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THE GROUP OF ORDER PRESERVING AUTOMORPHISMS OF THE RING OF DIFFERENTIAL OPERATORS ON A LAURENT POLYNOMIAL ALGEBRA IN PRIME CHARACTERISTIC

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Abstract. Let K be a field of characteristic $p > 0$. It is proved that the group $\mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ of order preserving automorphisms of the ring $\mathcal{D}(L_n)$ of differential operators on a Laurent polynomial algebra $L_n := K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is isomorphic to a skew direct product of groups $\mathbb{Z}_p^n \rtimes \mathrm{Aut}_K(L_n)$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Moreover, the group $\mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ is found explicitly. Similarly, $\mathrm{Aut}_{ord}(\mathcal{D}(P_n)) \simeq \mathrm{Aut}_K(P_n)$, where $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra.

1. Introduction

Throughout, ring means an associative ring with 1; p is a prime number; $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is the field that contains p elements; $\mathbb{Z}_p$ is the ring of p-adic integers; K is an arbitrary field of characteristic $p > 0$ (if it is not stated otherwise); $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra; $\mathcal{D}(P_n) = \bigoplus_{\alpha \in \mathbb{N}^n} P_n \partial^{[\alpha]}$ is the ring of differential operators on P_n where $\partial^{[\alpha]} := \prod_{i=1}^n \frac{\partial_i^{\alpha_i}}{\alpha_i!}$; $L_n := K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a Laurent polynomial algebra and $\mathcal{D}(L_n) = \bigoplus_{\alpha \in \mathbb{N}^n} L_n \partial^{[\alpha]}$ is the ring of differential operators on the algebra L_n; $\{\mathcal{D}(L_n)_i := \bigoplus_{|\alpha| \le i} L_n \partial^{[\alpha]}\}_{i \ge 0}$ is the *order filtration* on $\mathcal{D}(L_n)$; and

$$\mathrm{Aut}_{ord}(\mathcal{D}(L_n)) := \{\sigma \in \mathrm{Aut}_K(\mathcal{D}(L_n)) \mid \sigma(\mathcal{D}(L_n)_i) = \mathcal{D}(L_n)_i,\ i \ge 0\}$$

is the group of *order preserving* automorphisms of the algebra $\mathcal{D}(L_n)$. Similarly the order filtration on $\mathcal{D}(P_n)$ and the group $\mathrm{Aut}_{ord}(\mathcal{D}(P_n))$ are defined.

In arbitrary characteristic, it is a difficult problem to find generators for the groups $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$. The results are known for $\mathrm{Aut}_K(P_1)$ (easy), $\mathrm{Aut}_K(P_2)$ (Jung [8] and van der Kulk [10]) and $\mathrm{Aut}_K(\mathcal{D}(P_1))$ when $\mathrm{char}(K) = 0$ (Dixmier [7]). Little is known about the groups of automorphisms in the remaining cases.

In characteristic zero, there is a strong connection between the groups $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$ as, for example, the (essential) equivalence of the Jacobian Conjecture for P_n and the Dixmier Problem/Conjecture for $\mathcal{D}(P_n)$ shows (see [1], [9],

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[6]; see also [4]). Moreover, conjectures such as the two mentioned conjectures make sense only for the algebras $P_m \otimes \mathcal{D}(P_n)$ as was proved in [2] (the two conjectures can be reformulated in terms of locally nilpotent derivations that satisfy certain conditions, and the algebras $P_m \otimes \mathcal{D}(P_n)$ are the only associative algebras that have such derivations). This general conjecture is true iff either the JC or the DC is true; see [2].

In prime characteristic, relations between the two groups $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$ are even tighter, as the following result shows.

