THE GROUP OF ORDER PRESERVING AUTOMORPHISMS OF THE RING OF DIFFERENTIAL OPERATORS ON A LAURENT POLYNOMIAL ALGEBRA IN PRIME CHARACTERISTIC

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Abstract. Let $K$ be a field of characteristic $p > 0$. It is proved that the group $\text{Aut}_{ord}(D(L_n))$ of order preserving automorphisms of the ring $D(L_n)$ of differential operators on a Laurent polynomial algebra $L_n := K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is isomorphic to a skew direct product of groups $\mathbb{Z}_p \times \text{Aut}_K(L_n)$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. Moreover, the group $\text{Aut}_{ord}(D(L_n))$ is found explicitly. Similarly, $\text{Aut}_{ord}(D(P_n)) \simeq \text{Aut}_K(P_n)$, where $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra.

1. Introduction

Throughout, ring means an associative ring with 1; $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is the field that contains $p$ elements; $\mathbb{Z}_p$ is the ring of $p$-adic integers; $K$ is an arbitrary field of characteristic $p > 0$ (if it is not stated otherwise); $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra; $D(P_n) = \bigoplus_{\alpha \in \mathbb{N}_n} P_n \partial^{[\alpha]}$ is the ring of differential operators on $P_n$ where $\partial^{[\alpha]} := \prod_{i=1}^n \frac{\partial x_i}{x_i^{\alpha_i}}$, $L_n := K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial algebra and $D(L_n) = \bigoplus_{\alpha \in \mathbb{N}^n} L_n \partial^{[\alpha]}$ is the ring of differential operators on the algebra $L_n$: $\{D(L_n)_i := \bigoplus_{|\alpha| \leq i} L_n \partial^{[\alpha]}\}_{i \geq 0}$ is the order filtration on $D(L_n)$; and

$$\text{Aut}_{ord}(D(L_n)) := \{ \sigma \in \text{Aut}_K(D(L_n)) \mid \sigma(D(L_n)_i) = D(L_n)_i, \ i \geq 0 \}$$

is the group of order preserving automorphisms of the algebra $D(L_n)$. Similarly the order filtration on $D(P_n)$ and the group $\text{Aut}_{ord}(D(P_n))$ are defined.

In arbitrary characteristic, it is a difficult problem to find generators for the groups $\text{Aut}_K(P_n)$ and $\text{Aut}_K(D(P_n))$. The results are known for $\text{Aut}_K(P_1)$ (easy), $\text{Aut}_K(P_2)$ (Jung [8] and van der Kulk [10]) and $\text{Aut}_K(D(P))$ when $\text{char}(K) = 0$ (Dixmier [7]). Little is known about the groups of automorphisms in the remaining cases.

In characteristic zero, there is a strong connection between the groups $\text{Aut}_K(P_n)$ and $\text{Aut}_K(D(P_n))$ as, for example, the (essential) equivalence of the Jacobian Conjecture for $P_n$ and the Dixmier Problem/Conjecture for $D(P_n)$ shows (see [1], [9], [11], [12], [13]).

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Moreover, conjectures such as the two mentioned conjectures make sense only for the algebras $P_m \otimes D(P_n)$ as was proved in [2] (the two conjectures can be reformulated in terms of locally nilpotent derivations that satisfy certain conditions, and the algebras $P_m \otimes D(P_n)$ are the only associative algebras that have such derivations). This general conjecture is true iff either the JC or the DC is true; see [2].

In prime characteristic, relations between the two groups $\text{Aut}_K(P_n)$ and $\text{Aut}_K(D(P_n))$ are even tighter, as the following result shows.

**Theorem 1.1** ([3], Rigidity of the group $\text{Aut}_K(D(P_n))$). Let $K$ be a field of characteristic $p > 0$ and $\sigma, \tau \in \text{Aut}_K(D(P_n))$. Then $\sigma = \tau$ iff $\sigma(x_1) = \tau(x_1), \ldots, \sigma(x_n) = \tau(x_n)$.

