A CLASS OF $\mathbb{Z}^d$ SHIFTS OF FINITE TYPE WHICH FACTORS ONTO LOWER ENTROPY FULL SHIFTS

ANGELA DESAI

(Communicated by Jane M. Hawkins)

Abstract. We prove that if a $\mathbb{Z}^d$ shift of finite type with entropy greater than $\log N$ satisfies the corner gluing mixing condition of Johnson and Madden, then it must factor onto the full $N$-shift.

1. Introduction

A basic question in symbolic dynamics is the question of when one shift space can factor onto another. There are well-known results addressing this question for $\mathbb{Z}$ shifts of finite type (SFTs). In particular, any $\mathbb{Z}$ SFT with entropy at least $\log N$ factors onto the full $N$-shift.

The situation is more complicated for $d > 1$. Often, we must impose further requirements, such as mixing conditions, to achieve similar results. Robinson and Sahin [RS] extended Krieger’s universal model results to $d > 1$ for SFTs with the uniform filling property. Lightwood [L1, L2] extended the Krieger Embedding Theorem [Kr] to $\mathbb{Z}^d$ subshifts with $d > 1$ for a class of SFTs called square-filling-mixing.

Introducing a new mixing condition called corner gluing, Johnson and Madden [JM] proved that any $\mathbb{Z}^d$ corner gluing SFT with entropy greater than $\log N$ has a finite extension which factors onto the full $N$-shift. They then posed the question of whether the extension is necessary. We prove that it is not.

Theorem 1.1. Let $X$ be a corner gluing $\mathbb{Z}^d$ SFT, and suppose $h(X) > \log N$. Then there exists a factor map $\varphi : X \to X_{[N]}$.

2. Definitions and notation

Let $A = \{0, 1, ..., N\}$, and let $X_{[N]} = A^{\mathbb{Z}^d}$, $d \in \mathbb{N}$. Give $A$ the discrete topology, and then give $X_{[N]}$ the product topology. A point $x \in X_{[N]}$ can be viewed as an infinite $d$-dimensional array of symbols: for $w \in \mathbb{Z}^d$, let $x_w$ be the symbol in location $w$.

For each $v \in \mathbb{Z}^d$, define a shift map $\sigma_v : x \mapsto y$ by $y_w = x_{v+w}$, and let $\sigma$ be the $\mathbb{Z}^d$ action $\{\sigma_v\}_{v \in \mathbb{Z}^d}$. The system $(X_{[N]}, \sigma)$ is the full $\mathbb{Z}^d$ $N$-shift. For $R \subset \mathbb{Z}^d$, a configuration on $R$ is some $\mathcal{M} \in A^R$. For $x \in X_{[N]}$, denote the configuration occurring at $R$ by $x_R$. 

Received by the editors March 28, 2007, and, in revised form, September 22, 2007.

2000 Mathematics Subject Classification. Primary 37B10; Secondary 37B40.

Key words and phrases. Shift of finite type, entropy.

@2009 American Mathematical Society
Reverts to public domain 28 years from publication.
If $X$ is a closed, shift-invariant subset of $X_{[N]}$, then $(X, \sigma|_X)$ is called a $\mathbb{Z}^d$ shift space, or subshift. Let $A_X$ be the symbol set of $X$. A configuration $M \in A^R$ is allowed in $X$ if there is some $x \in X$ such that $x_R = M$. Then we say that $M$ occurs in $x$.

The most important subshifts are the shifts of finite type. A $\mathbb{Z}^d$ subshift $X$ is a shift of finite type (SFT) if it can be defined by forbidding a finite set of configurations $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ occurring in $(A_X)^{\mathbb{Z}^d}$. $X$ is a one-step shift of finite type if a point $x \in (A_X)^{\mathbb{Z}^d}$ is allowed in $X$ whenever $x_{(m,n)}$ is not in $\mathcal{F}$ for all $m, n \in \mathbb{Z}^d$ with $\|m - n\| = 1$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$. Every SFT may be recoded to be a one-step shift of finite type. We will assume below that all shifts of finite type are one-step.

Let $c = (1, 1, \ldots, 1) \in \mathbb{Z}^d$. Let $\Lambda(n) = \{v = (v_1, \ldots, v_d) \in \mathbb{Z}^d : 0 \leq v_i < n\}$, the square of length $n$ with lower left corner at the origin. Let $\overline{\Lambda}(2n-1) = \{v = (v_1, \ldots, v_d) \in \mathbb{Z}^d : -n < v_i < n\}$, the square of length $2n - 1$ centered at the origin. An $n$-block is a configuration on $\Lambda(n)$. Let $B_n(X)$ be the set of $n$-blocks allowed in $X$. Let $B(X) = \bigcup_n B_n(X)$.

