannot be preserved in our model). This amounts to transforming the nonlinear term proposed in [14] (see equation (TNS_1) above) in such a way as to preserve both the positivity conservation property in Fourier space *and* the incompressibility condition. This is in fact a technicality which may be handled by explicit computations in Fourier space; actually the more interesting aspect of the result we obtain is that the initial data constructed in [2] to show the possibility of global solutions associated with arbitrarily large initial data actually generates a blow-up solution for (TNS_3). This, together with the fact that we are also able to obtain blow-up solutions in the two-dimensional case, indicates that proving a global existence result for arbitrarily large data for (NS) requires using the energy estimate or the specific structure of the nonlinear term – two properties which are discarded in our model.

Let us state the result proved in this paper.

Theorem 1. *Let the dimension d be equal to 2 or 3. There is a bilinear operator Q, which is a d-dimensional matrix of Fourier multipliers of order one, such that the equation*

$$(\mathrm{TNS}_d) \quad \begin{cases} \partial_t u - \Delta u = Q(u,u) \quad \textit{in}\, \mathbb{R}^+ \times \mathbb{R}^d \\ \quad\quad\quad \textit{div}\, u = 0 \\ \quad\quad\quad u_{|t=0} = u_0 \end{cases}$$

satisfies properties (I) *and* (S) *and such that there is a global, unique solution if the initial data is small enough in BMO^{-1}. Moreover there is a family of smooth initial data u_0, which may be chosen arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$, such that the associated solution of* (TNS_d) *blows up in all Besov norms, whereas the associated solution of* (NS) *exists globally in time.*

The proof of the theorem is given in the sections below. In Section 2 we deal with the two-dimensional case, while the three-dimensional case is treated in Section 3: in both cases we present an alternative to the bilinear term of (NS), which preserves scaling and the divergence-free property while giving rise to solutions that blow up in finite time for some classes of initial data. The fact that some of those initial data in fact generate a global solution for the three-dimensional Navier-Stokes equations is addressed in Section 4.

Remark 1.1. We note that the method of the proof allows one to construct blow-up solutions for the hyper-viscous case, i.e. for equations of the form

$$\begin{cases} \partial_t u - \Delta^\alpha u = Q(u,u) \quad \text{in}\, \mathbb{R}^+ \times \mathbb{R}^d \\ \quad\quad\quad \text{div}\, u = 0 \\ \quad\quad\quad u_{|t=0} = u_0 \end{cases}$$

where $\alpha \geq 1$ and $\Delta^\alpha f = \mathcal{F}^{-1}(|\xi|^{2\alpha}\hat{f}(\xi))$. Indeed, the only important feature for constructing blow-up solutions by the method of [14] is that the system, written in the Fourier variable, preserves the positivity of the symbol and hence the positivity of $\hat{u}^j(t,\xi)$ if $\hat{u}_0^j(\xi) > 0$, for any $j \in \{1,\dots,d\}$.

Remark 1.2. In the two-dimensional case, it might seem more natural to work on the vorticity formulation of the equation: in 2D it is well known that the vorticity satisfies a transport-diffusion equation, which easily provides the existence of global solutions for any sufficiently smooth initial data. An example where the vorticity equation (rather than (NS)) is modified is provided at the end of Section 2 below.

2. Proof of the theorem in the two-dimensional case

In this section we shall construct the quadratic form Q, as given in the statement of Theorem 1, which allows the construction of blow-up solutions for the (TNS_2) system.