**Remark.** Theorem 1.1 does not hold in characteristic zero, and, in general, in prime characteristic it does not hold for localizations of the polynomial algebra $P_n$ (Theorem 1.3). Note that the $K$-algebra $D(P_n)$ is not finitely generated.

As a direct consequence of Theorem 1.1 there is the corollary (see Section 2 for details).

**Corollary 1.2.** $\text{Aut}_{ord}(D(P_n)) \simeq \text{Aut}_K(P_n)$.

The situation is completely different for the Laurent polynomial algebra $L_n$.

**Theorem 1.3.** $\text{Aut}_{ord}(D(L_n)) \simeq \mathbb{Z}_p^n \rtimes \text{Aut}_K(L_n)$. Moreover, each automorphism $\sigma \in \text{Aut}_{ord}(D(L_n))$ is a unique product of two automorphisms $\sigma = \sigma_s \tau$ where $\tau \in \text{Aut}_K(L_n), s = (s_i) \in \mathbb{Z}_p^n, \sigma_s(x_i) = x_i$ for all $i$, and

$$\sigma_s(\partial^{(\alpha)}) = \prod_{i=1}^n \frac{(\partial_i + s_i x_i^{-1})^{a_i}}{a_i!}, \quad \alpha = (a_i) \in \mathbb{N}^n.$$

The meaning of the RHS of (1) is explained in Section 2. Theorem 1.3 gives all the elements of the group $\text{Aut}_{ord}(D(L_n))$ explicitly since the group $\text{Aut}_K(L_n)$ is known: it is isomorphic to the semidirect product $\text{GL}_n(\mathbb{Z}) \rtimes K^*$. The main idea of the proof of Theorem 1.3 is to show that the group $\text{Aut}_{ord}(D(L_n))$ is equal to the group

$$\text{St}(L_n) := \{\sigma \in \text{Aut}_K(D(L_n)) \mid \sigma(L_n) = L_n\},$$

which is a semidirect product $\text{St}(L_n) = \text{st}(L_n) \rtimes \text{Aut}_K(L_n)$, where

$$\text{st}(L_n) := \{\sigma \in \text{St}(L_n) \mid \sigma|_{L_n} = \text{id}\}$$

is the normal subgroup of $\text{St}(L_n)$. The group $\text{st}(L_n)$ consists of all the automorphisms $\sigma_s, s \in \mathbb{Z}_p^n$ (see 1.3), and the map $\mathbb{Z}_p^n \to \text{st}(L_n), s \mapsto \sigma_s$, is a group isomorphism $(\sigma_{s+t} = \sigma_s \sigma_t)$; see Theorem 2.4.

2. **The Group $\text{Aut}_{ord}(D(L_n))$**

Let $R$ be a commutative $K$-algebra and $D(R) = \bigcup_{i \geq 0} D(R)_i$ be the ring of $K$-linear differential operators on the algebra $R$, where $\{D(R)_i\}_{i \geq 0}$ is the *order filtration* on $D(R)$. In more detail, $D(R) \subseteq \text{End}_K(R), D(R)_0 := \text{End}_R(R) \simeq R$, and

$$D(R)_i := \{f \in \text{End}_K(R) \mid rf - fr \in D(R)_{i-1} \text{ for all } r \in R\}, \quad i \geq 1.$$
Let \( \text{Aut}_{\text{ord}}(\mathcal{D}(R)) \) be the subgroup of \( \text{Aut}_K(\mathcal{D}(R)) \) of order preserving \( K \)-automorphisms, i.e.

\[
\text{Aut}_{\text{ord}}(\mathcal{D}(R)) := \{ \sigma \in \text{Aut}_K(\mathcal{D}(R)) \mid \sigma(\mathcal{D}(R)_i) = \mathcal{D}(R)_i, \; i \geq 0 \}.
\]

