MIYAOKA-YAU INEQUALITY FOR MINIMAL PROJECTIVE MANIFOLDS OF GENERAL TYPE

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Abstract. In this short paper, we prove the Miyaoka-Yau inequality for minimal projective \( n \)-manifolds of general type by using Kähler-Ricci flow.

1. Introduction

If \( M \) is a projective \( n \)-manifold with ample canonical bundle \( K_M \), there exists a Kähler-Einstein metric \( \omega \) with negative scalar curvature by Yau’s theorem on the Calabi conjecture ([14]), which was obtained by Aubin independently ([1]). As a consequence, there is an inequality for Chern numbers, the Miyaoka-Yau inequality,

\[
\frac{2(n+1)}{n} c_2(M) - c_1^2(M) \cdot (-c_1(M))^{n-2} \geq 0,
\]

where \( c_1(M) \) and \( c_2(M) \) are the first and the second Chern classes of \( M \) (cf. [13]). Furthermore, if the equality in (1.1) holds, the Kähler-Einstein metric \( \omega \) is a complex hyperbolic metric; i.e. the holomorphic sectional curvature of \( \omega \) is a negative constant. If \( n = 2 \), (1.1) even holds for algebraic surfaces of general type (cf. [4], [8], [9]), which may not admit any Kähler-Einstein metric. In [12], the inequality (1.1) is proved for any dimensional minimal projective manifold of general type by using conic Kähler-Einstein metrics. In this short paper, we give a different proof of (1.1) for minimal projective \( n \)-manifolds of general type by using Kähler-Ricci flow and study the extremal case of (1.1).

Let \( M \) be a minimal projective manifold of general type with \( \dim \mathbb{C} M = n \geq 2 \). The canonical bundle \( K_M \) of \( M \) is big, and semi-ample, i.e. \( K_M^n > 0 \), and, for a positive integer \( m \gg 1 \), the linear system \( |mK_M| \) is base point free (as quoted in [11]). For \( m \gg 1 \), the complete linear system \( |mK_M| \) defines a holomorphic map \( \Phi : M \to \mathbb{CP}^N \), which is birational onto its image \( M_{can} \). \( M_{can} \) is called the canonical model of \( M \), and \( \Phi \) is called the contraction map. Note that \( M \) may not admit any Kähler-Einstein metric. The Kähler-Ricci flow is an evolution equation of a family of Kähler metrics \( \omega_t, t \in [0, T) \), on \( M \),

\[
\partial_t \omega_t = -\text{Ric}(\omega_t) - \omega_t,
\]

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where \( \text{Ric}(\omega_t) \) is the Ricci form of \( \omega_t \). By [11], [10], [3], and [15], for any Kähler metric as initial metric, the solution \( \omega_t \) of the Kähler-Ricci flow equation exists for all time \( t \in [0, \infty) \), and the scalar curvature of \( \omega_t \) is uniformly bounded. Thus we can prove (1.1) by using the technique developed in [6], where a Hitchin-Thorpe type inequality was proved for 4-manifolds which admit a long time solution to a normalized Ricci flow equation with bounded scalar curvature. Before proving the Miyaoka-Yau inequality, we show that the \( L^2 \)-norm of the Einstein tensor tends to zero along a subsequence of a solution of the Kähler-Ricci flow equation (1.2).

**Theorem 1.1.** Let \( M \) be a minimal projective manifold of general type with \( \dim_{\mathbb{C}} M = n \geq 2 \), and let \( \omega_t, t \in [0, \infty) \), be a solution of the Kähler-Ricci flow equation (1.2). Then there exists a sequence of times \( t_k \to \infty \), when \( k \to \infty \), such that
\[
\lim_{k \to \infty} \int_M |\rho_{t_k}|^2 \omega_{n t_k} = 0,
\]
where \( \rho_{t_k} = \text{Ric}_{t_k} - \frac{R_{t_k}}{n} \omega_{t_k} \) denotes the Einstein tensor of \( \omega_{t_k} \) and \( R_{t_k} \) denotes the scalar curvature of \( \omega_{t_k} \).

As a corollary of this theorem, we obtain the Miyaoka-Yau inequality for minimal projective manifolds of general type.

**Corollary 1.2.** If \( M \) is a minimal projective manifold of general type with \( \dim_{\mathbb{C}} M = n \geq 2 \), then
\[
\left( \frac{2(n + 1)}{n} c_2(M) - c_1^2(M) \right) \cdot (-c_1(M))^{n-2} \geq 0.
\]
Furthermore, if the equality holds, there is a complex hyperbolic metric on the smooth part \( M_0 \) of the canonical model \( M_{can} \) of \( M \).

