INTEGRAL REPRESENTATION FOR NEUMANN SERIES OF BESSEL FUNCTIONS

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ABSTRACT. A closed integral expression is derived for Neumann series of Bessel functions — a series of Bessel functions of increasing order — over the set of real numbers.

1. Introduction and motivation

The series

\[ \mathcal{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}(z), \quad z \in \mathbb{C}, \]

where \( \nu, \alpha_n \) are constants and \( J_{\mu} \) signifies the Bessel function of the first kind of order \( \mu \), is called a Neumann series [21, Chapter XVI]. Such series owe their name to the fact that they were first systematically considered (for integer \( \mu \)) by Carl Gottfried Neumann in his important book [15] in 1867; subsequently, in 1877, Leopold Bernhard Gegenbauer extended such series to \( \mu \in \mathbb{R} \) (see [21, p. 522]).

Neumann series of Bessel functions arise in a number of application areas. For example, in connection with random noise, Rice [18, Eqs. (3.10–3.17)] applied Bennett’s result,

\[ \sum_{n=1}^{\infty} \left( \frac{u}{v} \right)^n J_n(2i\sqrt{uv}) = e^{u^2/2} \int_0^u x e^{-x^2/2} J_0(2i\sqrt{ux}) \, dx. \]

Luke [8, pp. 271–288] proved that

\[ 1 - \int_0^v e^{-(u+v)} J_0(2i\sqrt{uv}) \, dx = \begin{cases} e^{-(u+v)} \sum_{n=0}^{\infty} \left( \frac{u}{v} \right)^{n/2} J_n(2i\sqrt{uv}), & u < v, \\ 1 - e^{-(u+v)} \sum_{n=1}^{\infty} \left( \frac{v}{u} \right)^{n/2} J_n(2i\sqrt{uv}), & u > v; \end{cases} \]

cf. also [16, Eq. (2a)]. In both of these applications \( \mathcal{N}_0 \) plays a key role. The function \( \mathcal{N}_0 \) also appears as a relevant technical tool in the solution of the infinite dielectric wedge problem by Kontorovich–Lebedev transforms [20, §§4, 5]. It also

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arises in the description of internal gravity waves in a Boussinesq fluid [14], as well as in the study of the propagation properties of diffracted light beams; see, for example, [12] Eqs. (6a,b), (7b), (10a,b)].

Expanding a given function \( f \), say, into a Neumann series of the form

\[
\mathcal{N}_\nu(x) = \sum_{n=0}^{\infty} a_{n\nu} J_{\nu+2n+1}(x), \quad \nu \geq -1/2,
\]

where

\[
a_{n\nu} = 2(\nu + 2n + 1) \int_{0}^{\infty} t^{-1} f(t) J_{\nu+2n+1}(t) dt,
\]

Wilkins discussed the question of existence of an integral representation for \( \mathcal{N}_\nu(x) \), as well as the conditions under which the Neumann series \( \mathcal{N}_\nu(x) \) converges uniformly in \( x \) to the 'input' function \( f \) [22, §§11–13].

By modifying a result of Watson [21, p. 23, footnote], Maximon represented a simple Neumann series appearing in the literature in connection with physical problems [11, Eq. (4)] as an indefinite integral expression containing Bessel functions. Meligy expanded into a Neumann series \( \mathcal{N}_{L+1/2} \) of arbitrary argument, containing Bessel functions of order \( L + 1/2 + n/2 \), where \( L \) is the orbital angular momentum quantum number, the wave functions that describe the states of motion of charged particles in a Coulomb field [13, Eqs. (8), (9)]. The inversion probability of a large spin is found via modified Neumann series of Bessel functions \( J_{(2N+1)(2n-1)+1} \) for integer \( N \geq 2 \); see, [5, Theorem].

The evaluation of the capacitance matrix of a system of finite-length conductors [2] uses \( \mathcal{N}_p \), with \( p \) integer; in [10], free vibrations of a wooden pole were modelled by a coupled system of ordinary differential equations and solved by Neumann series; we note in passing that the analysis of an isotropic medium containing a cylindrical borehole by Love’s auxiliary function and the analytical and numerical study of Neumann series of Bessel functions [18] are two further areas in which the unknown coefficients of \( \mathcal{N}_\nu \) are derived and computed from boundary and initial conditions of the problem under consideration.

2. STATEMENT OF THE MAIN RESULT

In this short note our main goal is to establish a closed integral representation formula for the series \( \mathcal{N}_\nu(z) \). This will be achieved by using the Laplace integral representation of the associated Dirichlet series. Thus, we replace \( z \in \mathbb{C} \) with \( x \in \mathbb{R}_+ \) and assume in what follows that the behaviour of \( (\alpha_n)_{n \in \mathbb{N}} \) ensures the convergence of the series \( \sum \) over \( \mathbb{R}_+ \).

Throughout the paper, \([a]\) and \( \{a\} = a - [a] \) will denote the integer and fractional part of a real number \( a \), respectively, while \( \chi_S \) will signify the characteristic function of the set \( S \subset \mathbb{R} \).

