

LIPSCHITZNESS OF THE LEMPERS AND GREEN FUNCTIONS

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ABSTRACT. Necessary and sufficient conditions for Lipschitzness of the Lempert and Green functions are found in terms of their boundary behaviors.

1. INTRODUCTION AND RESULTS

By \mathbb{D} we denote the unit disc in \mathbb{C} . Let D be a domain in \mathbb{C}^n . Recall first the definitions of the Lempert function and the Kobayashi–Royden pseudometric of D :

$$l_D(z, w) := \inf\{\alpha \in [0, 1] : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\},$$
$$\kappa_D(z; X) := \inf\{\alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha\varphi'(0) = X\}.$$

We point out that both functions are upper semicontinuous and l_D is symmetric. The Kobayashi–Buseman pseudometric $\hat{\kappa}_D(z; \cdot)$ (the Kobayashi pseudodistance k_D) is the largest pseudonorm (pseudodistance) which does not exceed $\kappa_D(z; \cdot)$ ($\tanh^{-1} l_D$). Note that if D is a *taut* domain, i.e., $\mathcal{O}(\mathbb{D}, D)$ is a normal family, then κ_D and $\hat{\kappa}_D$ are the infinitesimal forms of l_D and k_D , respectively (see [9], Theorem 1 for a more general result). Moreover, recall (cf. [7], Proposition 3.2) that D is a taut domain if and only if

$$\lim_{z \in K, w \rightarrow \partial D} l_D(z, w) = 1 \text{ for any } K \Subset D.$$

(Note that for an unbounded D the point ∞ belongs, by definition, to ∂D .)

The main result in [6] (see Theorem 6 there) is that κ_D is locally a Hölder function of order $2/3$ on any C^6 -smooth strongly pseudoconvex domain D in \mathbb{C}^n (see also [5], where it is claimed that κ_D is locally Lipschitz but the proof there seems to be not correct).

Our first goal in the present note is to generalize this result showing that l_D and κ_D are Lipschitz functions under a natural assumption about the boundary behavior of l_D . In fact we have the following result.¹

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¹Proofs for this and the next results will be presented in section 2.

Proposition 1. *Let $D \subset \mathbb{C}^n$ be a hyperbolic domain (i.e., k_D is a distance) and $K \Subset D$ be such that $\sup_{z \in K, w \in D} \frac{1 - l_D(z, w)}{\text{dist}(w, \partial D)} < \infty$. Then:*

- (i) l_D is a Lipschitz function on $K \times D$;
- (ii) there is a $C > 0$ such that if $z, w \in K, X, Y \in \mathbb{C}^n$, then

$$|\kappa_D(z; X) - \kappa_D(w; Y)| \leq C((\|X\| + \|Y\|) \cdot \|z - w\| + \|X - Y\|).$$

Remark A. (a) By symmetry, l_D is a Lipschitz function on $D \times K$, too. On the other hand, $l_{\mathbb{D}}$ is not a Lipschitz function on $\mathbb{D} \times \mathbb{D}$.

(b) For the Carathéodory–Reiffen pseudometric, the same estimate as in (ii) remains true for any domain in \mathbb{C}^n (see [3], Proposition 2.5.1(c)).

(c) Any compact subset of a strongly pseudoconvex domain satisfies the assumption of Proposition 1 (cf. [3], Theorem 10.2.1).

(d) If $D \subset \mathbb{C}^n$ is a hyperbolic domain, $K \Subset D, L \Subset \mathbb{C}^n$, and $\sup_{z \in K, w \in D} \frac{1 - l_D(z, w)}{\text{dist}^\alpha(w, \partial D)} < \infty, \alpha \in (0, 1)$, then obvious modifications in the proof of Proposition 1 imply that l_D and κ_D are Hölder functions with exponent α on $K \times D$ and $K \times L$, respectively. On the other hand, α cannot be taken larger than 1; one can show that for any domain $D \subsetneq \mathbb{C}^n$ and any point $z \in D$ we have $\limsup_{w \rightarrow \partial D} \frac{1 - l_D(z, w)}{\text{dist}(w, \partial D)} > 0$.

We point out that for a taut domain D the assumption of Proposition 1 is also necessary for l_D to be a Lipschitz function.

Corollary 2. *Let $D \subset \mathbb{C}^n$ be a taut domain and $K \Subset D$. Then $\sup_{z \in K, w \in D} \frac{1 - l_D(z, w)}{\text{dist}(w, \partial D)} < \infty$ if and only if l_D is a Lipschitz function on $K \times D$.*

To prove the Lipschitzness of $\hat{\kappa}_D$ under the assumption of Proposition 1, we shall need the following result.