Each automorphism \( \sigma \in \text{Aut}_K(R) \) can be naturally extended (by change of variables) to a \( K \)-automorphism, say \( \sigma \), of the ring \( \mathcal{D}(R) \) of differential operators on the algebra \( R \) by the rule

\[
\sigma(a) := \sigma a \sigma^{-1}, \; a \in \mathcal{D}(R).
\]

Then the group \( \text{Aut}_K(R) \) can be seen as a subgroup of \( \text{Aut}_K(\mathcal{D}(R)) \) via \( \mathfrak{2} \) via \( \mathfrak{3} \). Then it follows from a definition of the ring \( \mathcal{D}(R) \) and the equality \( [r, \sigma a \sigma^{-1}] = \sigma [\sigma^{-1}(r), a] \sigma^{-1} \), where \( r \in R, \; a \in \mathcal{D}(R) \), that

\[
\text{Aut}_K(R) \subseteq \text{Aut}_{\text{ord}}(\mathcal{D}(R)).
\]

The stabilizer of the algebra \( R \),

\[
\text{St}(R) := \{ \sigma \in \text{Aut}_K(\mathcal{D}(R)) \mid \sigma(R) = R \},
\]

is a subgroup of \( \text{Aut}_K(\mathcal{D}(R)) \). It contains the normal subgroup

\[
\text{st}(R) := \{ \sigma \in \text{St}(R) \mid \sigma|_R = \text{id} \},
\]

which is the kernel of the restriction epimorphism

\[
\text{St}(R) \rightarrow \text{Aut}_K(R), \; \sigma \mapsto \sigma|_R.
\]

By \( \mathfrak{3} \), the stabilizer of \( R \),

\[
\text{St}(R) = \text{st}(R) \rtimes \text{Aut}_K(R)
\]

is the semi-direct product of its subgroups. It is obvious that

\[
\text{Aut}_{\text{ord}}(\mathcal{D}(R)) \subseteq \text{St}(R).
\]

The ring \( R \) is a left \( \mathcal{D}(R) \)-module. The action of an element \( \delta \in \mathcal{D}(R) \) on an element \( r \in R \) is denoted either by \( \delta(a) \) or \( \delta \ast r \) (in order to avoid multiple brackets).

**Proof of Corollary \( \mathfrak{1,2} \)** By Theorem \( \mathfrak{1,1} \) \( \text{st}(P_n) = \{ \text{id} \} \). Now the result follows from \( \mathfrak{3, 4} \) and \( \mathfrak{5} \):

\[
\text{Aut}_K(P_n) \subseteq \text{Aut}_{\text{ord}}(\mathcal{D}(P_n)) \subseteq \text{St}(P_n) = \text{st}(P_n) \rtimes \text{Aut}_K(P_n) = \text{Aut}_K(P_n).
\]

\[ \square \]

**The rings of differential operators** \( \mathcal{D}(P_n) \) and \( \mathcal{D}(L_n) \). The ring \( \mathcal{D}(P_n) \) of differential operators on a polynomial algebra \( P_n := K[x_1, \ldots, x_n] \) is a \( K \)-algebra generated by the elements \( x_1, \ldots, x_n \) and *commuting* higher derivations \( \partial_i^{[k]} := \frac{\partial^K_i}{\partial x_i^{k}} \), \( i = 1, \ldots, n, \; k \geq 1 \), that satisfy the following defining relations:

\[
[x_i, x_j] = 0, \quad [\partial_i^{[k]}, \partial_j^{[l]}] = 0, \quad \partial_i^{[k]} [\partial_j^{[l]}] = \binom{k + l}{k} \partial_i^{[k+l]}, \quad [\partial_i^{[k]}, x_j] = \delta_{ij} \partial_i^{[k-1]},
\]