Theorem 1.1 ([3], Rigidity of the group $\mathrm{Aut}_K(\mathcal{D}(P_n))$)**.** *Let K be a field of characteristic $p > 0$ and $\sigma, \tau \in \mathrm{Aut}_K(\mathcal{D}(P_n))$. Then $\sigma = \tau$ iff $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.*

Remark. Theorem 1.1 does not hold in characteristic zero, and, in general, in prime characteristic it does not hold for localizations of the polynomial algebra P_n (Theorem 1.3). Note that the K-algebra $\mathcal{D}(P_n)$ is not finitely generated.

As a direct consequence of Theorem 1.1 there is the corollary (see Section 2 for details).

Corollary 1.2. $\mathrm{Aut}_{ord}(\mathcal{D}(P_n)) \simeq \mathrm{Aut}_K(P_n)$.

The situation is completely different for the Laurent polynomial algebra L_n.

Theorem 1.3. $\mathrm{Aut}_{ord}(\mathcal{D}(L_n)) \simeq \mathbb{Z}_p^n \rtimes \mathrm{Aut}_K(L_n)$. *Moreover, each automorphism $\sigma \in \mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ is a unique product of two automorphisms $\sigma = \sigma_s\tau$ where $\tau \in \mathrm{Aut}_K(L_n)$, $s = (s_i) \in \mathbb{Z}_p^n$, $\sigma_s(x_i) = x_i$ for all i, and*

$$\sigma_s(\partial^{[\alpha]}) = \prod_{i=1}^{n} \frac{(\partial_i + s_i x_i^{-1})^{\alpha_i}}{\alpha_i!}, \quad \alpha = (\alpha_i) \in \mathbb{N}^n. \tag{1}$$

The meaning of the RHS of (1) is explained in Section 2. Theorem 1.3 gives all the elements of the group $\mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ explicitly since the group $\mathrm{Aut}_K(L_n)$ is known: it is isomorphic to the semidirect product $\mathrm{GL}_n(\mathbb{Z}) \rtimes K^{*n}$.

The main idea of the proof of Theorem 1.3 is to show that the group $\mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ is equal to the group

$$\mathrm{St}(L_n) := \{\sigma \in \mathrm{Aut}_K(\mathcal{D}(L_n)) \mid \sigma(L_n) = L_n\},$$

which is a semidirect product $\mathrm{St}(L_n) = \mathrm{st}(L_n) \rtimes \mathrm{Aut}_K(L_n)$, where

$$\mathrm{st}(L_n) := \{\sigma \in \mathrm{St}(L_n) \mid \sigma|_{L_n} = \mathrm{id}\}$$

is the normal subgroup of $\mathrm{St}(L_n)$. The group $\mathrm{st}(L_n)$ consists of all the automorphisms σ_s, $s \in \mathbb{Z}_p^n$ (see (1)), and the map $\mathbb{Z}_p^n \to \mathrm{st}(L_n)$, $s \mapsto \sigma_s$, is a group isomorphism ($\sigma_{s+t} = \sigma_s\sigma_t$); see Theorem 2.4.

2. The group $\mathrm{Aut}_{ord}(\mathcal{D}(L_n))$

Let R be a commutative K-algebra and $\mathcal{D}(R) = \bigcup_{i\geq 0} \mathcal{D}(R)_i$ be the ring of K-linear differential operators on the algebra R, where $\{\mathcal{D}(R)_i\}_{i\geq 0}$ is the *order filtration* on $\mathcal{D}(R)$. In more detail, $\mathcal{D}(R) \subseteq \mathrm{End}_K(R)$, $\mathcal{D}(R)_0 := \mathrm{End}_R(R) \simeq R$, and

$$\mathcal{D}(R)_i := \{f \in \mathrm{End}_K(R) \mid rf - fr \in \mathcal{D}(R)_{i-1} \text{ for all } r \in R\}, \quad i \geq 1.$$

Let $\mathrm{Aut}_{ord}(\mathcal{D}(R))$ be the subgroup of $\mathrm{Aut}_K(\mathcal{D}(R))$ of order preserving K-automorphisms, i.e.