Consider the real-valued function \( x \mapsto a_x = a(x) \) and suppose that \( a \in \mathcal{C}^1[k,m] \), \( k, m \in \mathbb{Z}, k < m \). The classical Euler–Maclaurin summation formula states that

\[
\sum_{j=k}^{m} a_j = \int_{k}^{m} a(x)dx + \frac{1}{2} \left( a_k + a_m \right) + \int_{k}^{m} \left( x - [x] - \frac{1}{2} \right) a'(x)dx.
\]

On introducing the operator

\[
d_x := 1 + \{x\} \frac{d}{dx},
\]
obvious transformations yield the following condensed form of the Euler–Maclaurin formula:

\[ \sum_{j=k+1}^{m} a_j = \int_{k}^{m} (a(x) + \{x\}a'(x)) \, dx = \int_{k}^{m} \delta_x a(x) \, dx. \]  

**Theorem.** Let \( \alpha \in C^1(\mathbb{R}+) \) and let \( \alpha_{|\mathbb{N}} = (\alpha_n)_{n \in \mathbb{N}} \). Then, for all \( x, \nu \) such that \( 0 < x < 2 \min \left( 1, \left( e \lim_{n \to \infty} \sqrt[n]{|\alpha_n|} \right)^{-1} \right) \), \( \nu > -1/2 \), we have that

\[ \mathfrak{N}_\nu(x) = -\int_1^\infty \frac{\partial}{\partial \omega} \left( \Gamma(\nu + \omega + 1/2) J_{\nu+\omega}(x) \right) \int_0^{[\omega]} d\eta \left( \frac{\alpha(\eta)}{\Gamma(\nu + \eta + 1/2)} \right) d\eta \, d\omega. \]

**Proof.** Consider the integral representation formula \([8.411, \text{Eq.(10)}]\)

\[ J_{\nu}(z) = \frac{(x/2)^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_{-1}^1 \cos(zt)(1-t^2)^{\nu-1/2} \, dt, \quad z \in \mathbb{C}, \, \Re\{\nu\} > -1/2. \]

Applying (5) to (1) taking \( x > 0 \), we get

\[ \mathfrak{N}_\nu(x) = \sqrt{\frac{2x}{\pi}} \int_0^1 \cos(xt) \left( \frac{x(1-t^2)}{2} \right)^{\nu-1/2} D_\alpha(t) \, dt \]

with the Dirichlet series

\[ D_\alpha(t) := \sum_{n=1}^{\infty} \alpha_n \left( x(1-t^2)/2 \right)^n = \sum_{n=1}^{\infty} \frac{\alpha_n \exp \left\{ -n \ln \left( 2(1-t^2) \right) \right\}}{\Gamma(n + \nu + 1/2)}. \]

Recalling that \( \Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} (1 + O(s^{-1})) \), \( |s| \to \infty \), we see that the Dirichlet series \( D_\alpha(t) \) is absolutely convergent for all \( x \in \mathbb{R}^+ \) and \( t \in (-1,1) \) such that

\[ |x|(1-t^2) \leq \left| \frac{x}{2} \right| \left( \lim_{n \to \infty} \sqrt[n]{|\alpha_n|} \right)^{-1}. \]

Furthermore, \( D_\alpha(t) \) has a Laplace integral representation when \( \ln 2/(x(1-t^2)) > 0 \). In this case we can take \( x \in (0,2) \) and \( t \in (-1,1) \), since the required positivity condition is satisfied when

\[ \frac{2}{x(1-t^2)} \geq \frac{2}{x} > 1. \]

Hence, the \( x \)-domain becomes

\[ 0 < x < 2 \min \left( 1, \left( e \lim_{n \to \infty} \sqrt[n]{|\alpha_n|} \right)^{-1} \right). \]

