Let $X$ be an infinite-dimensional Banach space, $\{S(t)\}_{t \geq 0}$ be a $C_0$-semigroup with the generator $(A, D(A))$ on $X$, $K$ be a compact operator on $X$, and $\{S_K(t)\}_{t \geq 0}$ be the $C_0$-semigroup generated by $(A + K, D(A))$.

In [3] we discussed the non-exponential stabilization for isometric $C_0$-semigroups under compact perturbation, but a slip has occurred in the proof of Theorem 3. More precisely, we have used the isometricity of the adjoint semigroup $\{S^*(t)\}_{t \geq 0}$ while the isometricity of the $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ is assumed. However, there are some isometric $C_0$-semigroups whose adjoint semigroups are not isometric; for a concrete example, we refer to [1] and [2]. Therefore, the statement of Theorem 3 in [3] must be changed as follows.

**Theorem 1.** Let $\{S(t)\}_{t \geq 0}$ be an isometric $C_0$-semigroup on an infinite-dimensional reflexive Banach space $X$. Then, the $C_0$-semigroup $\{S_K(t)\}_{t \geq 0}$ cannot be uniformly exponentially stable.

**Proof.** Let $U := \{x \in X : \|x\| \leq 1\}$ be the unit ball in $X$. Since $X$ is an infinite-dimensional Banach space, then $U$ is not a compact set in $X$. Thus, there exist $\epsilon_0 > 0$ and $y_j \in U, j = 1, 2, \ldots$, such that

$$\|y_i - y_j\| \geq \epsilon_0 > 0$$

for all $i \neq j$.

For the compactness of operator $K$, we deduce that $K^*$ is compact and $R(K^*)$, the range of operator $K^*$, is separable, and consequently, $\{S^*(\tau)K^*x^* : x^* \in X^*\}$ is separable for each $\tau \geq 0$. Letting $Q^+$ be the set of non-negative rational numbers, we have that $Q^+$ is denumerable, and hence

$$\bigcup_{\tau \in Q^+} \{S^*(\tau)K^*x^* : x^* \in X^*\}$$

is also separable. Therefore, $V^* := \text{span}\{S^*(\tau)K^*x^* : x^* \in X^*, \\tau \geq 0\}$ is separable, or equivalently, there exists a countable subset $\{v_{\tau}^1, v_{\tau}^2, \ldots, v_{\tau}^m, \ldots\}$ which is dense in $V^*$. From the uniform boundedness of $\{y_k\}_{k=1}^\infty$ and the diagonal method, there is a subsequence $\{y_{k_n}\}_{n=1}^\infty$ of $\{y_j\}_{j=1}^\infty$ such that $\{(v_{\tau}^m(y_{k_n}))\}_{n=1}^\infty$ is convergent.
for any \( v_m^* \). Therefore, from \( \|y_{j_n}\| \leq 1 \) and the denseness of \( \{v_m^*\}_{m=1}^{\infty} \) in \( V^* \), it follows that \( \{x^*(y_{j_n})\}_{n=1}^{\infty} \) is convergent for any \( x^* \in \overline{V^*} \).

Defining a map \( f_0: \overline{V^*} \to C \) by

\[
f_0(x^*) := \lim_{n \to \infty} x^*(y_{j_n})
\]

for \( x^* \in \overline{V^*} \), we can easily see that \( f_0 \) is a continuous and linear functional on \( \overline{V^*} \). From the Hahn-Banach Theorem and the reflexivity of \( X \), there exists a bounded linear functional \( f \in X^{**}(=X) \) such that

\[
f|_{\overline{V^*}} = f_0 \quad \text{and} \quad \|f\| = \|f_0\|.
\]

Furthermore, we can deduce that

1. \( KS(\tau)y_{j_n} \to KS(\tau)f \) as \( n \to \infty \)

for each \( \tau \geq 0 \). In fact, assume that there exist \( \eta > 0 \), \( \tau_0 > 0 \) and a subsequence \( \{y_{j_{n_k}}\}_{k=1}^{\infty} \) of \( \{y_{j_n}\}_{n=1}^{\infty} \) such that

\[
\|KS(\tau_0)y_{j_{n_k}} - KS(\tau_0)f\| > \eta, \quad k = 1, 2, \ldots
\]

Hence, there exists \( x_k^* \in X^* \) with \( \|x_k^*\| = 1 \) such that

2. \( |x_k^*(KS(\tau_0)y_{j_{n_k}}) - x_k^*(KS(\tau_0)f)| > \frac{\eta}{2}, \quad k = 1, 2, \ldots \)

Since \( K^* \) is compact, without loss of generality, there exists \( y_0^* \in X^* \) such that \( K^*x_k^* \to y_0^* \) as \( k \to \infty \), and then \( S^*(\tau_0)y_0^* \in \overline{V^*} \). Therefore, we deduce that

\[
|x_k^*(KS(\tau_0)y_{j_{n_k}}) - x_k^*(KS(\tau_0)f)| \\
= |(S^*(\tau_0)K^*x_k^*)(y_{j_{n_k}}) - (S^*(\tau_0)K^*x_k^*)(f)| \\
\leq |(S^*(\tau_0)K^*x_k^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(y_{j_{n_k}})| + |(S^*(\tau_0)y_0^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(f_0)| \\
+ |(S^*(\tau_0)y_0^*)(f) - (S^*(\tau_0)K^*x_k^*)(f)| \\
\leq (1 + \|f_0\|)|K^*x_k^* - y_0^*| + |(S^*(\tau_0)y_0^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(f_0)| \\
\to 0 \quad (k \to \infty),
\]

where we have used \( f|_{\overline{V^*}} = f_0 \), the isometricity of \( \{S^*(\tau)\}_{\tau \geq 0} \) and the reflexivity of \( X \). There is a contradiction with (2) which shows the validity of (1).