Let us consider a system of the following form:

$$\left\{ \begin{array}{c} \partial_t u - \Delta u = \mathcal{Q}(u,u) - \nabla p \\ \operatorname{div} u = 0. \end{array} \right.$$

Taking the Leray projection of this equation, we obtain

$$\partial_t u - \Delta u = \mathbb{P}\mathcal{Q}(u,u).$$

We wish to follow the idea of the proof of [14], that is, to construct $Q = \mathbb{P}\mathcal{Q}$ as a matrix of Fourier multipliers of order 1 such that the product $\widehat{\mathbb{P}\mathcal{Q}}$ preserves the positivity of the Fourier transform. We define $Q(u,u)$ as the vector whose j-component is, for $j \in \{1,2\}$,

$$(Q(u,u))^j = \sum_i q_{i,j}(D)(u^i u^j), \tag{2.1}$$

and we require that $q_{i,j}(D)$ be Fourier multipliers of order 1. For example, let us simply choose

$$\widehat{\mathcal{Q}}(\xi) = |\xi| \mathbf{1}_{\xi_1\xi_2<0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Recalling that

$$\widehat{\mathbb{P}}(\xi) = \begin{pmatrix} 1 - \frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} \\ -\frac{\xi_1\xi_2}{|\xi|^2} & 1 - \frac{\xi_2^2}{|\xi|^2} \end{pmatrix},$$

we easily obtain

$$\widehat{\mathbb{P}\mathcal{Q}}(\xi) = \mathbf{1}_{\xi_1\xi_2<0} \frac{1}{|\xi|} \begin{pmatrix} \xi_2^2 - \xi_1\xi_2 & \xi_2^2 - \xi_1\xi_2 \\ \xi_1^2 - \xi_1\xi_2 & \xi_1^2 - \xi_2\xi_1 \end{pmatrix},$$

so all the elements of this matrix are positive.

The Duhamel formulation of (TNS_2) reads

$$\widehat{u}^j(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0^j(\xi) + \sum_i \int_0^t e^{-(t-s)|\xi|^2} q_{i,j}(\xi)(\widehat{u}^i(s) * \widehat{u}^j(s))\, ds,$$

where we have denoted by $q_{i,j}(\xi)$ the matrix elements of $\widehat{\mathbb{P}\mathcal{Q}}(\xi)$.

It is not difficult to see that all the usual results on the Cauchy problem for the Navier-Stokes equations hold for this system (namely the results of [6], [1] and [10] as recalled in the introduction). Moreover it is clear that if the Fourier transform of $\widehat{u}_0$ is positive, then this positivity property holds for all times.

Now let us construct initial data that will generate a solution which blows up in finite time. We will be following closely the argument of [14], and we refer to that article for all the computational details. We start by choosing the initial data $u_0 = (u_0^1, u_0^2)$ so that $\widehat{u}_0^1 \geq 0$ and the support of $\widehat{u}_0^1$ lies in the second and fourth sectors of the complex plane, that is, the zone where $\xi_1\xi_2 < 0$. We also suppose that this spectrum is symmetric with respect to zero (and, to fix notation, that the support of $\widehat{u}_0^1$ intersects the set $|\xi_j| \geq 1/2$, for $j \in \{1,2\}$). Taking into account the divergence free condition which states that $\widehat{u}^2(\xi) = -\dfrac{\xi_1\xi_2}{\xi_2^2}\widehat{u}^1(\xi)$, we deduce that $\widehat{u}_2$ is supported in the same region as $\widehat{u}_1$ and is also nonnegative.

Let us denote by A the L^1 norm of u_0 (which will be assumed to be large enough at the end), and let us write $u_0 = Aw_0$. The idea, as in [14], is to prove that for any $k \in \mathbb{N}$ and $j \in \{1,2\}$,

$$\widehat{u}^j(t,\xi) \geq A^{2^k} e^{-2^k t} 2^{k-4(2^k-1)} \mathbf{1}_{t \geq t_k} \widehat{w}_0^{k,j}(\xi) \tag{2.2}$$

where we have written $w_0^{k,j} = (w_0^{0,j})^{2^k}$ and $\widehat{w}_0^0$ is the restriction of $\widehat{w}_0\, \mathbf{1}_{|\xi_j| \geq 1/2}$ to the second sector of the plane. Finally the time t_k is chosen so that $t_0 = 0$ and $t_k - t_{k-1} \geq 2^{-2k} \log 2$. Notice that $\lim\limits_{k\to\infty} t_k = \log 2^{1/3}$. The result (2.2) is proved by induction. Suppose that (2.2) is true for $k-1$ (it is clearly true for $k=0$). Due to the positivity of $\widehat{u}_0$, we can write