for all \( i, j = 1, \ldots, n, \; k, l \geq 1 \), where \( \delta_{ij} \) is the Kronecker delta, \( \partial_i^{[0]} := 1, \; \partial_i^{[-1]} := 0 \), and \( \partial_i^{[1]} = \partial_i = \frac{\partial}{\partial x_i} \in \text{Der}_K(P_n), \; i = 1, \ldots, n \). The action of the higher derivation \( \partial_i^{[k]} \) on the polynomial algebra

\[
P_n = K \otimes_{\mathbb{Z}} \mathbb{Z} [x_1, \ldots, x_n] \cong K \otimes_{\mathbb{Z}} \mathbb{Z}_p [x_1, \ldots, x_n]
\]

should be understood as the action of the element \( 1 \otimes_{\mathbb{Z}} \partial_i^{[k]} \).
The algebra $D(P_n)$ is a simple algebra. Note that the algebra $D(P_n)$ is not finitely generated, is not (left or right) Noetherian, and does not satisfy finitely many defining relations. This is in contrast to the characteristic zero case. We have

$$D(P_n) = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K x^\alpha \partial^{[\beta]} \subset D(L_n) = \bigoplus_{\alpha \in \mathbb{Z}^n, \beta \in \mathbb{N}^n} K x^\alpha \partial^{[\beta]}.$$  

For each $i = 1, \ldots, n$ and $j \in \mathbb{N}$ written $p$-adically as $j = \sum_k j_k p^k$, $0 \leq j_k < p$,

$$\partial_i^{[j]} = \prod_k \partial_i^{[jp^k]} = \prod_k \frac{\partial_i^{[jp^k]_k}}{j_k!}, \quad \partial_1^{[jp^k]} = \frac{\partial_i^{[jp^k]}}{j_k!},$$

where $\partial_i^{[jp^k]_k} := (\partial_i^{[jp^k]})^k$. For $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{Z}^n$,

$$\partial_\alpha \ast x^\beta = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} x^\beta - \alpha, \quad \begin{pmatrix} \beta \\ \alpha \end{pmatrix} := \prod_{i} \begin{pmatrix} \beta_i \\ \alpha_i \end{pmatrix}.$$  

For $\alpha, \beta \in \mathbb{N}^n$,

$$\partial_\alpha \partial_\beta = \begin{pmatrix} \alpha + \beta \\ \beta \end{pmatrix} \partial^{[\alpha + \beta]}.$$  

The binomial differential operators. For each natural number $i$, the binomial polynomial

$$\binom{t}{i} := \frac{t(t-1) \cdots (t-i+1)}{i!}, \quad \binom{t}{0} := 1,$$

can be seen as a function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. For each $p$-adic integer $s \in \mathbb{Z}_p$ and a natural number $i \in \mathbb{N}$, we have the differential operator on the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[x^{\pm 1}] := \mathbb{Z}_p[x, x^{-1}]$ of Laurent polynomials with coefficients from $\mathbb{Z}_p$:

$$\frac{(\partial + sx^{-1})^i}{i!} = x^{-i} (x\partial + s)(x\partial + s - 1) \cdots (x\partial + s - i + 1)$$

$$= x^{-i} \begin{pmatrix} x\partial + s \\ i \end{pmatrix} \in D(\mathbb{Z}_p[x^{\pm 1}]),$$

where $\partial := \frac{d}{dx} \in \text{Der}_{\mathbb{Z}_p}(\mathbb{Z}_p[x^{\pm 1}])$. Let $\mathcal{L}_n := \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring in $n$ variables with coefficients from $\mathbb{Z}_p$ and let $D(\mathcal{L}_n)$ be the ring of $\mathbb{Z}_p$-linear differential operators on the ring $\mathcal{L}_n$. Then, for any elements $\alpha = (\alpha_i) \in \mathbb{N}^n$ and $s = (s_i) \in \mathbb{Z}_p^n$, there is the following differential operator on the algebra $\mathcal{L}_n$:

$$b_i^{[\alpha]} := \prod_{i=1}^n x_i^{-\alpha_i} \begin{pmatrix} x_i\partial_i + s_i \\ \alpha_i \end{pmatrix} = \prod_{i=1}^n \frac{\partial_i + s_i x_i^{-1} \alpha_i}{\alpha_i!} \in D(\mathcal{L}_n).$$