Proposition 3. *Let $D \subset \mathbb{C}^n$ be a hyperbolic domain and let $K \Subset D, c > 0$ be such that*

$$|\kappa_D(z; X) - \kappa_D(w; X)| \leq c\|X\| \cdot \|z - w\|, \quad z, w \in K, X \in \mathbb{C}^n.$$

Then there is a $C > 0$ such that if $z, w \in K, X, Y \in \mathbb{C}^n$, then

$$|\hat{\kappa}_D(z; X) - \hat{\kappa}_D(w; Y)| \leq C((\|X\| + \|Y\|) \cdot \|z - w\| + \|X - Y\|).$$

The next corollary is an immediate consequence of Propositions 1 and 3.

Corollary 4. *Let $D \subset \mathbb{C}^n$ and $K \Subset D$ be as in Proposition 1. Then there is a $C > 0$ such that if $z, w \in K, X, Y \in \mathbb{C}^n$, then*

$$|\hat{\kappa}_D(z; X) - \hat{\kappa}_D(w; Y)| \leq C((\|X\| + \|Y\|) \cdot \|z - w\| + \|X - Y\|).$$

The second aim of our paper is to find a necessary and sufficient condition for the exponential of the pluricomplex Green function to be Lipschitz (similar to that for the Lempert function).

Recall first the definitions of the pluricomplex Green function and the Azukawa pseudometric of a domain D in \mathbb{C}^n :

$$g_D(z, w) := \sup\{u(w) : u \in PSH(D), u < 0, \limsup_{\zeta \rightarrow z} (u(\zeta) - \log \|\zeta - z\|) < \infty\},$$

$$A_D(z; X) := \limsup_{t \rightarrow 0} \frac{\tilde{g}_D(z, z + tX)}{|t|},$$

where $\tilde{g}_D := \exp g_D$. We point out that both functions are upper semicontinuous (cf. [4], page 10) and $\tilde{g}_D \leq l_D$. Note also that, in general, g_D is not symmetric.

Recall also that a domain $D \subset \mathbb{C}^n$ is called *hyperconvex* if it has a negative plurisubharmonic exhaustion function. The next proposition is a consequence of the proof of Theorem 3.1 in [2] (see also [1], Theorem 2, for a weaker version).

Proposition 5. *Let $D \subset \mathbb{C}^n$ be a bounded domain. Then the following conditions are equivalent:*

- (i) *there is $u \in PSH(D)$ with $u < 0$ and $\inf_{z \in D} u(z)/\text{dist}(z, \partial D) > -\infty$;*
- (ii) *D is hyperconvex and there are $z_0 \in D$ and $C > 0$ such that if $w_1, w_2 \in D \setminus \{z_0\}$, then*

$$|g_D(z_0, w_1) - g_D(z_0, w_2)| \leq C \frac{\|w_1 - w_2\|}{\min\{\|z_0 - w_1\|, \|z_0 - w_2\|\}};$$

- (iii) *D is hyperconvex and for any $K \Subset D$ there is a $C > 0$ such that if $z \in K$ and $w_1, w_2 \in D \setminus \{z\}$, then*

$$|g_D(z, w_1) - g_D(z, w_2)| \leq C \frac{\|w_1 - w_2\|}{\min\{\|z - w_1\|, \|z - w_2\|\}}.$$

As a simple consequence we get the following result for \tilde{g}_D .

Corollary 6. *Let D and u be as in Proposition 5(i) and let $K \Subset D$. Then there is $C > 0$ such that*

$$|\tilde{g}_D(z, w_1) - \tilde{g}_D(z, w_2)| \leq C\|w_1 - w_2\|, \quad z \in K, w_1, w_2 \in D.$$

Remark B. Let D be a hyperconvex domain (not necessary bounded) and u be as in Proposition 5 (if D is bounded, then (i) implies that u is an exhaustion function of D and hence D is hyperconvex). Then, for an arbitrary $K \Subset D$, the assumptions of Proposition 1 are satisfied. Indeed, it follows from (5) below that

$$1 - l_D(z, w) \leq 1 - \tilde{g}_D(z, w) \leq -g_D(z, w) \leq c \text{dist}(w, \partial D),$$

$$z \in K, w \in D, \text{ near } \partial D;$$

hence the inequality in the assumption of Proposition 1 is fulfilled. It remains to use that $l_D \geq \tilde{g}_D$ and that D is hyperconvex. Hence D is taut (cf. [7], page 607) and therefore hyperbolic.

From Corollary 6 we get that, under the same assumptions, \tilde{g}_D and A_D are Lipschitz functions (in both arguments).