$$\begin{aligned} \widehat{u}^j(t,\xi) &\geq \sum_i \int_0^t e^{-(t-s)|\xi|^2} q_{i,j}(\xi) (\widehat{u}^i(s,\xi) * \widehat{u}^j(s,\xi))\, ds \\ &\geq \int_0^t e^{-(t-s)|\xi|^2} q_{j,j}(\xi) (\widehat{u}^j(s,\xi) * \widehat{u}^j(s,\xi))\, ds \end{aligned}$$

and, using the induction assumption along with the support restriction of $w_0^{k-1,j}$, we find that

$$\begin{aligned} \widehat{u}^j(t,\xi) &\geq \int_0^t e^{-(t-s)|\xi|^2} q_{j,j}(\xi) (A^{2^k-1}\alpha_{k-1}(s))^2\, ds\; \widehat{w}_0^{k-1,j} * \widehat{w}_0^{k-1,j}(\xi) \\ &\geq \int_0^t e^{-(t-s)2^{2k}} q_{j,j}(\xi) (A^{2^k-1}\alpha_{k-1}(s))^2\, ds\; \widehat{w}_0^{k-1,j} * \widehat{w}_0^{k-1,j}(\xi), \end{aligned}$$

where $\alpha_k(t) = 2^{k-4(2^k-1)}\mathbf{1}_{t\geq t_k}$. But $\widehat{w}_0^{k-1,j} * \widehat{w}_0^{k-1,j} = \widehat{w}_0^{k,j}$, and on the support of $\widehat{w}_0^{k,j}$ we have $q_{j,j}(\xi) \geq C2^k$. The induction then follows exactly as in [14].

Once (2.2) is obtained, the blow-up of all $\dot{B}^s_{\infty,\infty}$ norms follows directly, upon noticing that $u^j(t_\infty)$ can be bounded from below in $\dot{B}^s_{\infty,\infty}$ by $C(Ae^{-t_\infty}2^{-4})^{2^k}2^{(s+1)k}$, which goes to infinity with k as soon as $Ae^{-t_\infty}2^{-4} > 1$. This lower bound is simply due to the fact that (denoting by Δ_k the usual Littlewood-Paley truncation operator that appears in the definition of Besov norms)

$$\begin{aligned} \|u(t_\infty)\|_{\dot{B}^s_{\infty,\infty}} &= \sup_k 2^{ks}\|\Delta_k u(t_\infty)\|_{L^\infty} \\ &\geq \sup_k 2^{ks}|\Delta_k u(t_\infty,0)| = \sup_k 2^{ks}\|\widehat{\Delta_k u}(t_\infty)\|_{L^1} \end{aligned}$$

since $\widehat{\Delta_k u}(t_\infty)$ is nonnegative.

Remark 2.1. One may notice that as soon as the matrix Q has been defined, the computation turns out to be identical to the case studied in [14]. In particular the important fact is that $\widehat{u}_0$ is nonnegative (and that its support intersects, say, the set $|\xi_j| \geq 1/2$).

Remark 2.2. As explained in the introduction, it seems natural to try to improve the previous example by perturbing the vorticity equation, since that equation is special in two space dimensions. Let us therefore consider the vorticity $\omega = \partial_1 u^2 - \partial_2 u^1$. As is well known, the two-dimensional Navier-Stokes equations can simply be written as a transport-diffusion equation in ω:

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = 0,$$

which can also be written, since u is divergence free, as

$$\partial_t \omega + \partial_1(u^1 \omega) + \partial_2(u^2 \omega) - \Delta \omega = 0.$$