The inclusions of abelian monoids $\mathbb{N}^n \subset \mathbb{Z}^n \subset \mathbb{Z}_p^n$ yield the inclusions of their monoid rings

$$\mathcal{N}_n := \bigoplus_{\alpha \in \mathbb{N}^n} \mathbb{Z}_p x^{\alpha} \subset \mathcal{L}_n \subset \mathcal{M}_n := \bigoplus_{\alpha \in \mathbb{Z}_p^n} \mathbb{Z}_p x^{\alpha},$$

and the inclusions of rings

$$\mathcal{A}_n := \bigoplus_{\beta \in \mathbb{N}^n} \mathcal{N}_n \partial^{[\beta]} \subset \mathcal{B}_n := \bigoplus_{\beta \in \mathbb{N}^n} \mathcal{L}_n \partial^{[\beta]} \subset \mathcal{C}_n := \bigoplus_{\beta \in \mathbb{N}^n} \mathcal{M}_n \partial^{[\beta]} \subset \text{End}_{\mathbb{Z}_p}(\mathcal{M}_n),$$
where $\partial^{[\beta]} = \prod_{i=1}^{n} \frac{\partial^{[\beta_i]}}{\beta_i}$. For any elements $\alpha \in \mathbb{Z}_p^n$ and $\beta \in \mathbb{N}^n$,

$$\partial^{[\beta]} \cdot x^\alpha = \left( \frac{\alpha}{\beta} \right) x^{\alpha-\beta}, \text{ where } \left( \frac{\alpha}{\beta} \right) := \prod_{i=1}^{n} \left( \frac{\alpha_i}{\beta_i} \right).$$

The prime number $p$ belongs to the centre of each of the rings $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{C}_n$.

Taking factor rings modulo the ideals generated by the element $p$ in each of the rings, we obtain the inclusions of $\mathbb{F}_p$-algebras:

$$\bigoplus_{\alpha, \beta \in \mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \bigoplus_{\alpha \in \mathbb{Z}_p^n, \beta \in \mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \bigoplus_{\alpha \in \mathbb{Z}_p^n, \beta \in \mathbb{N}^n} \mathbb{F}_p x^\alpha \partial^{[\beta]} \subset \text{End}_p \left( \bigoplus \mathbb{F}_p x^\alpha \right).$$

Then, applying $K \otimes_{\mathbb{Z}_p} -$, we obtain the inclusions of rings

$$\mathcal{D}(P_n) \subset \mathcal{D}(L_n) \subset \mathcal{D}_n := \bigoplus_{\alpha \in \mathbb{Z}_p^n, \beta \in \mathbb{N}^n} K x^\alpha \partial^{[\beta]}.$$

For each element $s \in \mathbb{Z}_p^n$, the inner automorphism $\omega_{x^{-1}} : a \mapsto x^{-s}ax^s$ of the ring $\mathcal{C}_n$ acts trivially on the ring $\mathcal{M}_n$, and, for each element $\beta \in \mathbb{N}^n$,

$$\omega_{x^{-1}}(\partial^{[\beta]}) = b_s^{[\beta]}.$$

Indeed,

$$\omega_{x^{-1}}(\partial^{[\beta]}) = \prod_{i=1}^{n} x_i^{-s_i} \partial_{i}^{[\beta]} x_i^{s_i} = \prod_{i=1}^{n} x_i^{-s_i} x_i^{-\beta_i} \frac{\partial^{[\beta_i]}}{\beta_i} x_i^{s_i} = \prod_{i=1}^{n} x_i^{-\beta_i} x_i^{-s_i} \left( x_i \partial_{i}^{[\beta_i]} \right) x_i^{s_i} = \prod_{i=1}^{n} x_i^{-\beta_i} \left( x_i \partial_{i}^{[\beta_i]} \right) x_i^{s_i} = b_s^{[\beta]}.$$