$$\mathrm{Aut}_{ord}(\mathcal{D}(R)) := \{\sigma \in \mathrm{Aut}_K(\mathcal{D}(R)) \,|\, \sigma(\mathcal{D}(R)_i) = \mathcal{D}(R)_i,\ i \geq 0\}.$$

Each automorphism $\sigma \in \mathrm{Aut}_K(R)$ can be naturally extended (by change of variables) to a K-automorphism, say σ, of the ring $\mathcal{D}(R)$ of differential operators on the algebra R by the rule

$$\sigma(a) := \sigma a \sigma^{-1}, \quad a \in \mathcal{D}(R). \tag{2}$$

Then the group $\mathrm{Aut}_K(R)$ can be seen as a subgroup of $\mathrm{Aut}_K(\mathcal{D}(R))$ via (2). Then it follows from a definition of the ring $\mathcal{D}(R)$ and the equality $[r, \sigma a \sigma^{-1}] = \sigma[\sigma^{-1}(r), a]\sigma^{-1}$, where $r \in R$, $a \in \mathcal{D}(R)$, that

$$\mathrm{Aut}_K(R) \subseteq \mathrm{Aut}_{ord}(\mathcal{D}(R)). \tag{3}$$

The stabilizer of the algebra R,

$$\mathrm{St}(R) := \{\sigma \in \mathrm{Aut}_K(\mathcal{D}(R)) \,|\, \sigma(R) = R\},$$

is a subgroup of $\mathrm{Aut}_K(\mathcal{D}(R))$. It contains the normal subgroup

$$\mathrm{st}(R) := \{\sigma \in \mathrm{St}(R) \,|\, \sigma|_R = \mathrm{id}\},$$

which is the kernel of the restriction epimorphism

$$\mathrm{St}(R) \to \mathrm{Aut}_K(R), \quad \sigma \mapsto \sigma|_R.$$

By (3), the stabilizer of R,

$$\mathrm{St}(R) = \mathrm{st}(R) \rtimes \mathrm{Aut}_K(R) \tag{4}$$

is the semi-direct product of its subgroups. It is obvious that

$$\mathrm{Aut}_{ord}(\mathcal{D}(R)) \subseteq \mathrm{St}(R). \tag{5}$$

The ring R is a left $\mathcal{D}(R)$-module. The action of an element $\delta \in \mathcal{D}(R)$ on an element $r \in R$ is denoted either by $\delta(a)$ or $\delta * r$ (in order to avoid multiple brackets).

Proof of Corollary 1.2. By Theorem 1.1, $\mathrm{st}(P_n) = \{\mathrm{id}\}$. Now the result follows from (3), (4) and (5):

$$\mathrm{Aut}_K(P_n) \subseteq \mathrm{Aut}_{ord}(\mathcal{D}(P_n)) \subseteq \mathrm{St}(P_n) = \mathrm{st}(P_n) \rtimes \mathrm{Aut}_K(P_n) = \mathrm{Aut}_K(P_n).$$

□

The rings of differential operators $\mathcal{D}(P_n)$ and $\mathcal{D}(L_n)$. The ring $\mathcal{D}(P_n)$ of differential operators on a polynomial algebra $P_n := K[x_1, \ldots, x_n]$ is a K-algebra generated by the elements $x_1, \ldots, x_n$ and *commuting* higher derivations $\partial_i^{[k]} := \frac{\partial_i^k}{k!}$, $i = 1, \ldots, n$, $k \geq 1$, that satisfy the following defining relations:

$$[x_i, x_j] = 0, \ \ [\partial_i^{[k]}, \partial_j^{[l]}] = 0, \quad \partial_i^{[k]}\partial_i^{[l]} = \binom{k+l}{k}\partial_i^{[k+l]}, \quad [\partial_i^{[k]}, x_j] = \delta_{ij}\partial_i^{[k-1]}, \tag{6}$$