If $X$ and $Y$ are subshifts, then a map $\phi : X \to Y$ is a block code if for $x \in X$, $\phi(x)_v$ depends on some finite block configuration occurring in $x$, centered at $v$ for all $v \in \mathbb{Z}^d$. That is, if there is a map $\Phi : B_{2n-1}(X) \to A_Y$ such that $\phi(x)_v = \Phi(x_{\overline{\Lambda}(2n-1)+v})$ for all $v \in \mathbb{Z}^d$. The block codes from $X$ to $Y$ are exactly the continuous, shift-commuting maps. If $\phi$ is one-to-one, then it is called an embedding. If $\phi$ is onto, it is called a factor code, or a factor map. If $\phi$ is both one-to-one and onto, then it is a conjugacy. A subshift $X$ is a sofic shift if there exists an SFT $Y$ and a factor code $\pi : Y \to X$.

The topological entropy of a $d$-dimensional subshift $X$ is defined to be

$$h(X) = \lim_{n \to \infty} \frac{1}{n^d} \log |B_n(X)|.$$  

3. PROOF OF THEOREM

For $k = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$, let $R_k = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : 0 \leq a_i < k_i \text{ for } 1 \leq i \leq d\}$.

**Definition 3.1 (JM).** A $\mathbb{Z}^d$ SFT $X$ is corner gluing if there exists a gluing constant $g > 0$ such that given any two finite subsets $E_1, E_2 \subset \mathbb{Z}^d$ as defined below and any two allowable configurations $C_1$ and $C_2$ on these subsets, there exists a point $x \in X$ with $x_{E_1} = C_1$ and $x_{E_2} = C_2$. Here $E_1 = R_k + (k' - k)$ for some $k \in \mathbb{N}^d$ and some $k' \in \mathbb{N}^d$ with $k' > k + gc$, and $E_2 = R_{k'} \setminus R_{k+gc}$ (see Figure 4 for the case where $d = 2$).

We can also think about this in terms of creating a larger rectangular configuration on $R_{k'}$ containing $C_1$ and $C_2$ and some uncontrolled gluing symbols between them. Then we say we are gluing $C_1$ to $C_2$. We refer to the configurations used to glue them together as gluing strips.

In the proof of Theorem 4.1 we will need to make use of the following result:

**Theorem 3.2 (D).** Let $X$ be an SFT with $h(X) > 0$. Then there exists a family of SFT subsystems of $X$ whose entropies are dense in $[0, h(X)]$.

We also need the following lemma, which constructs a marker square $M$ that is aperiodic for low periods. For $R \subset \mathbb{Z}^d$ and $v \in \mathbb{Z}^d \setminus \{0\}$, a configuration $C$ on $R$
Figure 1. Corner gluing

is said to be \( v \)-periodic if for every pair \( w, w + v \in R \) we have \( C_w = C_{w+v} \). For simplicity, throughout this section we will give arguments only for the case where \( d = 2 \). The proofs for \( d \neq 2 \) are similar.

Lemma 3.3. Let \( X \) be a corner gluing \( \mathbb{Z}^d \) SFT with \( h(X) > 0 \), and let \( g \) be the gluing constant. Then for \( f, c \in \mathbb{N} \), if \( F \in B_f(X) \), then there exists a square configuration \( M \in B(X) \) as in Figure 2 such that \( M \) is not \( v \)-periodic whenever \( \|v\|_\infty < c \).

Figure 2. Marker square \( M \)

Proof. First we will construct a rectangular configuration \( Q \) such that \( Q \) is not \( v \)-periodic whenever \( \|v\|_\infty < c \). Choose some \( Q_0 \in B_c(X) \). Consider the \( v \in \mathbb{Z}^2 \) such that \( \|v\|_\infty < c \) and \( Q_0 \) is \( v \)-periodic. Enumerate these as \( v_1, v_2, \ldots, v_p \).