The inner automorphism $\omega_{x^{-1}} : b \mapsto x^{-s}bx^s$ of the ring $\mathcal{D}_n$ is the restriction modulo $p$ of the inner automorphism $\omega_{x^{-1}} : a \mapsto x^{-s}ax^s$ of the ring $\mathcal{C}_n$. It is obvious that

$$\omega_{x^{-1}}(\mathcal{D}(L_n)) = \mathcal{D}(L_n).$$

Let $\sigma_s$ be the restriction of the inner automorphism $\omega_{x^{-1}}$ of the ring $\mathcal{D}_n$ to the subring $\mathcal{D}(L_n)$. Then the automorphism $\sigma_s$ is as in (11) and $\sigma_s \in \text{st}(L_n)$.

It is obvious that the map

$$\mathbb{Z}_p^n \rightarrow \text{st}(L_n), \ s \mapsto \sigma_s,$$

is a group monomorphism. In fact, this map is an isomorphism; see Theorem 2.4.

Before giving the proof of Theorem 2.4 we need to prove some more results.

**Lemma 2.1.** The kernel of the $\mathbb{F}_p$-linear map $\partial_{i}^{p-1} + F$ acting on the ring $L_n$ is $\mathbb{F}_p x_i^{p-1}$, where $F : a \mapsto ap^p$ is the Frobenius map on $L_n$.

**Proof.** First, let us consider the case when $i = n = 1$. We drop the subscript 1 in this case. Let $s = \lambda x^j + \cdots \in K[x^p]$ be a nonzero element of the kernel of the $\mathbb{F}_p$-linear map $\partial^{p-1} + F$, where $\lambda x^j$ is its least term, $0 \neq \lambda \in K$ (the map $\partial_{i}^{p-1} + F$ is $\mathbb{F}_p$-linear since $F(\mu) = \mu^p = \mu$ for all $\mu \in \mathbb{F}_p$). Then

$$0 = (\partial^{p-1} + F)(\lambda x^j) = \lambda j(j-1) \cdots (j-p+2)x^{j-p+1} + \lambda^p p^{pj},$$

and so $j-p+1 = p^j$ and $\lambda j(j-1) \cdots (j-p+2) + \lambda^p = 0$. The first equality yields $j = -1$. Then the second can be written as $\lambda^p - \lambda = 0$ since $(p-1)! \equiv -1 \mod p$. Therefore, $\lambda \in \mathbb{F}_p$ and the set $\mathbb{F}_p x^{-1}$ belongs to the kernel of the map $\partial^{p-1} + F$. The least term of the element $s - \lambda x^{-1}$ of the kernel of the map $\partial^{p-1} + F$
has to be zero by the above argument, and so \( s - \lambda x^{-1} = 0 \). This proves that \( \ker(\partial^{p-1} + F) = \mathbb{F}_p x^{-1} \). Now, the general case follows from this special one since \( L_n \subset K(x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_n)[x_i^{\pm 1}] \).

Let \( R \) be a ring. The first Weyl algebra \( A_1(R) \) over \( R \) is a ring generated over \( R \) by two elements \( x \) and \( \partial \) that satisfy the defining relation \( \partial x - x \partial = 1 \).

**Theorem 2.2** ([5]). Let \( K \) be a reduced commutative \( \mathbb{F}_p \)-algebra and \( A_1(K) \) be the first Weyl algebra over \( K \). Then

\[
(\partial + f)^p = \partial^p + \frac{\partial^{p-1} f}{dx^{p-1}} + f^p
\]

for all \( f \in K[x] \). In more detail, \( (\partial + f)^p = \partial^p - \lambda_{p-1} + f^p \), where \( f = \sum_{i=0}^{p-1} \lambda_i x^i \in K[x] = \bigoplus_{i=0}^{p-1} K[x^p]x^i, \lambda_i \in K[x^p] \).