2. **Proof of Theorem 1.1**

Let \( M \) be a minimal projective manifold of general type with \( \dim_{\mathbb{C}} M = n \geq 2 \), \( M_{can} \) be the canonical model of \( M \), and \( \Phi : M \to M_{can} \) be the contraction map. Consider the Kähler-Ricci flow equation on \( M \),
\[
\partial_t \omega_t = -\text{Ric}(\omega_t) - \omega_t,
\]
with initial metric \( \omega_0 \). In [7], the short time existence of the solution of (2.1) is proved. Then, in [11], [10], and [3], it is proved that the solution \( \omega_t \) of (2.1) exists for all time, i.e. \( t \in [0, +\infty) \), and there exists a unique semi-positive current \( \omega_\infty \) on \( M \) which satisfies that:

1. \( \omega_\infty \) represents \( -2\pi c_1(M) \).
2. \( \omega_\infty \) is a smooth Kähler-Einstein metric with negative scalar curvature on \( \Phi^{-1}(M_0) \), where \( M_0 \) is the smooth part of \( M_{can} \).
3. On any compact subset \( K \subset \Phi^{-1}(M_0), \omega_t C^\infty \)-converges to \( \omega_\infty \) when \( t \to \infty \).

In [15], it is shown that there is a constant \( C > 0 \) depending only on \( \omega_0 \) such that
\[
|R_t| < C,
\]
where \( R_t \) is the scalar curvature of \( \omega_t \).
First, we need evolution equations for volume forms and scalar curvatures as follows:

\[(2.3) \quad \partial_t \omega_t^n = -(R_t + n)\omega_t^n\]

and

\[(2.4) \quad \partial_t R_t = \Delta_t R_t + |Ric_t|^2 + R_t = \Delta_t R_t + |Ric_t^n|^2 - (R_t + n),\]

where \(Ric_t^n = Ric_t + \omega_t\) and \(|Ric_t^n|^2 = |Ric_t|^2 + 2R_t + n\) (cf. Lemma 2.38 in [5]).

**Lemma 2.1.** There are two constants \(t_0 > 0\) and \(c > 0\) independent of \(t\) such that, for \(t > t_0\),

\[\bar{R}_t = \inf_{x \in M} R_t(x) \leq -n + e^{-t}c < -\frac{n}{2} < 0.\]

**Proof.** If we define \(\alpha_t = [\omega_t] \in H^{1,1}(M, \mathbb{R})\), from (2.1) we have

\[\partial_t \alpha_t = -2\pi c_1(M) - \alpha_t\]

and

\[(2.5) \quad \alpha_t = -2\pi c_1(M) + e^{-t}(2\pi c_1(M) + \alpha_0).\]

Thus

\[(2.6) \quad [\omega_\infty] = \alpha_\infty = \lim_{t \to -\infty} \alpha_t = -2\pi c_1(M).\]

Since

\[\bar{R}_t \int_M \omega_t^n \leq \int_M R_t \omega_t^n = n \int_M Ric_t \wedge \omega_t^{n-1} = n2\pi c_1(M) \cdot \alpha_t^{n-1},\]

we obtain

\[\bar{R}_t \leq n \frac{2\pi c_1(M) \cdot \alpha_t^{n-1}}{\alpha_t^n} = n \frac{2\pi c_1(M) \cdot \alpha_t^{n-1}}{-2\pi c_1(M) \cdot \alpha_t^{n-1} + e^{-t}(2\pi c_1(M) + \alpha_0) \cdot \alpha_t^{n-1}} \leq \frac{n}{1 + e^{-t}A_t},\]

where \(A_t = \frac{2\pi c_1(M) + \alpha_0}{2\pi c_1(M) \cdot \alpha_t^{n-1}}\). Note that \((-c_1(M))^n > 0\). Thus there is a \(t_1 > 0\) such that if \(t > t_1\), \(A_t < \left|\frac{\alpha_\infty + \alpha_0}{\alpha_\infty}\right| + 1 = A\), and we obtain that

\[\bar{R}_t \leq \frac{-n}{1 + e^{-t}A} < -n + e^{-t}c,\]

where \(c = -n(\frac{A}{1 + e^{-t}A})\). By taking \(t_0 > t_1\) such that \(e^{-t_0}c < \frac{n}{2}\), we obtain the conclusion. \(\square\)

**Lemma 2.2.**

\[\int_0^\infty \int_M |\bar{R}_t + n|\omega_t^n dt < \infty.\]