Proposition 7. *Let D and u be as in Proposition 5(i) and let $K \subset D$ be compact. Then:*

- (i) *\tilde{g}_D is a Lipschitz function on $K \times D$;*
- (ii) *there is a $C > 0$ such that if $z, w \in K$, $X, Y \in \mathbb{C}^n$, then*

$$|A_D(z; X) - A_D(w; Y)| \leq C((\|X\| + \|Y\|) \cdot \|z - w\| + \|X - Y\|).$$

It remains an open question whether \tilde{g}_D is a Lipschitz function on $D \times K$.

Remark C. Let $D \subset \mathbb{C}^n$ be a pseudoconvex balanced domain with Minkowski function h_D . Recall that (cf. [3], Propositions 3.1.10 and 4.2.7 (b))

$$l_D(0, \cdot) = \kappa_D(0; X) = g_D(0, \cdot) = A_D(0; \cdot) = h_D.$$

Note also that (cf. [7], Proposition 4.4) D is taut $\Leftrightarrow D$ is hyperconvex $\Leftrightarrow D$ is bounded and h_D is continuous.

By Corollary 2 or Corollary 6, for a taut balanced domain D the following are equivalent:

- (i) there is $c > 0$ such that $1 - h_D(z) \leq c \cdot \text{dist}(z, \partial D)$, $z \in D$;
- (ii) there is $c' > 0$ such that $|h_D(z) - h_D(w)| \leq c' \|z - w\|$, $z, w \in D$.

(Taking $h_D(z) = |z_1| + |z_2| + \sqrt{|z_1 z_2|}$ provides an example of a taut balanced domain $D \subset \mathbb{C}^2$ which does not have the above properties.)

We point out that (i) \Leftrightarrow (ii) with $c = c'$ for any balanced domain D in \mathbb{C}^n .

Indeed, assume that (i) holds. Then for any z, w with $1 > h_D(z) > h_D(w)$ we have

$$\begin{aligned} h_D(z) - h_D(w) &= h_D(z)(1 - h_D(w/h_D(z))) \leq h_D(z)c \cdot \text{dist}(w/h_D(z), \partial D) \\ &\leq h_D(z)c \|w/h_D(z) - z/h_D(z)\| = c \|z - w\|. \end{aligned}$$

Conversely, assume that (ii) is true. Fix a $z \in D$. If $\|u\| < r_z := (1 - h_D(z))/c'$, then $h_D(z + u) \leq h_D(z) + c'\|u\| < 1$, which shows that $\mathbb{B}_n(z, r_z) \subset D$. Hence $\text{dist}(z, \partial D) \geq r_z$; that is, (i) holds with $c = c'$.

2. PROOFS

Proof of Proposition 1. The assumption of Proposition 1 means that there is a $c > 0$ such that for any $r \in (0, 1)$ and $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) \in K$ one has that

$$c \cdot \text{dist}(\varphi(r\mathbb{D}), \partial D) \geq 1 - r.$$

Note that there is a $c_1 > 0$ such that

$$c_1 \|z - w\| \geq l_D(z, w), \quad z \in K, w \in D.$$

On the other hand, if D is unbounded, then, by hyperbolicity,

$$m^* = \liminf_{z \in K, w \rightarrow \infty} l_D(z, w) > 0$$

(use e.g. [7], Proposition 3.1). Fix an $m \in (0, \min\{1/2, m^*\})$. Then, again by hyperbolicity, we find a $c_2 > 0$ such that

$$l_D(z, w) \leq m, z \in K, w \in D \Rightarrow l_D(z, w) \geq c_2 \|z - w\|$$

(apply e.g. [3], Theorem 7.2.2; if D is bounded, the last inequality holds even on $K \times D$ with suitable $c_2 > 0$; no other assumptions are needed in this situation). We may assume that $c_1 > 1 > c_2$. Set $c_3 = c_1(1 + c/(mc_2))$. To prove (i), it suffices to show that if

$$(1) \quad |l_D(z, w_1) - l_D(z, w_2)| \leq c_3 \|w_1 - w_2\|, \quad z \in K, w_1, w_2 \in D,$$

then

$$(2) \quad |l_D(w_1, z) - l_D(w_2, z)| \leq 2c_3 \|w_1 - w_2\|, \quad w_1, w_2 \in K, z \in D.$$

To prove (1), we may assume that $\alpha := l_D(z, w_1) \leq l_D(z, w_2)$ and $z \neq w_1$. Then, by hyperbolicity, $\alpha > 0$. Set $r = 1 - c\|w_1 - w_2\|/\alpha$. We shall consider three cases.