for all $i, j = 1, \ldots, n$, $k, l \geq 1$, where δ_{ij} is the Kronecker delta, $\partial_i^{[0]} := 1$, $\partial_i^{[-1]} := 0$, and $\partial_i^{[1]} = \partial_i = \frac{\partial}{\partial x_i} \in \mathrm{Der}_K(P_n)$, $i = 1, \ldots, n$. The action of the higher derivation $\partial_i^{[k]}$ on the polynomial algebra

$$P_n = K \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \ldots, x_n] \simeq K \otimes_{\mathbb{Z}} \mathbb{Z}_p[x_1, \ldots, x_n]$$

should be understood as the action of the element $1 \otimes_{\mathbb{Z}} \frac{\partial_i^k}{k!}$.

The algebra $\mathcal{D}(P_n)$ is a *simple* algebra. Note that the algebra $\mathcal{D}(P_n)$ is not finitely generated, is not (left or right) Noetherian, and does not satisfy finitely many defining relations. This is in contrast to the characteristic zero case. We have

$$\mathcal{D}(P_n) = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} Kx^\alpha\partial^{[\beta]} \subset \mathcal{D}(L_n) = \bigoplus_{\alpha\in\mathbb{Z}^n,\beta\in\mathbb{N}^n} Kx^\alpha\partial^{[\beta]}.$$

For each $i = 1, \dots, n$ and $j \in \mathbb{N}$ written p-adically as $j = \sum_k j_k p^k$, $0 \le j_k < p$,

$$\partial_i^{[j]} = \prod_k \partial_i^{[j_k p^k]} = \prod_k \frac{\partial_i^{[p^k]j_k}}{j_k!}, \quad \partial_i^{[j_k p^k]} = \frac{\partial_i^{[p^k]j_k}}{j_k!}, \tag{7}$$

where $\partial_i^{[p^k]j_k} := (\partial_i^{[p^k]})^{j_k}$. For $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{Z}^n$,

$$\partial^{[\alpha]} * x^\beta = \binom{\beta}{\alpha} x^{\beta-\alpha}, \quad \binom{\beta}{\alpha} := \prod_i \binom{\beta_i}{\alpha_i}. \tag{8}$$

For $\alpha, \beta \in \mathbb{N}^n$,

$$\partial^{[\alpha]}\partial^{[\beta]} = \binom{\alpha+\beta}{\beta}\partial^{[\alpha+\beta]}. \tag{9}$$

The binomial differential operators. For each natural number i, the binomial polynomial

$$\binom{t}{i} := \frac{t(t-1)\cdots(t-i+1)}{i!}, \quad \binom{t}{0} := 1,$$

can be seen as a function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. For each p-adic integer $s \in \mathbb{Z}_p$ and a natural number $i \in \mathbb{N}$, we have the differential operator on the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[x^{\pm 1}] := \mathbb{Z}_p[x, x^{-1}]$ of Laurent polynomials with coefficients from $\mathbb{Z}_p$:

$$\begin{aligned} \frac{(\partial + sx^{-1})^i}{i!} &= x^{-i}\frac{(x\partial+s)(x\partial+s-1)\cdots(x\partial+s-i+1)}{i!} \\ &= x^{-i}\binom{x\partial+s}{i} \in \mathcal{D}(\mathbb{Z}_p[x^{\pm 1}]), \end{aligned} \tag{10}$$

where $\partial := \frac{d}{dx} \in \mathrm{Der}_{\mathbb{Z}_p}(\mathbb{Z}_p[x^{\pm 1}])$. Let $\mathcal{L}_n := \mathbb{Z}_p[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the Laurent polynomial ring in n variables with coefficients from $\mathbb{Z}_p$ and let $\mathcal{D}(\mathcal{L}_n)$ be the ring of $\mathbb{Z}_p$-linear differential operators on the ring $\mathcal{L}_n$. Then, for any elements $\alpha = (\alpha_i) \in \mathbb{N}^n$ and $s = (s_i) \in \mathbb{Z}_p^n$, there is the following differential operator on the algebra $\mathcal{L}_n$:

$$b_s^{[\alpha]} := \prod_{i=1}^n x_i^{-\alpha_i}\binom{x_i\partial_i + s_i}{\alpha_i} = \prod_{i=1}^n \frac{(\partial_i + s_i x_i^{-1})^{\alpha_i}}{\alpha_i!} \in \mathcal{D}(\mathcal{L}_n). \tag{11}$$