Let \( i \geq 1 \). Assume that \( Q_{i-1} \in B_l(X) \), for some \( l \in \mathbb{N} \), is not \( v_j \)-periodic for \( j < i - 1 \) and occurs with lower left corner at the origin. Let \( v_i = (a, b) \). By symmetry, we may assume \( a \geq 0 \). The block \( Q_{i-1} \) will be the corner of the block \( Q_i \), as pictured in Figure 3 according to the following cases: (i) \( a, b > 0 \), (ii) \( a = 0, b > 0 \), (iii) \( a > 0, b = 0 \), and (iv) \( a > 0, b < 0 \).

Consider case (i). Choose \( k \in \mathbb{N} \) large enough that \( ka, kb > l + g \) and suppose \( \alpha \) is the symbol occurring at the lower left corner of \( Q_{i-1} \). Since \( h(X) > 0 \), there is some \( \beta \in A_X \) with \( \beta \neq \alpha \). Extend \( Q_{i-1} \) to an L-shape shown by the dashed lines in Figure 3(i), then glue the symbol \( \beta \) in at position \( k \overrightarrow{v_i} \). Extend the resulting rectangle to a square \( Q_i \). \( Q_i \) is not \( v_i \)-periodic because \( v_i \)-periodicity would imply \( \alpha = \beta \). As \( Q_i \) has \( Q_{i-1} \) as a subblock, it is not \( v_j \)-periodic for \( j < i - 1 \) either. For the remaining three cases the argument is the same (see Figure 3(ii),(iii),(iv)). The construction of \( Q_i \) is the same, based on the corresponding figures. End this process with \( Q = Q_p \). Then \( Q \) will not be \( v \)-periodic for \( \|v\|_\infty < c \).
Now construct the marker square $M$ as follows. Extend $F$ to an $L$-shaped configuration as in Figure 4(i). Then glue in $Q$ as in Figure 4(ii), where the shaded region is the gluing region of width $g$ necessary in the definition of corner gluing.

Extend this configuration to another $L$-shaped configuration, represented in Figure 5(i) with dashed lines. Choose some rectangle extension of $F$ of the form seen in Figure 5(ii). Glue this rectangle to the $L$-shaped configuration to form a configuration as in Figure 5(iii).

Next, extend this rectangle to another $L$-shape as in Figure 6(i), and choose some rectangle as in Figure 6(ii) with an $F$ at the right and left ends. Note that such a configuration is allowed because there is a point which contains it in Figure 6(i). Glue these configurations together to form the square in Figure 6(iii). Take $M$ to be the subblock with an $F$ at each corner.

With this lemma, we are ready to prove Theorem 1.1 using methods similar to those used by Johnson and Madden in [JM].

**Proof of Theorem 1.1.** By Theorem 3.2 there is a proper subsystem $Y$ of finite type in $X$ with $h(Y) > \log N$. Choose some square configuration $F \in B(X)$ that
is forbidden in $Y$, and call its side length $f$. Construct a square configuration $M$ using $F$ as in Lemma 3.3 for $c = 2(f + g)$. Denote the side length of $M$ by $m$. 

**Figure 5.** Construction of $M$, step 2

**Figure 6.** Construction of $M$, step 3
Given the marker square $M$ and any rectangular configuration $G$ allowed in $Y$ of height $m$ and arbitrary length, first extend $M$ to an $L$-shaped configuration as in Figure 7(i). Then glue this configuration to $G$ to get the new configuration seen in Figure 7(ii).

![Figure 7. Gluing M and G](image)

Continue this process to construct a configuration $L$ of the form seen in Figure 8(i). The blocks labeled $G$ can be filled in with any configuration of the appropriate size allowed in $Y$ (we think of these as ‘good’ blocks), and the shaded regions are the necessary gluing strips (which may depend on the choice of $G$-blocks). By the inside corner of $L$, we mean the upper right corner of the block $M$ in the lower left corner of $L$.

Glue a block of the type in Figure 8(ii) to $L$ to get a legal block $D$ of the type in Figure 8. The complement of $L$ in $D$, will be called a follower of $L$. We do not control the symbols in the gluing strips, but all $G$-configurations of the correct size will appear in follower blocks for some choice of gluing strip configuration. Let $l$ be the side length of the central block $G$ in $D$, and $J = l + 2g + m$; then $C \in B_J(X)$. Each $L$ has at least $|B_l(Y)|$ followers and because $h(Y) > \log N$, we have $|B_l(Y)| > N^{J^2}$ for large enough $l$. For each $L$, partition its followers into $N^{J^2}$ nonempty sets, $P(L)_1, P(L)_2, \ldots, P(L)_{N^{J^2}}$, depending only on the follower’s central $G$-block.