**Corollary 2.3.** For each element \( f \in L_n \) and \( i = 1, \ldots, n \), the equality

\[
(\partial_i + f)^p = \frac{\partial^{p-1} f}{\partial x_i^{p-1}} + f^p
\]

holds in the ring \( D(L_n) \).

**Proof.** Let \( L \) be the subalgebra \( K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \) of the algebra \( L_n \). Then \( L_n = \bigoplus_{i=0}^{p-1} Lx_i^{p} \). Consider the \( L \)-algebra homomorphism

\[
A_1(L) := L(x, \partial) \to D(L_n), \ x \mapsto x_i, \ \partial \mapsto \partial_i, \ l \mapsto l,
\]

where \( l \in L \). Now, the result follows from Theorem 2.2 since \( \partial_i^p = 0 \). \( \square \)

\( \Delta_n := \bigoplus_{a \in \mathbb{N}_0} K \partial^{[a]} \) is the algebra of scalar differential operators on \( P_n \).

**Theorem 2.4.** The map \([13]\) is an isomorphism.

**Proof.** It suffices to show that, for each automorphism \( \sigma \in \text{st}(L_n) \),

\[
\cdots \sigma_{p \cdot s_k} \cdots \sigma_{p \cdot s_1} \sigma_{s_0} \sigma = 1
\]

for some elements \( s_k = (s_{k1}, \ldots, s_{kn}) \in \{0, 1, \ldots, p - 1\}^n \). Note that for all \( k \geq 0 \) and \( i = 1, \ldots, n \),

\[
\sigma_{p \cdot s_k}(\partial_i^{[p^k]}) = \partial_i^{[p^k]}, \ s < k; \ \sigma_{p \cdot s_k}(\partial_i^{[p^k]}) = \partial_i^{[p^k]}, \ s_k x_i^{-p^k}.
\]

Note that the centralizer \( C(D(L_n), x_1, \ldots, x_n) \) of the elements \( x_1, \ldots, x_n \) in the ring \( D(L_n) \) satisfies

\[
C(D(L_n), x_1, \ldots, x_n) := \bigcap_{i=1}^n \ker(\text{ad}(x_i)) = L_n,
\]

where \( \text{ad}(x_i) \) is the inner derivation of the ring \( D(L_n) \) determined by the element \( x_i \). For all indices \( i \) and \( j \),

\[
[\sigma(\partial_i) - \partial_j, x_j] = [\sigma(\partial_i), x_j] - [\partial_i, x_j] = \delta_{ij} - \delta_{ij} = 0;
\]

hence \( a_i := \sigma(\partial_i) - \partial_i \in L_n \). Using Corollary 2.3 we see that

\[
0 = \sigma(\partial_i^p) = (\partial_i + a_i)^p = (\partial_i^{p-1} + F)(a_i),
\]

and so \( a_i = -s_0 a_i^{-1} \) for some elements \( s_{0i} \in \mathbb{F}_p \), by Lemma 2.2. Abusing the notation we assume in this proof that \( \mathbb{F}_p = \{0, 1, \ldots, p-1\} \). Let \( s_0 := (s_{01}, \ldots, s_{0n}) \).
Then $\sigma_n \sigma(\partial_i) = \partial_i$ for all $i$. Suppose that $k > 0$, and we have found vectors $s_0, \ldots, s_{k-1} \in \mathbb{F}_p^n$ such that

$$\sigma_{p^k}s_{k-1} \cdots \sigma_{p^2}s_{1}\sigma_{p}s_{0}\sigma(\partial_i) = \partial_i^{[p^r]} \quad 1 \leq i \leq n, \quad 0 \leq s \leq k-1.$$ 