Thus, for all such \( x \) we deduce that

\[ D_\alpha(t) = \ln \frac{2}{x(1-t^2)} \int_0^\infty \left( \frac{x(1-t^2)}{2} \right)^{\omega} \left( \sum_{j=1}^{[\omega]} \frac{\alpha_j}{\Gamma(j + \nu + 1/2)} \right) d\omega; \]
see, for example, [4, V] or [17, §§4, 6]. Now, it remains to sum the so-called counting function
\[ A_\alpha(\omega) := \sum_{j=1}^{[\omega]} \frac{\alpha_j}{\Gamma(j + \nu + 1/2)}; \]
The Euler–Maclaurin summation formula gives us
\[ A_\alpha(\omega) = \int_0^{[\omega]} \mathcal{D}_\alpha(\eta) \frac{\alpha(\eta)}{\Gamma(\nu + \eta + 1/2)} \, d\eta; \]
cf. [17, Lemma 1]. Substituting \( A_\alpha(\omega) \) and \( \mathcal{D}_\alpha(t) \) from (9) and (8) into (6), we get
\[ N_\nu(x) = -\sqrt{x^2/\pi} \int_0^{[\omega]} \mathcal{D}_\alpha(\eta) \frac{\alpha(\eta)}{\Gamma(\nu + \eta + 1/2)} \]
\[ \times \left( \int_0^1 \cos(xt) \left(\frac{x(1-t^2)}{2}\right)^{\nu+\omega-1/2} \ln \left(\frac{x(1-t^2)}{2}\right) \, dt \right) \, d\omega \, d\eta. \]
(10)
However, the innermost \((t-\text{integral})\) in (10),
\[ I_x(\kappa) := 2 \int_0^1 \cos(xt) \left(\frac{x(1-t^2)}{2}\right)^{\kappa} \ln \left(\frac{x(1-t^2)}{2}\right) \, dt, \quad \kappa := \nu + \omega - 1/2, \]
can be expressed in terms of the Gamma function and the Bessel function of the first kind by legitimate indefinite integration with respect to \( \kappa \), as follows. To begin, we define the Fourier cosine transform of a certain function \( f \) by
\[ F_c(f; x) := 2 \int_0^{\infty} \cos(xt) \, f(t) \, dt. \]
Now, we have that
\[ \int I_x(\kappa) \, d\kappa = 2 \left(\frac{x}{2}\right)^\kappa \int_0^1 \cos(xt)(1-t^2)^\kappa \, dt \]
\[ = \left(\frac{x}{2}\right)^\kappa F_c((1-t^2)^\kappa \chi_{[0,1]}(t); x) = \sqrt{\frac{2\pi}{x}} \cdot \Gamma(\kappa + 1) J_{\kappa+1/2}(x), \]
where we applied the Fourier cosine transform table [3, 17.34, Eq. (10)]. On observing that \( d\kappa = d\omega \), we deduce that
\[ I_x(\nu + \omega - 1/2) = \sqrt{\frac{2\pi}{x}} \cdot \frac{\partial}{\partial \omega} \left( \Gamma(\nu + \omega + 1/2) J_{\nu+\omega}(x) \right). \]
(11)
Substituting (11) into (10) we arrive at the asserted integral expression (4), remarking that the integration domain \( \mathbb{R}_+ \) changes into \([1, \infty)\) because \([\omega] \) equals zero for all \( \omega \in [0, 1) \).

3. Concluding remarks

To conclude, we mention some related integral representation formulæ for Neumann-type series, corresponding to special \( \alpha \)'s. Bivariate Lommel functions of order
The Neumann series of Bessel functions are defined by Neumann-type series [15, Eqs. (5), (6)] as follows:

\[ U_\nu(y, x) := \sum_{m=0}^{\infty} (-1)^m \left( \frac{y}{x} \right)^{\nu+2m} J_{\nu+2m}(x), \]

\[ V_\nu(y, x) := \cos \left( \frac{\nu\pi}{2} + \frac{\nu\pi}{2} + \frac{\nu\pi}{2} \right) + U_{-\nu+2}(y, x), \quad x, y \in \mathbb{R}. \]

These series converge for unrestricted values of \( \nu \).

Now, assuming that \( \Re\{\nu\} > 0 \), by the formulæ [15] 16.53, Eqs. (1), (2) [1] we easily deduce that

\[ U_{\nu, c}(x) := U_\nu(cx, x) = c^\nu x \int_0^1 t^\nu J_{\nu-1}(xt) \cos \left( \frac{\nu\pi}{2} x(1-t^2) \right) \, dt, \]

\[ U_{\nu+1, c}(x) = c^\nu x \int_0^1 t^\nu J_{\nu-1}(xt) \sin \left( \frac{\nu\pi}{2} x(1-t^2) \right) \, dt. \]

Similarly, by [15] 16.53, Eqs. (11), (12) [1] we also have that

\[ V_{\nu, c}(x) := V_\nu(cx, x) = -c^{-\nu} x \int_1^{\infty} t^{2-\nu} J_{1-\nu}(xt) \cos \left( \frac{\nu\pi}{2} x(1-t^2) \right) \, dt, \]

\[ V_{\nu-1, c}(x) = -c^{-\nu} x \int_1^{\infty} t^{2-\nu} J_{1-\nu}(xt) \sin \left( \frac{\nu\pi}{2} x(1-t^2) \right) \, dt, \]

provided \( x, c > 0, \Re\{\nu\} > 1/2 \).

The integral expressions developed above can be easily adapted to Neumann-type series of the form

\[ \sum_{m=0}^{\infty} \gamma^m J_{\nu+2m}(x), \quad x > 0, \gamma < 0. \]

An interesting open problem, worthy of further study, is the construction of examples with specific coefficients \( \alpha_n \), with known explicit forms of Neumann-type series, that can be derived directly from the representation formula [1].

References


1 Watson remarked that all four formulæ that were cited by him [15] 16.53, Eqs. (1), (2), (11), (12) had been derived by von Lommel (cf. von Lommel’s memoirs [6], [7] for further details).


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