**Claim.** Let $x \in X$. If blocks $D$ and $D'$ of the form in Figure 9 occur at different places in $x$, then their follower portions, $C$ and $C'$, do not overlap.

**Proof of claim.** Without loss of generality, assume $D$ occurs with lower left corner at the origin, and $D'$ occurs with lower left corner at $v$. Suppose $C$ and $C'$ do overlap. We know that $v$ is not such that $\|v\|_\infty < c$, as that would contradict the lack of small periodicity of $M$ assured by Lemma 3.3. But the lower left corner $M$ of $D'$ cannot overlap too much with any other $M$ in $D$ either, and we are assuming
that \( C \) and \( C' \) overlap. Therefore \( \|v\|_\infty \leq J - c \). Now since \( M \) was constructed with an \( F \) at each corner and \( c = 2(f + g) \), at least one subblock \( F \) of \( D' \) must occur entirely in a ‘good’ block \( G \) of \( D \). However these blocks were chosen from the blocks allowed in \( Y \) and so cannot contain \( F \) as a subblock. Thus \( C \) and \( C' \) cannot overlap. \( \square \)

Consider the \( J \)-blocks of \( X^\left\lfloor N \right\rfloor \). Enumerate them as \( E_1, E_2, \ldots, E_{NJ^2} \). Now we are ready to construct a factor map \( \varphi : X \to X^\left\lfloor N \right\rfloor \). We will define \( \varphi \) so that it essentially maps blocks from \( P(\mathcal{L})_i \) to \( E_i \) for each configuration \( \mathcal{L} \) and \( i = 1, 2, \ldots, NJ^2 \).

We make this precise as follows. For \( x \in X \), suppose a configuration \( D \) as in Figure 9 occurs in \( x_{\Lambda(2J-1)+v-Jc} \), the \( (2J-1) \)-block centered at \( v \), and \( x_v \) is in the follower portion of \( D \). By the claim, \( x_v \) occurs in the follower portion of no other such block \( D' \). Therefore, there exist unique \( u, w \in \mathbb{Z}^2 \) such that \( v = u + w \), where \( \mathcal{L} \) has its inside corner at \( u \), and \( 0 < w_i \leq J \) for \( i = 1, 2 \). If \( x_v \) occurs in \( \mathcal{C} \in P(\mathcal{L})_j \), then we define \( \varphi(x)_v \) to be the symbol from coordinate \( w \) of \( E_j \). If \( x_v \) is not in a follower, then \( \varphi(x)_v = 0 \).

Claim. \( \varphi \) is onto.

Proof of claim. Let \( E \in B_{kJ}(X^\left\lfloor N \right\rfloor) \) be as in Figure 10. Choose a configuration \( R \) of the form shown in Figure 11(i) whose height and width are both \( kJ + m + g \). We will glue configurations to \( R \) that will result in a square configuration which maps to \( E \). Consider the configuration \( \mathcal{L} \) in the lower left corner of \( R \).
If \( E(0,0) = E_i \), then choose a configuration \( B_{(0,0)} \in P(L_i) \) as in Figure 8(ii) to glue to \( L \). This new block \( B_{(0,0)} \) together with \( R \) forms two new \( L \)-configurations as shown in Figure 11(ii). One will be above \( B_{(0,0)} \) and one will be to the right of it. Glue in followers of each \( L \) from the partition elements corresponding to \( E(1,0) \) and \( E(0,1) \). Continuing in this manner, complete a block \( B \in B_{k,j}(X) \) that maps to \( E \) under the block map. \( \square \)

Johnson and Madden give the following example of a \( \mathbb{Z}^2 \) SFT \( X \), defined by the matrices below, that is corner gluing with \( h(X) > \log 2 \). Johnson and Madden’s theorem tells us only that \( X \) is the finite-to-one factor of an SFT that factors onto the full shift, and they ask whether \( X \) itself can factor onto \( X_{[2]} \). Theorem 1.1 tells us that it does.

\[
A_h = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad A_v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

It is still not known whether every \( \mathbb{Z}^d \) SFT with \( h(X) > \log N \) factors onto the full \( N \)-shift.

REFERENCES


DEPARTMENT OF BIOLOGY, CHEMISTRY, AND MATHEMATICS, UNIVERSITY OF MONTEVALLO, MONTEVALLO, ALABAMA 35115

Current address: Department of Mathematics, Anne Arundel Community College, 101 College Parkway, Arnold, Maryland 21012

E-mail address: avdesai@aacc.edu