Changing the places of the derivatives, and noticing that a derivative of u has the same scaling as ω, a model equation for the vorticity equation is simply

$$\partial_t \omega + \omega^2 - \Delta \omega = 0.$$

This simplified model is a semilinear heat equation for which the blow-up of the solution is well known (see [4], [5]). It is also easy to see that the argument of [14] is true for this system, which therefore blows up in finite time for large enough initial data with negative Fourier transform. Note that the equation on u becomes

$$\partial_t u + \nabla^\perp \Delta^{-1} \big((\operatorname{curl} u)^2 \big) - \Delta u = -\nabla p \quad , \quad \operatorname{div} u = 0,$$

which blows up but does not preserve the sign of the Fourier transform.

3. Proof of the theorem in the three-dimensional case

The three-dimensional situation follows the lines of the two-dimensional case studied above, though it is slightly more technical. The main step, as in the previous section, consists of finding a three-dimensional matrix $\mathcal{Q}$ such that the Fourier transform of the product $\mathbb{P}\mathcal{Q}$ has positive coefficients (we recall that $\mathbb{P}$ denotes the L^2 projection onto divergence free vector fields). Let us define, similarly as in the previous section, the matrix

$$\widehat{\mathcal{Q}}(\xi) = |\xi| \mathbf{1}_{\xi \in \mathcal{E}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

where $\mathcal{E} \stackrel{\text{def}}{=} \big\{ \xi \in \mathbb{R}^3 : \ \xi_1 \xi_2 < 0, \ \xi_1 \xi_3 < 0, \ |\xi_2| < \min(|\xi_1|, |\xi_3|) \big\}$. We compute easily that

$$\widehat{\mathbb{P}\mathcal{Q}}(\xi) = \mathbf{1}_{\xi \in \mathcal{E}} |\xi|^{-1} \begin{pmatrix} \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_1\xi_3 & \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_1\xi_3 & \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_1\xi_3 \\ \xi_1^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 & \xi_1^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 & \xi_1^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 \\ \xi_1^2 + \xi_2^2 - \xi_1\xi_3 - \xi_2\xi_3 & \xi_1^2 + \xi_2^2 - \xi_1\xi_3 - \xi_2\xi_3 & \xi_1^2 + \xi_2^2 - \xi_1\xi_3 - \xi_2\xi_3 \end{pmatrix}.$$

Let us consider the sign of the matrix elements of $\widehat{\mathbb{P}\mathcal{Q}}(\xi)$. The first line of the above matrix is clearly made up of positive scalars, due to the sign condition imposed on the components of ξ. The components of the second line may be written as

$$\xi_1^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 = \xi_1^2 - \xi_1\xi_2 + \xi_3(\xi_3 - \xi_2),$$

which is also positive since either ξ_2 and ξ_3 are both positive, in which case $\xi_3 > \xi_2$, or they are both negative, in which case $\xi_3 < \xi_2$. Similarly one has

$$\xi_1^2 + \xi_2^2 - \xi_1\xi_3 - \xi_2\xi_3 = \xi_1^2 + \xi_2^2 - \xi_3(\xi_1 + \xi_2),$$

and either $\xi_1 > 0$, $\xi_2 < 0$, $\xi_3 < 0$ and $\xi_1 + \xi_2 > 0$ or $\xi_1 < 0$, $\xi_2 > 0$, $\xi_3 > 0$ and $\xi_1 + \xi_2 < 0$. So the third line is also made up of positive real numbers.

Now that it has been checked that all coefficients are positive, we just have to follow again the proof of the two-dimensional case to obtain the expected result showing the blow-up of solutions to

$$\partial_t u - \Delta u = Q(u,u), \quad u_{|t=0} = u_0,$$

where $Q(u,u) = \mathbb{P}\mathcal{Q}(u,u)$ is the vector defined in (2.1). We will not write out all the details, which are identical to those in the two-dimensional case, but simply give the form of the initial data, which is summarized in the next proposition.