Case 1. $r > \max\{\alpha, m\}$. Then for any $\alpha' \in (\alpha, r)$ there is $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(\alpha') = w_1$. Set $\psi(\zeta) = \varphi(r\zeta) + (w_2 - w_1)r\zeta/\alpha'$, $\zeta \in \mathbb{D}$. Then $\psi \in \mathcal{O}(\mathbb{D}, D)$ and $\psi(\alpha'/r) = w_2$ ($\alpha' < r$). It follows that $l_D(z, w_2) \leq \alpha'/r$ and hence

$$\begin{aligned} l_D(z, w_2) - l_D(z, w_1) &\leq \alpha(1 - r)/r \\ &= c\|w_2 - w_1\|/r \leq c\|w_2 - w_1\|/m \leq c_3\|w_2 - w_1\|. \end{aligned}$$

Case 2. $\alpha \geq \max\{r, m\}$. Then

$$\begin{aligned} l_D(z, w_2) - l_D(z, w_1) &< 1 - \alpha \leq 1 - r \\ &= c\|w_1 - w_2\|/\alpha \leq c\|w_1 - w_2\|/m < c_3\|w_1 - w_2\|. \end{aligned}$$

Case 3. $m \geq \max\{r, \alpha\}$. Then

$$\|w_1 - w_2\| = (1 - r)\alpha/c \geq (1 - m)\alpha/c \geq (1 - m)c_2\|z - w_1\|/c,$$

and, by the triangle inequality, $\|z - w_2\| \leq (1 + c/((1 - m)c_2))\|w_1 - w_2\|$. Since $m \leq 1/2$, it follows that

$$l_D(z, w_2) - l_D(z, w_1) < l_D(z, w_2) \leq c_1\|z - w_2\| \leq c_3\|w_1 - w_2\|.$$

This completes the proof of (1).

The proof of (2) is similar to that of (1) and we sketch it. We may assume that $0 < \beta := l_D(w_1, z) \leq l_D(w_2, z)$ and then set $s = 1 - 2c\|w_1 - w_2\|/\beta$. We get as above that:

Case 1. If $\beta \geq \max\{s, m\}$, then $l_D(w_2, z) - l_D(w_1, z) < 2c\|w_1 - w_2\|/\beta$;

Case 2. If $m \geq \max\{s, \beta\}$, then $l_D(w_2, z) - l_D(w_1, z) < c_3\|w_1 - w_2\|$.

Case 3. In the remaining case $s > \max\{\beta, m\}$, for any $\beta' \in (\beta, s)$ we may find $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = w_1$ and $\varphi(\beta') = z_1$. Set $\psi(\zeta) = \varphi(s\zeta) + (w_2 - w_1)(1 - s\zeta/\beta')$, $\zeta \in \mathbb{D}$. Then $\psi \in \mathcal{O}(\mathbb{D}, D)$, $\psi(0) = w_2$ and $\psi(\beta'/s) = z$. It follows that $l_D(w_2, z) \leq \beta/s$ and hence

$$l_D(w_2, z) - l_D(w_1, z) \leq 2c\|w_2 - w_1\|,$$

which completes the proof of (2).

Next, we shall prove (ii). It is enough to show that

$$(3) \quad |\kappa_D(z; X) - \kappa_D(w; X)| \leq 4cc_4\|X\| \cdot \|z - w\|$$

and

$$(4) \quad |\kappa_D(z; X) - \kappa_D(z; Y)| \leq c_5\|X - Y\|,$$

for any $z, w \in K$, $X, Y \in \mathbb{C}^n$, where $c_4 := \sup_{u \in K, \|U\|=1} \kappa_D(u; U)$, $c_5 := c_4(1 + 2c/c_6)$ and $c_6 := \inf_{u \in K, \|U\|=1} \kappa_D(u; U)$ ($c_6 > 0$ by hyperbolicity; cf. [3], Theorem 7.2.2).

For proving (3), observe that

$$|\kappa_D(z; X) - \kappa_D(w; X)| \leq 2c_4\|X\|.$$

So (3) is trivial if $p = 1 - c\|z - w\| \leq 1/2$. Otherwise, we may assume that $\kappa_D(z; X) \leq \kappa_D(w; X)$. For any $\varphi \in (\mathbb{D}, D)$ set $\psi(\zeta) = \varphi(p\zeta) + w - z$, $\zeta \in \mathbb{D}$. Then $\psi \in \mathcal{O}(\mathbb{D}, D)$, which shows that $\kappa_D(w; X) \leq \kappa_D(z; X)/p$. This implies (3) with $2cc_4$ instead of $4cc_4$.

To get (4), we may assume that $\gamma = \kappa_D(z; X) \leq \kappa_D(z; Y)$ and $X \neq 0$. Then $\gamma > 0$. For $q = 1 - c\|X - Y\|/\gamma$ we have two cases.