**Proof.** By (2.4) and the maximal principle, \(\partial_t \bar{R}_t \geq -(\bar{R}_t + n)\), and so

\[(2.7) \quad n + \bar{R}_t \geq Ce^{-t},\]
for a constant $C$ independent of $t$. Note that by Lemma 2.1, (2.7) and (2.3), when $t > t_0$,

\[
\int_M |R_t + n|\omega_t^n| \leq \int_M (R_t - \bar{R}_t)\omega_t^n + \int_M |n + \bar{R}_t|\omega_t^n
\]

\[
\leq \int_M (R_t + n)\omega_t^n + 2\int_M |n + \bar{R}_t|\omega_t^n
\]

\[
\leq \int_M (R_t + n)\omega_t^n + C_3e^{-t}
\]

\[
= n(2\pi c_1 \cdot \alpha_t^{n-1} + \alpha_t^n) + C_3e^{-t}
\]

\[
= ne^{-t}(2\pi c_1 + \alpha_0) \cdot \alpha_t^{n-1} + C_3e^{-t}
\]

\[
\leq C_4e^{-t}
\]

for two constants $C_3$ and $C_4$ independent of $t$. Thus

\[
\int_0^\infty \int_M |R_t + n|\omega_t^n| dt = \int_0^{t_0} \int_M |R_t + n|\omega_t^n| dt + \int_{t_0}^\infty \int_M |R_t + n|\omega_t^n| dt < \infty.
\]

**Proof of Theorem 1.1.** From (2.4), (2.3), (2.6), (2.2), and Lemma 2.2, we obtain

\[
\int_0^\infty \int_M |Rc_t|^2\omega_t^n| dt = \int_0^\infty \int_M \left(\frac{\partial}{\partial t}Rc_t\right)\omega_t^n| dt + \int_0^\infty \int_M (R_t + n)\omega_t^n| dt
\]

\[
= \int_0^\infty \frac{d}{dt}\int_M (Rc_t)\omega_t^n| dt + \int_0^\infty \int_M (R_t + 1)(R_t + n)\omega_t^n| dt
\]

\[
\leq n\alpha_t^n - \int_0^\infty \int_M R_0\omega_0^n + C\int_0^\infty \int_M |R_t + n|\omega_t^n| dt
\]

\[
< \infty.
\]

If $\rho_t = Rc_t - \frac{R_t}{n}\omega_t$ is the Einstein tensor of $\omega_t$, then $|\rho_t|^2 = |Rc_t|^2 - \frac{1}{n}(R_t + n)^2$, and from the above estimation,

\[
\int_0^\infty \int_M |\rho_t|^2\omega_t^n| dt \leq \int_0^\infty \int_M |Rc_t|^2\omega_t^n| dt < \infty.
\]

Thus there is a sequence $t_k \rightarrow \infty$ such that

\[
\lim_{k \rightarrow \infty} \int_M |\rho_{t_k}|^2\omega_{t_k}^n = 0.
\]

**Proof of Corollary 1.2.** Note that the Kähler curvature tensor has a decomposition

\[
Rm_t = \frac{R_t}{2n^2}\omega_t \otimes \omega_t + \frac{1}{n}\omega_t \otimes \rho_t + \frac{1}{n}\rho_t \otimes \omega_t + B_t
\]

(cf. (2.63) and (2.38) in [2]). By Chern-Weil theory,

\[
\frac{2(n + 1)}{n}c_2(M) - c_1^2(M) \cdot |\omega_t|^{n-2} = \frac{(n - 2)!}{4\pi^2 n^2} \int_M \left(\frac{n + 1}{n}|B_{0,t}|^2 - \frac{(n^2 - 2)}{n^2}|\rho_t|^2\right)|\omega_t|^n
\]

(cf. (2.82a) and (2.67) in [2]), where $B_{0,t} = B_t - \frac{\omega_t}{n^2 - 1}\text{Id}$ is the tensor given by (2.64) in [2] corresponding to $\omega_t$. By Theorem 1.1, there is a sequence $t_k \rightarrow \infty$ such that

\[
\lim_{k \rightarrow \infty} \int_M |\rho_{t_k}|^2\omega_{t_k}^n = 0.
\]
Hence
\[ \frac{2(n+1)}{n} c_2(M) - c_1^2(M) \cdot (-2\pi c_1(M))^{n-2} = \lim_{k \to \infty} \frac{2(n+1)}{n} c_2(M) - c_1^2(M) \cdot [\omega_k]^{n-2} \]
\[ = \lim_{k \to \infty} \frac{2(n+1)}{n} c_2(M) - c_1^2(M) \cdot [\omega_k]^{n-2} \]
\[ = \lim_{k \to \infty} \frac{(n-2)!}{4\pi^2 n!} \int_M \left( \frac{n+1}{n} |B_{0,t_k}|^2 \right) \omega_k^n \]
\[ \geq 0. \]

If the equality holds, on any compact subset \( K \subset \Phi^{-1}(M_0) \),
\[ \int_K |B_{0,\infty}|^2 \omega_\infty^n \leq \lim_{k \to \infty} \int_M |B_{0,t_k}|^2 \omega_k^n = 0, \]
by the smooth convergence of \( \omega_k \) to \( \omega_\infty \). Thus \( B_{0,\infty} \equiv 0 \). Since \( \omega_\infty \) is a Kähler-Einstein metric with negative scalar curvature on \( \Phi^{-1}(M_0) \), the holomorphic sectional curvature is a negative constant by Section 2.66 in [2]: i.e. \( \omega_\infty \) is a complex hyperbolic metric.

\[ \square \]

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