Noting that \( \|y_i - y_j\| \geq \epsilon_0 \) for \( i \neq j \), we may assume that \( \|y_{j_n} - f\| > \frac{\epsilon_0}{2} \); and letting \( x_n = \frac{y_{j_n} - f}{\|y_{j_n} - f\|} \), we deduce from (1) and the definition of \( x_n \) that

3. \( \|x_n\| = 1 \) and \( \|KS(\tau)x_n\| \to 0 \quad (n \to \infty) \)

for each \( \tau \geq 0 \).

Let \( f_{n,m}(\tau): [0, \infty) \to X \) be defined by

\[
f_{n,m}(\tau) = \begin{cases} 
S_K(\tau)KS(m - \tau)x_n, & \tau \in [0, m], \\
0, & \tau \in (m, \infty);
\end{cases}
\]

thus, from (3), we have

\[
\|f_{n,m}(\tau)\| \to 0 \quad (n \to \infty)
\]
Hence, we conclude that

Theorem 3.

If have been proved in [3].

exponentially stabilized by a compact operator? Actually, the following results
for all \( t \) \( \geq 0 \). Thus,

\[
\| g_m(\tau) \| \leq \begin{cases} 
M_K \| e^{-w_K \tau} \|, & \tau \in [0, m], \\
0, & \tau \in (m, \infty).
\end{cases}
\]

Hence, we conclude that

\[
\lim_{m \to \infty} \int_0^\infty \| g_m(\tau) \| d\tau = \lim_{m \to \infty} \int_0^m \| S_K(\tau) KS(\tau)x_m \| d\tau = 0
\]

by Lebesgue’s dominated convergence theorem.

From the bounded perturbation theorem of \( C_0 \)-semigroups, it follows that

\[
S_K(t)x = S(t)x + \int_0^t S_K(\tau) KS(\tau)x d\tau, \quad x \in X, \ t \geq 0,
\]

which yields that

\[
\lim_{m \to \infty} \| S(m)x_m \| \leq \lim_{m \to \infty} \left( \| S_K(m)x_m \| + \int_0^m \| S_K(\tau) KS(\tau)x_m d\tau \| \right)
\]

\[
= 0.
\]

On the other hand, \( \{ S(t) \} \) \( \geq 0 \) is an isometric \( C_0 \)-semigroup. Then we have \( \| S(m)x_m \| = \| x_m \| = 1 \) for all \( m \). There is a contradiction with (4), which ends the proof of Theorem 1.

Finally, we recall the following result in Luo, Weng, and Feng [4].

**Theorem 2.** Let \( X \) be a reflexive Banach space, and let the adjoint semigroup \( \{ S^*(t) \} \) \( \geq 0 \) be strongly stable. If \( \{ S_K(t) \} \) \( \geq 0 \) is uniformly exponentially stable, then \( \{ S(t) \} \) \( \geq 0 \) is also uniformly exponentially stable.

It is well known that a \( C_0 \)-semigroup \( \{ S(t) \} \) \( \geq 0 \) is uniformly exponentially stable if and only if \( \| S(t_0) \| < 1 \) for some \( t_0 > 0 \). Therefore, the following natural question arises: Can a \( C_0 \)-semigroup whose adjoint semigroup is isometric be uniformly exponentially stabilized by a compact operator? Actually, the following results have been proved in [3].

**Theorem 3.** If \( \{ S(t) \} \) \( \geq 0 \) is a \( C_0 \)-semigroup on an infinite-dimensional Banach space \( X \) and its adjoint semigroup \( \{ S^*(t) \} \) \( \geq 0 \) is an isometric semigroup, then the \( C_0 \)-semigroup \( \{ S_K(t) \} \) \( \geq 0 \) cannot be uniformly exponentially stable for any compact operator \( K \) on \( X \).
Proof. Suppose that \( \{S_K(t)\}_{t \geq 0} \) is uniformly exponentially stable. Then there exist constants \( M_K \geq 1 \) and \( w_K > 0 \) such that
\[
\|S_K(t)\| \leq M_K e^{-w_K t}, \quad t \geq 0.
\]
□

From the proof of Theorem 3 in [3], as the notation in [3], there exists a sequence \( \{x^*_n\}_{m=1}^\infty \subset X^* \) with \( \|x^*_n\| = 1 \) such that
\[
\lim_{m \to \infty} \int_0^m \|S^*_K(\tau)K^*S^*(m - \tau)x^*_n\|d\tau = 0.
\]

From the bounded perturbation theorem of \( C_0 \)-semigroups, we deduce that
\[
\lim_{m \to \infty} \|S^*(m)x^*_n\| \\
\leq \lim_{m \to \infty} \left( \|S^*_K(m)x^*_n\| + \| \int_0^m S^*_K(\tau)K^*S^*(m - \tau)x^*_n\|d\tau \right) \\
\leq \lim_{m \to \infty} \left( M_K e^{-w_K m} + \int_0^m \|S^*_K(\tau)K^*S^*(m - \tau)x^*_n\|d\tau \right) \\
= 0.
\]

There is a contradiction with the isometricity of \( \{S^*(t)\}_{t \geq 0} \).

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