Case 1. $q > 1/2$. Let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ be such that $\varphi(0) = z$ and $\gamma'\varphi'(0) = X$ for some γ' . Set $\psi(\zeta) = \varphi(q\zeta) + (Y - X)q\zeta/\gamma'$. Then $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $\gamma'\psi'(0) = qY$. It follows that $\kappa_D(z; Y) \leq \gamma/q$ and hence

$$\kappa_D(z; Y) - \kappa_D(z; X) \leq \gamma(1 - q)/q = c\|X - Y\|/q \leq c_5\|X - Y\|.$$

Case 2. $q \leq 1/2$. Then $\|X - Y\| = (1 - q)\gamma/c \geq c_6\|X\|/(2c)$ and, by the triangle inequality, $\|Y\| \leq (1 + 2c/c_6)\|X - Y\|$. It follows that

$$\kappa_D(z; Y) - \kappa_D(z; X) < \kappa_D(z; Y) \leq c_4\|Y\| \leq c_5\|X - Y\|.$$

This completes the proof of Proposition 1. □

Proof of Corollary 2. By Proposition 1, it is enough to show that if

$$|l_D(z, w_1) - l_D(z, w_2)| \leq c\|w_1 - w_2\|, \quad z \in K, w_1, w_2 \in D,$$

then

$$\sup_{z \in K, w \in D} \frac{1 - l_D(z, w)}{\text{dist}(w, \partial D)} < \infty.$$

Suppose this is not true. Then there are sequences $(z_j)_j \subset K$ and $(w_j)_j \subset D$ such that

$$1 - l_D(z_j, w_j) \geq j \text{dist}(w_j, \partial D), \quad j \in \mathbb{N}.$$

Choose $b_j \in \partial D$ with $\|w_j - b_j\| = \text{dist}(w_j, \partial D)$ and sequences $(b_{j,k})_k \subset D$ with $b_{j,k} \rightarrow b_j$ if $k \rightarrow \infty$, $j \in \mathbb{N}$. Then

$$\begin{aligned} j &\leq \frac{1 - l_D(z_j, b_{j,k}) + l_D(z_j, b_{j,k}) - l_D(z_j, w_j)}{\|w_j - b_j\|} \\ &\leq \frac{1 - l_D(z_j, b_{j,k})}{\|w_j - b_j\|} + c \frac{\|w_j - b_{j,k}\|}{\|w_j - b_j\|} \leq 1 + 2c \end{aligned}$$

if $k = k_j$ is sufficiently large. Recall that D is taut; therefore such a k_j always exists, a contradiction. □

Proof of Proposition 3. Since $\hat{\kappa}_D(z; \cdot)$ is a norm, it is enough to show that for any $K \Subset D$ there is $C' > 0$ such that

$$|\hat{\kappa}_D(z; X) - \hat{\kappa}_D(w; X)| \leq C'\|X\| \cdot \|z - w\|, \quad z, w \in K, X \in \mathbb{C}^n.$$

We may assume that $z \neq w$, $X \neq 0$ and $\hat{\kappa}_D(z; X) \leq \hat{\kappa}_D(w; X)$. Then there are vectors $X_1, \dots, X_{2n-1} \in \mathbb{C}^n$ with sum X such that (see [8], Theorem 1)

$$\sum_{j=1}^{2n-1} \kappa_D(z; X_j) \leq \hat{\kappa}_D(z; X) + \|X\| \cdot \|z - w\|.$$

It follows that

$$\begin{aligned} 0 \leq \hat{\kappa}_D(w; X) - \hat{\kappa}_D(z; X) &\leq \sum_{j=1}^{2n-1} (\kappa_D(w; X_j) - \kappa_D(z; X_j)) + \|X\| \cdot \|z - w\| \\ &\leq \|z - w\|(\|X\| + c \sum_{j=1}^{2n-1} \|X_j\|). \end{aligned}$$

It remains to use that $\sum_{j=1}^{2n-1} \|X_j\| \leq \frac{c_4}{c_6}\|X\|$, where c_4 and c_6 are as in the proof of Proposition 1. □

Proof of Proposition 5. (ii) \Rightarrow (i). Put $u = \tilde{g}_D(z_0, \cdot) - 1$. Since D is a hyperconvex domain, then $\lim_{w \rightarrow \partial D} u(w) = 0$. Now a similar argument as in the proof of Corollary 2 implies that u has the required property.

Since (iii) \Rightarrow (ii) is trivial, it remains to prove:

(i) \Rightarrow (iii). Since u is an exhaustion function of D , it follows that D is hyperconvex.