The inclusions of abelian monoids $\mathbb{N}^n \subset \mathbb{Z}^n \subset \mathbb{Z}_p^n$ yield the inclusions of their monoid rings

$$\mathcal{N}_n := \bigoplus_{\alpha\in\mathbb{N}^n} \mathbb{Z}_p x^\alpha \subset \mathcal{L}_n \subset \mathcal{M}_n := \bigoplus_{\alpha\in\mathbb{Z}_p^n} \mathbb{Z}_p x^\alpha,$$

and the inclusions of rings

$$\mathcal{A}_n := \bigoplus_{\beta\in\mathbb{N}^n} \mathcal{N}_n\partial^{[\beta]} \subset \mathcal{B}_n := \bigoplus_{\beta\in\mathbb{N}^n} \mathcal{L}_n\partial^{[\beta]} \subset \mathcal{C}_n := \bigoplus_{\beta\in\mathbb{N}^n} \mathcal{M}_n\partial^{[\beta]} \subset \mathrm{End}_{\mathbb{Z}_p}(\mathcal{M}_n),$$

where $\partial^{[\beta]} = \prod_{i=1}^n \frac{\partial_i^{\beta_i}}{\beta_i!}$. For any elements $\alpha \in \mathbb{Z}_p^n$ and $\beta \in \mathbb{N}^n$,

$$\partial^{[\beta]} * x^\alpha = \binom{\alpha}{\beta} x^{\alpha-\beta}, \text{ where } \binom{\alpha}{\beta} := \prod_{i=1}^n \binom{\alpha_i}{\beta_i}.$$

The prime number p belongs to the centre of each of the rings $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$. Taking factor rings modulo the ideals generated by the element p in each of the rings, we obtain the inclusions of $\mathbb{F}_p$-algebras:

$$\bigoplus_{\alpha,\beta\in\mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \bigoplus_{\alpha\in\mathbb{Z}^n,\beta\in\mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \bigoplus_{\alpha\in\mathbb{Z}_p^n,\beta\in\mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \mathrm{End}_{\mathbb{F}_p}(\bigoplus_{\alpha\in\mathbb{Z}^n} \mathbb{F}_p x^\alpha).$$

Then, applying $K \otimes_{\mathbb{Z}} -$, we obtain the inclusions of rings

$$\mathcal{D}(P_n) \subset \mathcal{D}(L_n) \subset \mathcal{D}_n := \bigoplus_{\alpha\in\mathbb{Z}_p^n,\beta\in\mathbb{N}^n} K x^\alpha \partial^{[\beta]}.$$

For each element $s \in \mathbb{Z}_p^n$, the inner automorphism $\omega_{x^{-s}} : a \mapsto x^{-s} a x^s$ of the ring $\mathcal{C}_n$ acts trivially on the ring $\mathcal{M}_n$, and, for each element $\beta \in \mathbb{N}^n$,

$$\omega_{x^{-s}}(\partial^{[\beta]}) = b_s^{[\beta]}. \tag{12}$$

Indeed,

$$\begin{aligned}\omega_{x^{-s}}(\partial^{[\beta]}) &= \prod_{i=1}^n x_i^{-s_i} \partial_i^{[\beta_i]} x_i^{s_i} = \prod_{i=1}^n x_i^{-s_i} x_i^{-\beta_i} \frac{x_i^{\beta_i}\partial_i^{\beta_i}}{\beta_i!} x_i^{s_i} = \prod_{i=1}^n x_i^{-\beta_i} x_i^{-s_i} \binom{x_i\partial_i}{\beta_i} x_i^{s_i} \\ &= \prod_{i=1}^n x_i^{-\beta_i} \binom{x_i\partial_i + s_i}{\beta_i} x_i^{-s_i} x_i^{s_i} = b_s^{[\beta]}.\end{aligned}$$