Proposition 3.1. *Let u_0 be a smooth, divergence-free vector field such that the components of $\widehat{u}_0$ are even, nonnegative functions and such that the support of $\widehat{u}_0$ intersects the set $|\xi_j| \geq 1/2$, for $j \in \{1,2,3\}$. Then the unique solution to* (TNS$_3$) *associated with u_0 blows up in finite time in all Besov spaces.*

We will not detail the proof of this proposition, as it is identical to that in the two-dimensional case (and thus to [14]).

Of course one must check that such initial data exists. The simplest way to construct such initial data is simply to suppose that it only has two nonvanishing components, say u_0^1 and u_0^2, and that the Fourier transform of u_0^1 is supported in $\mathbf{1}_{\xi_1\xi_2<0}$ while intersecting the set $|\xi_j| \geq 1/2$. The divergence free condition ensures that the same properties hold for u_0^2 (and u_0^3 is assumed to vanish identically). An explicit example is provided in the next section.

This ends the proof of the "blowing up" part of the theorem.

Remark 3.1. Notice that in the example, the energy inequality (E) cannot be satisfied, as it would require that $(Q(u,u)|u)_{L^2} \geq 0$, which cannot hold in our situation if the Fourier transform of $\widehat{u}$ is nonnegative.

4. Examples of arbitrarily large initial data providing a blow-up solution to (TNS$_3$) and a global solution to (NS)

In this short section, we check that the initial data provided in [2], which allows us to obtain large, global solutions for the Navier-Stokes equations, gives rise to a solution that blows up in finite time for the modified three-dimensional Navier-Stokes equation constructed in the previous section.

More precisely we have the following result.

Proposition 4.1. *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^3)$ such that $\widehat{\phi} \geq 0$ and such that $\widehat{\phi}$ is even and has its support in the region $\mathbf{1}_{\xi_1\xi_2<0}$ while intersecting the set $|\xi_j| \geq 1/2$, for $j \in \{1,2,3\}$. Let ε and α be given in $]0,1[$, and consider the family of initial data*

$$u_{0,\varepsilon}(x) = (\partial_2\varphi_\varepsilon(x), -\partial_1\varphi_\varepsilon(x), 0)$$

where

$$\varphi_\varepsilon(x) = \frac{(-\log\varepsilon)^{\frac{1}{5}}}{\varepsilon^{1-\alpha}} \cos\left(\frac{x_3}{\varepsilon}\right)(\partial_1\phi)\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\Big).$$

Then for $\varepsilon > 0$ small enough, the unique solution of (NS) *associated with $u_{0,\varepsilon}$ is smooth and global in time, whereas the unique solution of* (TNS_3) *associated with $u_{0,\varepsilon}$ blows up in finite time in all Besov norms.*

Remark 4.1. It is proved in [2] that such initial data has a large $\dot{B}^{-1}_{\infty,\infty}$ norm, in the sense that there is a constant C such that

$$C^{-1}(-\log\varepsilon)^{\frac{1}{5}} \leq \|u_{0,\varepsilon}\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C(-\log\varepsilon)^{\frac{1}{5}}.$$

To prove Proposition 4.1, we notice that the initial data given in the proposition is a particular case of the family of initial data presented in [2], Theorem 2, which generates a unique, global solution as soon as ε is small enough (in [2] there is no restriction on the support of the Fourier transform and $\partial_1\phi$ is simply ϕ). So we just have to check that the initial data fits with the requirements of Section 3 above, and more precisely that it satisfies the assumptions of Proposition 3.1. Notice that

$$\widehat{\varphi}_\varepsilon(\xi) = \frac{(-\log\varepsilon)^{\frac{1}{5}}}{2\varepsilon^{1-2\alpha}}\left(i\xi_1\widehat{\phi}(\xi_1,\varepsilon^\alpha\xi_2,\xi_3+\frac{1}{\varepsilon}) + i\xi_1\widehat{\phi}(\xi_1,\varepsilon^\alpha\xi_2,\xi_3-\frac{1}{\varepsilon})\right).$$