Fix a $K \Subset D$. We shall show that if D is hyperconvex (not necessarily bounded) and u is as in (i), then

$$(5) \quad \liminf_{z \in K, w \rightarrow \partial D} g_D(z, w) / \text{dist}(z, \partial D) > -\infty.$$

Indeed, let \tilde{u} be an exhaustion function of D and $\hat{u} = \max\{u, \tilde{u}\}$. Then take a domain $G_1 \Subset D$, $K \Subset G_1$, and put $\varepsilon = \sup_{G_1} \hat{u} / 2 < 0$. Next we choose a domain $G_2 \Subset D$, $G_1 \Subset G_2$, such that $\inf_{\partial G_2} \hat{u} \geq \varepsilon$.

Fix a $z \in K$. Set $\varphi(z, \cdot) = \log(\|\cdot - z\| / \text{diam } G_2)$, $m = \inf_{K \times G_1} \varphi$ and

$$v_z = \begin{cases} \varphi(z, \cdot) + m & \text{on } G_1 \\ \max\{\varphi(z, \cdot) + m, m\hat{u}/\varepsilon\} & \text{on } G_2 \setminus G_1 \\ m\hat{u}/\varepsilon & \text{on } D \setminus G_2. \end{cases}$$

It is easy to check that $v_z \in PSH(D)$ for $z \in K$. Hence $g_D(z, \cdot) \geq v_z$, which implies (5).

Now let $r > 0$ be such that $\mathbb{B}(a, r) \subset D$ for any $z \in K$. For any $\varepsilon \in (0, r)$ we set

$$g_D^\varepsilon(z, w) = \sup\{u(w) : u \in PSH(D), u < 0, u|_{\mathbb{B}(z, \varepsilon)} \leq \log(\varepsilon/r)\}.$$

One can easily check that $g_D^\varepsilon(z, \cdot)$ is a maximal plurisubharmonic function on $D \setminus \mathbb{B}(z, \varepsilon)$ (cf. [3], page 383 for this notion),

$$(6) \quad \max\{\log(\varepsilon/r), g_D(z, w)\} \leq g_D^\varepsilon(z, w) \leq \log \frac{\max\{\|z - w\|, \varepsilon\}}{r}$$

and $g_D^\varepsilon(z, \cdot) \downarrow g_D(z, \cdot)$ as $\varepsilon \downarrow 0$ locally uniformly in $D \setminus \{z\}$ (cf. [2], page 338, and Proposition 2.2). Moreover, since D is hyperconvex, g_D^ε can be extended as a continuous function on $D \times \overline{D}$ by setting $g_D^\varepsilon|_{D \times \partial D} = 0$.

We shall find $c_1, c_2 > 0$ such that if $z \in K$, $w_1, w_2 \in D \setminus \{z\}$, and $\varepsilon > 0$ satisfy the inequality

$$(7) \quad \max\{\varepsilon, c_1\|w_1 - w_2\|\} < \min\{r/2, \|z - w_1\|, \|z - w_2\|\},$$

then

$$(8) \quad |g_D^\varepsilon(z, w_1) - g_D^\varepsilon(z, w_2)| \leq c_2 \frac{\|w_1 - w_2\|}{\min\{\|z - w_1\|, \|z - w_2\|\}}.$$

Assuming (8), take arbitrary points $w_1, w_2 \in D \setminus \{z\}$. To prove (iii), we may assume that $g_D^\varepsilon(z, w_1) \leq g_D^\varepsilon(z, w_2)$, where ε is as above. There is a semicircle with diameter $[w_1 w_2]$, say $\gamma : [0, \pi] \rightarrow \mathbb{C}^n$, $\gamma(0) = w_1, \gamma(\pi) = w_2$, such that $\text{dist}(z, \gamma) = \min\{\|z - w_1\|, \|z - w_2\|\}$. Let $t' \in (0, 1]$ be the largest number such that $\gamma(t) \in D$ for $t \in (0, t')$. If $t' = 1$, then an “integration along γ ” gives

$$g_D^\varepsilon(z, w_2) - g_D^\varepsilon(z, w_1) \leq \pi c_2 \frac{\|w_1 - w_2\|}{\min\{\|z - w_1\|, \|z - w_2\|\}}.$$

If $t' < 1$, then $\gamma(t') \in \partial D$. Since

$$\lim_{w \rightarrow \partial D} g_D^\varepsilon(z, w) = 0 > g_D^\varepsilon(z, w_2) > g_D^\varepsilon(z, w_1)$$

and g_D^ε is continuous, we may find a $t^* \in [0, t')$ with $g_D^\varepsilon(z, \gamma(t^*)) = g_D^\varepsilon(z, w_2)$. Then, similarly as above, we get the same estimates. Letting $\varepsilon \rightarrow 0$ gives the estimate in (iii) with $C = \pi c_2$.