The inner automorphism $\omega_{x^{-s}} : b \mapsto x^{-s} b x^s$ of the ring $\mathcal{D}_n$ is the restriction modulo p of the inner automorphism $\omega_{x^{-s}} : a \mapsto x^{-s} a x^s$ of the ring $\mathcal{C}_n$. It is obvious that

$$\omega_{x^{-s}}(\mathcal{D}(L_n)) = \mathcal{D}(L_n).$$

Let σ_s be the restriction of the inner automorphism $\omega_{x^{-s}}$ of the ring $\mathcal{D}_n$ to the subring $\mathcal{D}(L_n)$. Then the automorphism σ_s is as in (1) and $\sigma_s \in \mathrm{st}(L_n)$.

It is obvious that the map

$$\mathbb{Z}_p^n \to \mathrm{st}(L_n), \quad s \mapsto \sigma_s, \tag{13}$$

is a group monomorphism. In fact, this map is an isomorphism; see Theorem 2.4. Before giving the proof of Theorem 2.4, we need to prove some more results.

Lemma 2.1. *The kernel of the $\mathbb{F}_p$-linear map $\partial_i^{p-1} + F$ acting on the ring L_n is $\mathbb{F}_p x_i^{-1}$, where $F : a \mapsto a^p$ is the Frobenius map on L_n.*

Proof. First, let us consider the case when $i = n = 1$. We drop the subscript 1 in this case. Let $s = \lambda x^j + \cdots \in K[x^{\pm 1}]$ be a nonzero element of the kernel of the $\mathbb{F}_p$-linear map $\partial^{p-1} + F$, where λx^j is its least term, $0 \neq \lambda \in K$ (the map $\partial_i^{p-1} + F$ is $\mathbb{F}_p$-linear since $F(\mu) = \mu^p = \mu$ for all $\mu \in \mathbb{F}_p$). Then

$$0 = (\partial^{p-1} + F)(\lambda x^j) = \lambda j(j-1)\cdots(j-p+2)x^{j-p+1} + \lambda^p x^{pj},$$

and so $j - p + 1 = pj$ and $\lambda j(j-1)\cdots(j-p+2) + \lambda^p = 0$. The first equality yields $j = -1$. Then the second can be written as $\lambda^p - \lambda = 0$ since $(p-1)! \equiv -1 \mod p$. Therefore, $\lambda \in \mathbb{F}_p$ and the set $\mathbb{F}_p x^{-1}$ belongs to the kernel of the map $\partial^{p-1} + F$. The least term of the element $s - \lambda x^{-1}$ of the kernel of the map $\partial^{p-1} + F$

has to be zero by the above argument, and so $s - \lambda x^{-1} = 0$. This proves that $\ker(\partial^{p-1} + F) = \mathbb{F}_p x^{-1}$. Now, the general case follows from this special one since $L_n \subset K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)[x_i^{\pm 1}]$. $\square$

Let R be a ring. The *first Weyl algebra* $A_1(R)$ over R is a ring generated over R by two elements x and ∂ that satisfy the defining relation $\partial x - x\partial = 1$.

Theorem 2.2 ([5])**.** *Let K be a reduced commutative $\mathbb{F}_p$-algebra and $A_1(K)$ be the first Weyl algebra over K. Then*

$$(\partial + f)^p = \partial^p + \frac{d^{p-1} f}{dx^{p-1}} + f^p$$

for all $f \in K[x]$. In more detail, $(\partial + f)^p = \partial^p - \lambda_{p-1} + f^p$, where $f = \sum_{i=0}^{p-1} \lambda_i x^i \in K[x] = \bigoplus_{i=0}^{p-1} K[x^p] x^i$, $\lambda_i \in K[x^p]$.