In order to finish the proof of the induction on $k$, we have to find a vector $s_k \in \mathbb{F}_p^n$ such that the above equalities hold for $k$ rather than $k-1$. Let $\tau := \sigma_{p^{k-1}s_{k-1}} \cdots \sigma_{p^2}s_1\sigma_p s_0 \sigma$. For all $i$ and $j$,$$
\tau(\partial_i^{[p^k]}) = \partial_i^{[p^r]}, \quad \tau(\partial_j^{[p^k]}, x_j^{p^k}) = \tau(\partial_i^{[p^r]}, x_j^{p^k}) - \partial_i^{[p^r]}x_j^{p^k} = \delta_{ij} - \delta_{ij} = 0;$$

hence

$$b_i := \tau(\partial_i^{[p^k]}) - \partial_i^{[p^k]} \in C(D(L_n), x_1^{p^k}, \ldots, x_n^{p^k}) = L_n \otimes \Delta_{n,p^k}.$$ 

where $\Delta_{n,p^k} = \bigoplus_{\alpha_i < p^k} K \partial^{[\alpha]}$. For all $i \neq j$,

$$0 = \tau(\partial_i^{[p^k]}, x_j) = [\partial_i^{[p^r]} + b_i, x_j] = [b_i, x_j],$$

and so $b_i \in \sum_{s < p^k} L_n \partial^{[s]}$. Now,

$$\partial_i^{[p^{k-1}]} = \tau(\partial_i^{[p^{k-1}]}) = \tau(\partial_i^{[p^r]}, x_i) = [\partial_i^{[p^r]} + b_i, x_i] = [\partial_i^{[p^{r-1}]} + [b_i, x_i];$$

hence $b_i \in L_n$. For all $i$ and $j$, and $s < k$,

$$0 = \tau(\partial_i^{[p^k]}, \partial_j^{[p^r]}) = [\partial_i^{[p^k]} + b_i, \partial_j^{[p^r]}] = [b_i, \partial_j^{[p^r]}],$$

and so $b_i \in K[x_1^{p^k}, \ldots, x_n^{p^k}]$. Finally,

$$0 = \tau(\partial_i^{[p^k]}, p) = (\partial_i^{[p^k]} + b_i)p = (\partial_i^{[p^k]}(p-1) + F)(b_i);$$

hence $b_i = -s_i x_i^{p^k}$ for some elements $s_i \in \mathbb{F}_p$, by Lemma 2.1. Let $s_k = (s_{k1}, \ldots, s_{kn})$. Then

$$\sigma_{p}s_{k} \cdots \sigma_{p^2}s_1\sigma_p s_0 \sigma(\partial_i) = \partial_i^{[p^r]}, \quad 1 \leq i \leq n, \quad 0 \leq s \leq k.$$ 

This finishes the proof of the theorem. \hfill \Box

**Corollary 2.5.** $\text{Aut}_{\text{ord}}(D(L_n)) = \text{St}(L_n)$.

**Proof.** The inclusion $\text{Aut}_{\text{ord}}(D(L_n)) \subseteq \text{St}(L_n)$ is obvious; see [5]. The inverse inclusion follows from the facts that $\text{St}(L_n) = \text{st}(L_n) \rtimes \text{Aut}_K(L_n)$, $\text{St}(L_n) \subseteq \text{Aut}_{\text{ord}}(D(L_n))$ (by Theorem 2.4) and $\text{Aut}_K(L_n) \subseteq \text{Aut}_{\text{ord}}(D(L_n))$; see [3]. \hfill \Box

**Proof of Theorem 1.3.** This follows from Corollary 2.4 and Theorem 2.4.

$$\text{Aut}_{\text{ord}}(D(L_n)) = \text{St}(L_n) = \text{st}(L_n) \rtimes \text{Aut}_K(L_n) \simeq \mathbb{Z}_p^n \rtimes \text{Aut}_K(L_n).$$ 

\hfill \Box

**References**


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