To prove (8), we may assume that $g_D^\varepsilon(z, w_1) < g_D^\varepsilon(z, w_2)$. Now let $f = f_{z,w} \in \mathcal{O}(D, \mathbb{D})$ be an extremal function for the Carathéodory distance $c_D(z, w)$ (cf. [3], page 16). We may assume that $f(z) = 0$. For $z \neq w$ set $h_{z,w}(\zeta) = f(\zeta)/f(w)$, $\zeta \in D$. Then there are $c_1, c_3 > 0$ with

$$(9) \quad |h_{z,w}(\zeta)| \leq \frac{\tanh c_D(z, \zeta)}{\tanh c_D(z, w)} \leq c_1 \frac{\|\zeta - z\|}{\|w - z\|} \leq \frac{c_3}{\|w - z\|}, \quad z \in K, w \in D \setminus \{z\}$$

(use that $\mathbb{B}(z, r) \subset D \subset \mathbb{B}(z, R)$, $z \in K$, for certain r, R). Set

$$D' = \{\zeta \in D : \zeta + h_{z,w_1}(\zeta)(w_2 - w_1) \in D\}, \quad D'' = D' \setminus \overline{\mathbb{B}(z, \varepsilon)}$$

and

$$\hat{g}(\zeta) = g_D^\varepsilon(z, \zeta + (w_2 - w_1)h_{z,w_1}(\zeta)), \quad \zeta \in D'.$$

It follows by (7) and (9) that $\mathbb{B}(z, \varepsilon) \Subset \mathbb{B}(z, r/2) \Subset D'$ and $w_1 \in D''$. On the other hand, by (5), there is a $c_4 > 0$ such that

$$g_D(z, \zeta) \geq -c_4 \operatorname{dist}(\zeta, \partial D), \quad \zeta \in D \setminus \mathbb{B}(z, r/2).$$

Thus, (6) and (9) imply that

$$\min_{\zeta \in \partial D'} g_D^\varepsilon(z, \zeta) \geq \min_{\zeta \in \partial D'} g_D(z, \zeta) \geq -c_4 \max_{\zeta \in \partial D'} \operatorname{dist}(\zeta, \partial D) \geq -c_3 c_4 \frac{\|w_2 - w_1\|}{\|w_1 - z\|}.$$

Then for

$$v(\zeta) = \hat{g}(\zeta) - g_D^\varepsilon(z, \zeta)$$

we have that

$$\limsup_{\zeta \rightarrow \partial D'} v(\zeta) \leq c_3 c_4 \frac{\|w_2 - w_1\|}{\|w_1 - z\|}.$$

On the other hand, for $\zeta \in \partial \mathbb{B}(z, \varepsilon)$, it follows by (6) and (9) that

$$\begin{aligned} v(\zeta) &\leq \log \frac{\max\{\varepsilon, \|\zeta + (w_2 - w_1)h_{z,w_1}(\zeta) - z\|\}}{r} - \log \frac{\varepsilon}{r} \\ &= \log^+ \frac{\|\zeta + (w_2 - w_1)h_{z,w_1}(\zeta) - z\|}{\varepsilon} \leq \log \left(1 + c_1 \frac{\|w_2 - w_1\|}{\|w_1 - z\|} \right). \end{aligned}$$

Since $g_D^\varepsilon(z, \cdot)$ is a maximal plurisubharmonic function on D'' and it is continuous on $\overline{D''} \subset \overline{D'}$, the domination principle implies that

$$v(\zeta) \leq c_2 \frac{\|w_2 - w_1\|}{\|w_1 - z\|}, \quad \zeta \in D'',$$

where $c_2 = \max\{c_1, c_3 c_4\}$. Applying this for $\zeta = w_1$ gives (8). □

Proof of Corollary 6. Recall that there is a $c' > 0$ such that $\tilde{g}_D(z, w) \leq c'\|z - w\|$, $z \in K, w \in D$. Therefore, we may assume that $w_1, w_2 \neq z$ and $\|z - w_2\| \leq \|z - w_1\|$. Two cases are possible.

Case 1. $|g_D(z, w_1) - g_D(z, w_2)| < 1$. Then

$$\begin{aligned} |\tilde{g}_D(z, w_1) - \tilde{g}_D(z, w_2)| &= \tilde{g}_D(z, w_2) |\exp(g_D(z, w_1) - g_D(z, w_2)) - 1| \\ &< (e - 1)c'\|z - w_2\| \cdot |g_D(z, w_1) - g_D(z, w_2)| \leq (e - 1)c'C\|w_1 - w_2\|, \end{aligned}$$

where C is the constant from Proposition 5.