Corollary 2.3. *For each element $f \in L_n$ and $i = 1, \dots, n$, the equality*

$$(\partial_i + f)^p = \frac{\partial^{p-1} f}{\partial x_i^{p-1}} + f^p$$

holds in the ring $\mathcal{D}(L_n)$.

Proof. Let L be the subalgebra $K[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i^{\pm p}, x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ of the algebra L_n. Then $L_n = \bigoplus_{j=0}^{p-1} L x_i^j$. Consider the L-algebra homomorphism

$$A_1(L) := L\langle x, \partial \rangle \to \mathcal{D}(L_n), \quad x \mapsto x_i, \quad \partial \mapsto \partial_i, \quad l \mapsto l,$$

where $l \in L$. Now, the result follows from Theorem 2.2 since $\partial_i^p = 0$. $\square$

$\Delta_n := \bigoplus_{\alpha \in \mathbb{N}^n} K \partial^{[\alpha]}$ is the algebra of scalar differential operators on P_n.

Theorem 2.4. *The map* (13) *is an isomorphism.*

Proof. It suffices to show that, for each automorphism $\sigma \in \mathrm{st}(L_n)$,

$$\cdots \sigma_{p^k s_k} \cdots \sigma_{p s_1} \sigma_{s_0} \sigma = 1$$

for some elements $s_k = (s_{k1}, \dots, s_{kn}) \in \{0, 1, \dots, p-1\}^n$. Note that for all $k \geq 0$ and $i = 1, \dots, n$,

$$\sigma_{p^k s_k}(\partial_i^{[p^s]}) = \partial_i^{[p^s]}, \quad s < k; \quad \sigma_{p^k s_k}(\partial_i^{[p^k]}) = \partial_i^{[p^s k]} + s_{ki} x_i^{-p^k}. \tag{14}$$

Note that the centralizer $C(\mathcal{D}(L_n), x_1, \dots, x_n)$ of the elements $x_1, \dots, x_n$ in the ring $\mathcal{D}(L_n)$ satisfies

$$C(\mathcal{D}(L_n), x_1, \dots, x_n) := \bigcap_{i=1}^{n} \ker(\mathrm{ad}(x_i)) = L_n,$$

where $\mathrm{ad}(x_i)$ is the inner derivation of the ring $\mathcal{D}(L_n)$ determined by the element x_i. For all indices i and j,

$$[\sigma(\partial_i) - \partial_i, x_j] = \sigma([\partial_i, x_j]) - [\partial_i, x_j] = \delta_{ij} - \delta_{ij} = 0;$$

hence $a_i := \sigma(\partial_i) - \partial_i \in L_n$. Using Corollary 2.3, we see that

$$0 = \sigma(\partial_i^p) = (\partial_i + a_i)^p = (\partial_i^{p-1} + F)(a_i),$$

and so $a_i = -s_{0i} x_i^{-1}$ for some elements $s_{0i} \in \mathbb{F}_p$, by Lemma 2.1. Abusing the notation we assume in this proof that $\mathbb{F}_p = \{0, 1, \dots, p-1\}$. Let $s_0 := (s_{01}, \dots s_{0n})$.

Then $\sigma_{s_0}\sigma(\partial_i) = \partial_i$ for all i. Suppose that $k > 0$, and we have found vectors $s_0, \dots, s_{k-1} \in \mathbb{F}_p^n$ such that

$$\sigma_{p^{k-1}s_{k-1}} \cdots \sigma_{ps_1}\sigma_{s_0}\sigma(\partial_i^{[p^s]}) = \partial_i^{[p^s]}, \ \ 1 \le i \le n, \ \ 0 \le s \le k-1.$$