We need to check that $\widehat{u}^i_{0,\varepsilon} \geq 0$, for $i \in \{1,2,3\}$, and that the Fourier support intersects the set $|\xi_j| \geq 1/2$. We have

$$\widehat{u}_{0,\varepsilon}(\xi) = \frac{(-\log\varepsilon)^{\frac{1}{5}}}{2\varepsilon^{1-2\alpha}}\left(-\xi_1\xi_2\widehat{\phi}(\xi_1,\varepsilon^\alpha\xi_2,\xi_3\pm\frac{1}{\varepsilon}), \xi_1^2\widehat{\phi}(\xi_1,\varepsilon^\alpha\xi_2,\xi_3\pm\frac{1}{\varepsilon}), 0\right),$$

and so we clearly have the desired properties.

This ends the proof of the proposition and of the theorem.

References

[1] M. Cannone, Y. Meyer and F. Planchon, Solutions auto-similaires des équations de Navier-Stokes, *Séminaire sur les Équations aux Dérivées Partielles de l'École Polytechnique*, Palaiseau, Exposé VIII, 1993-1994. MR1300903 (95k:35157)

[2] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbf{R}^3$, accepted for publication, *Annales de l'Institut H. Poincaré, Analyse Non Linéaire.*

[3] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, accepted for publication, *Transactions of the AMS.*

[4] A. Friedman, Remarks on nonlinear parabolic equations. Proc. Sympos. Appl. Math., Vol. XVII, pages 3-23. Amer. Math. Soc., Providence, R.I., 1965. MR0186938 (32:4393)

[5] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I*, **13** (1966), pages 109-124. MR0214914 (35:5761)

[6] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16** (1964), pages 269–315. MR0166499 (29:3774)

[7] I. Gallagher and F. Planchon, On global infinite energy solutions to the Navier–Stokes equations in two dimensions, *Archive for Rational Mechanics and Analysis*, **161** (2002), pages 307–337. MR1891170 (2002m:35182)

[8] P. Germain, Équations de Navier-Stokes dans $\mathbb{R}^2$: existence et comportement asymptotique de solutions d'énergie infinie, *Bull. Sci. Math.*, **130** (2006), no. 2, pages 123–151. MR2200642 (2006k:35215)

[9] R. Grundy and R. McLaughlin, Three-dimensional blow-up solutions of the Navier-Stokes equations, *IMA J. Appl. Math.*, **63** (1999), no. 3, pages 287–306. MR1725742 (2000i:76035)

[10] H. Koch and D. Tataru, Well-posedness for the Navier–Stokes equations, *Advances in Mathematics*, **157** (2001), pages 22–35. MR1808843 (2001m:35257)

[11] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63** (1933), pages 193–248.

[12] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures. Appl.*, **12** (1933), pages 1–82.

[13] D. Li and Ya. Sinai, Blow ups of complex solutions of the 3D-Navier-Stokes system and renormalization group method, *J. Eur. Math. Soc.*, **10** (2008), no. 2, pages 267–313. MR2390325

[14] S. Montgomery-Smith, Finite-time blow up for a Navier-Stokes like equation, *Proc. Amer. Math. Soc.*, **129** (2001), no. 10, pages 3025–3029. MR1840108 (2002d:35164)

[15] M. Nagayama, H. Okamoto and J. Zhu, On the blow-up of some similarity solutions of the Navier-Stokes equations, *Topics in mathematical fluid mechanics*, pages 137–162, Quad. Mat., 10, Dept. Math., Seconda Univ. Napoli, Caserta, 2002. MR2051773 (2006a:76032)

Institut de Mathématiques de Jussieu, UMR 7586, Université Paris 7, 175 rue du Chevaleret, 75013 Paris, France

E-mail address: Isabelle.Gallagher@math.jussieu.fr

Département de Mathématiques, Université Paris 11, Bâtiment 425, 91405 Orsay Cedex, France

E-mail address: marius.paicu@math.u-psud.fr