Case 2. $|g_D(z, w_1) - g_D(z, w_2)| \geq 1$. Then, by Proposition 5,

$$C\|w_1 - w_2\| \geq \|z - w_2\| \geq \|z - w_1\| - \|w_1 - w_2\|,$$

and hence $(C + 1)\|w_1 - w_2\| \geq \|z - w_1\|$. It follows that

$$\begin{aligned} |\tilde{g}_D(z, w_1) - \tilde{g}_D(z, w_2)| &< \max\{\tilde{g}_D(z, w_1), \tilde{g}_D(z, w_2)\} \\ &\leq c'\|z - w_1\| \leq c'(C + 1)\|w_1 - w_2\|. \end{aligned}$$

□

Proof of Proposition 7. By (5), we may find $c > 0$ such that if

$$D_{z,\varepsilon} = \{u \in D : \tilde{g}_D(z, u) < \varepsilon\}, \quad z \in K, \varepsilon \in (0, 1),$$

then

$$\text{dist}(D_{z,\varepsilon}, \partial D) \geq c(1 - \varepsilon).$$

First, we shall prove (i). By virtue of Corollary 6, it is enough to find a $c_1 > 0$ such that

$$(10) \quad |\tilde{g}_D(z_1, w) - \tilde{g}_D(z_2, w)| \leq c_1\|z_1 - z_2\|, \quad z_1, z_2 \in K, w \in D.$$

We may assume that K is the closure of a smooth domain. Then there is a $c_2 > 0$ such that for any z_1, z_2 there is a smooth curve γ in K joining z_1 and z_2 with $l(\gamma) \leq c_2\|z_1 - z_2\|$. Set $\varepsilon = \varepsilon_{z_1, z_2} = 1 - \|z_1 - z_2\|/c$. Then, by integration over γ , it suffices to prove (10), fixing w and assuming that $1 - \|z_1 - z_2\|/c > \sup_{z \in K} \tilde{g}_D(z, w)$. Since $w \in D_{z_1, \varepsilon}$ and

$$D \supset \tilde{D} = \{z + z_2 - z_1 : z \in D_{z_1, \varepsilon}\},$$

then

$$\tilde{g}_D(z_1, w) = \varepsilon \tilde{g}_{D_{z_1, \varepsilon}}(z_1, w) = \varepsilon \tilde{g}_{\tilde{D}}(z_2, w + z_2 - z_1) \geq \tilde{g}_D(z_2, w + z_2 - z_1)$$

(cf. [10], Lemma 4.2.7, for the first equality). Hence

$$\begin{aligned} \tilde{g}_D(z_2, w) - \tilde{g}_D(z_1, w) &\leq \tilde{g}_D(z_2, w) - \varepsilon \tilde{g}_D(z_2, w + z_2 - z_1) \\ &\leq C\|z_2 - z_1\| + (1 - \varepsilon) = (C + 1/c)\|z_2 - z_1\|, \end{aligned}$$

where C is the constant from Corollary 6. By symmetry,

$$\tilde{g}_D(z_1, w) - \tilde{g}_D(z_2, w) \leq (C + 1/c)\|z_2 - z_1\|,$$

which implies (10).

To prove (ii), it is enough to show that:

- there is a $c_3 > 0$ such that for any $X, Y \in \mathbb{C}^n$,

$$(11) \quad |A_D(z; X) - A_D(z; Y)| \leq c_3\|X - Y\|;$$

- if $c_4 = \max_{z \in K, \|Z\|=1} A_D(z; Z)$, then

$$(12) \quad |A_D(z_1; X) - A_D(z_2; X)| \leq c_4\|X\| \cdot \|z - w\|/c$$

for any $z_1, z_2 \in K$ with $\varepsilon = \varepsilon_{z_1, z_2} > 0$ and any $X \in \mathbb{C}^n$.

Observe that (11) follows by choosing c_3 such that

$$|\tilde{g}_D(z, w_1) - \tilde{g}_D(z, w_2)| \leq c_3\|w_1 - w_2\|, \quad z \in K, w_1, w_2 \in D$$

and using that hyperconvexity implies

$$A_D(z; X) = \lim_{t \rightarrow 0} \frac{\tilde{g}_D(z, z + tX)}{|t|}.$$

To show (12), we may assume that $A_D(z_1; X) \leq A_D(z_2; Y)$. Since

$$A_D(z_1; X) = \varepsilon A_{D_{z_1, \varepsilon}}(z_1; X) = \varepsilon A_{\bar{D}}(z_2; X) \geq \varepsilon A_D(z_2; X)$$

(cf. [10], Lemma 4.2.7, for the first equality), then

$$0 \leq A_D(z_2; X) - A_D(z_1; X) \leq (1 - \varepsilon)A_D(z_2; X),$$

which implies (12). \square

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