In order to finish the proof of the claim by induction on k, we have to find a vector $s_k \in \mathbb{F}_p^n$ such that the above equalities hold for k rather than $k-1$. Let $\tau := \sigma_{p^{k-1}s_{k-1}} \cdots \sigma_{ps_1}\sigma_{s_0}\sigma$. For all i and j,

$$[\tau(\partial_i^{[p^k]}) - \partial_i^{[p^k]}, x_j^{p^k}] = \tau([\partial_i^{[p^k]}, x_j^{p^k}]) - [\partial_i^{[p^k]}, x_j^{p^k}] = \delta_{ij} - \delta_{ij} = 0;$$

hence

$$b_i := \tau(\partial_i^{[p^k]}) - \partial_i^{[p^k]} \in C(\mathcal{D}(L_n), x_1^{p^k}, \dots x_n^{p^k}) = L_n \otimes \Delta_{n,<p^k},$$

where $\Delta_{n,<p^k} = \bigoplus_{\text{all } \alpha_i < p^k} K\partial^{[\alpha]}$. For all $i \ne j$,

$$0 = \tau([\partial_i^{[p^k]}, x_j]) = [\partial_i^{[p^k]} + b_i, x_j] = [b_i, x_j],$$

and so $b_i \in \sum_{s<p^k} L_n\partial_i^{[s]}$. Now,

$$\partial_i^{[p^k-1]} = \tau(\partial_i^{[p^k-1]}) = \tau([\partial_i^{[p^k]}, x_i]) = [\partial_i^{[p^k]} + b_i, x_i] = \partial_i^{[p^k-1]} + [b_i, x_i];$$

hence $b_i \in L_n$. For all i and j, and $s < k$,

$$0 = \tau([\partial_i^{[p^k]}, \partial_j^{[p^s]}]) = [\partial_i^{[p^k]} + b_i, \partial_j^{[p^s]}] = [b_i, \partial_j^{[p^s]}],$$

and so $b_i \in K[x_1^{\pm p^k}, \dots, x_n^{\pm p^k}]$. Finally,

$$0 = \tau(\partial_i^{[p^k]p}) = (\partial_i^{[p^k]} + b_i)^p = (\partial_i^{[p^k](p-1)} + F)(b_i);$$

hence $b_i = -s_{ki}x_i^{p^k}$ for some elements $s_{ki} \in \mathbb{F}_p$, by Lemma 2.1. Let $s_k = (s_{k1}, \dots, s_{kn})$. Then

$$\sigma_{p^k s_k} \cdots \sigma_{ps_1}\sigma_{s_0}\sigma(\partial_i^{[p^s]}) = \partial_i^{[p^s]}, \ \ 1 \le i \le n, \ \ 0 \le s \le k.$$

This finishes the proof of the theorem. □

Corollary 2.5. $\mathrm{Aut}_{ord}(\mathcal{D}(L_n)) = \mathrm{St}(L_n)$.

Proof. The inclusion $\mathrm{Aut}_{ord}(\mathcal{D}(L_n)) \subseteq \mathrm{St}(L_n)$ is obvious; see (5). The inverse inclusion follows from the facts that $\mathrm{St}(L_n) = \mathrm{st}(L_n) \rtimes \mathrm{Aut}_K(L_n)$, $\mathrm{st}(L_n) \subseteq \mathrm{Aut}_{ord}(\mathcal{D}(L_n))$ (by Theorem 2.4) and $\mathrm{Aut}_K(L_n) \subseteq \mathrm{Aut}_{ord}(\mathcal{D}(L_n))$; see (3). □

Proof of Theorem 1.3. This follows from Corollary 2.5, (4) and Theorem 2.4:

$$\mathrm{Aut}_{ord}(\mathcal{D}(L_n)) = \mathrm{St}(L_n) = \mathrm{st}(L_n) \rtimes \mathrm{Aut}_K(L_n) \simeq \mathbb{Z}_p^n \rtimes \mathrm{Aut}_K(L_n).$$

□

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