

ON THE L_p NORM OF THE RADEMACHER PROJECTION AND RELATED INEQUALITIES

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ABSTRACT. The purpose of this paper is to find the exact norm of the Rademacher projection onto $\{r_1, r_2, r_3\}$. Namely, we prove

$$\|R_3\|_p = \frac{(3^{p/q} + 1)^{1/p}(3^{q/p} + 1)^{1/q}}{4}.$$

The same techniques also give the relative projection constant of $\ker\{1, \dots, 1\}$ in ℓ_p^n , that is,

$$\lambda(\ker\{1, \dots, 1\}, \ell_p^n) = \frac{((n-1)^{p/q} + 1)^{1/p}((n-1)^{q/p} + 1)^{1/q}}{n},$$

for $n = 2, 3, 4$. We discuss the relation of the above inequalities to the famous Khintchine and Clarkson inequalities. We conclude the paper by stating some conjectures that involve the geometry of the unit ball of ℓ_p^n .

1. INTRODUCTION

The projection $P : X \rightarrow V$ is called *minimal* if it has the smallest possible norm, that is, if

$$(1.1) \quad \|P\| = \lambda(V, X) = \inf\{\|Q\| : Q : X \rightarrow V \text{ and } Q \text{ is a projection}\}.$$

Observe that any projection with norm one is automatically minimal though, in general, a given subspace will not be the range of a projection of norm 1. The first problem is to find out whether a considered subspace is complemented, but even if that is the case, there could be no minimal projection (as the inf above does not have to be attained); for example, see [3]. In many cases the existence of a minimal projection is known a priori (see [21, 11]), which is the case when the subspace is finite-dimensional or finite-codimensional. Even in such cases, minimal projections will be difficult to find. As a result of the dramatic evidence on the difficult nature of such problems, one may cite the fact that the minimal projections of $\mathcal{C}([0, 1])$ onto the subspace of polynomials of degree > 2 remain unknown. Let us mention several papers that characterize minimal projections in general settings [21, 6, 5, 17, 22]. Still rarer is the situation in which the minimal projection is known to be unique. Even in L_p ($p \neq 2$) spaces there are examples of minimal projections that are not unique (see [24]).

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In two important cases we know minimal projections: the Fourier projection F_n and the Rademacher projection R_n are minimal in L_p [2, 18, 5]. The uniqueness of the Fourier projection in $C[0, 1]$ and $L_1[0, 1]$ has been settled in [7, 14]. In L_p ($p \neq 2$) the problem is open; the special case has been proved in [25].

Now we will discuss Rademacher projection. We will define the Rademacher functions, r_0, r_1, \dots by $r_j(t) = (-1)^{[2^j t]}$ for $0 \leq t \leq 1$, where $[\cdot]$ denotes the integer part of the argument. Since $r_0 \equiv 1$ many authors consider only $r_i, i \geq 1$ (r_i therefore being symmetric independent random variables). This is only a choice of convenience since

$$(1.2) \quad \|a_0 r_0 + a_1 r_1 + \dots + a_{n-1} r_{n-1}\|_p = \frac{1}{2} \|a_0 r_1 + a_1 r_2 + \dots + a_{n-1} r_n\|_p.$$

Put $Rad_n = \text{span}\{r_1, \dots, r_n\}$ and $\widetilde{Rad}_n = \text{span}\{r_0, \dots, r_{n-1}\}$. The Rademacher projection R_n (or \widetilde{R}_n) is an orthogonal projection onto Rad_n (\widetilde{Rad}_n correspondingly). Both spaces Rad_n and \widetilde{Rad}_n are symmetric. As a result (see [16]) both Rademacher projections are unique minimal in $L_p[0, 1]$ ($1 < p < \infty$). In $L_1[0, 1]$ and $L_\infty[0, 1]$, however, they are not unique minimal projections ([5, 26]).

The study of Rademacher functions came from the Khintchine inequality

$$(1.3) \quad A_p \left\| \sum_{i=1}^n a_i r_i \right\|_2 \leq \left\| \sum_{i=1}^n a_i r_i \right\|_p \leq B_p \left\| \sum_{i=1}^n a_i r_i \right\|_2.$$

One can easily compute that

$$(1.4) \quad \left\| \sum_{i=1}^n a_i r_i \right\|_p = (\text{Average} \{ \left| \sum_{i=1}^n \epsilon_i a_i \right|^p : \epsilon_i = \pm 1 \})^{1/p} = (E(\sum_{i=1}^n a_i r_i)^p)^{1/p}$$

and

$$(1.5) \quad \left\| \sum_{i=1}^n a_i r_i \right\|_2 = \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

The problem of finding the best possible constants appearing in (1.3) has a long history and has finally been solved in ([10]). There are also generalizations of the Khintchine inequality to Banach spaces. The Rademacher projection can also be generalized and its norm has been compared to the Banach-Mazur distance (see [23] for exact formulations and more information). In [4] the norm of the Rademacher projection $R_2 : L_p \rightarrow L_q$ has been computed. Paper [20] contains many interesting formulas involving integrals of Rademacher functions.

Unfortunately, R_3 may be the only Rademacher projection with a norm that can be succinctly expressed, though the author believes that, even in the general case, there could be an interesting formula to discover.

All of the above observations are also related to the relative projection constant of $\ker(1, \dots, 1)$ in ℓ_p^n . This is especially interesting due to the conjecture that $\ker(1, \dots, 1)$ is the worst complemented hyperplane in ℓ_p^n ($1 < p < \infty$). That is,

Conjecture 1.1. For any $f \in \ell_q^n$, $\lambda(\ker f, \ell_p^n) \leq \lambda(\ker(1, \dots, 1), \ell_p^n)$.

It is worth mentioning that for $p = 1$ and $p = \infty$ the relative projection constants $\lambda(\ker f, \ell_1^n)$ and $\lambda(\ker f, \ell_\infty^n)$ have been computed (see [3, 22]).

2. RESULTS

The Rademacher projection is defined by $R_n = \sum_{i=1}^n r_i \otimes r_i$ and $\tilde{R}_n = \sum_{i=0}^{n-1} r_i \otimes r_i$. We can write the above projections as

$$(2.1) \quad R_n(f) = \sum_{i=1}^n \left(\int_0^1 r_i(t)f(t) dt \right) r_i \quad \text{and} \quad \tilde{R}_n(f) = \sum_{i=0}^{n-1} \left(\int_0^1 r_i(t)f(t) dt \right) r_i.$$

Our main goal is to find $\|R_3\|_p$ (we will see that actually $\|R_3\|_p = \|\tilde{R}_3\|_p$).

Let S_{2^n} denote the space with a basis of simple functions of intervals $I_{l,2^k} = (\frac{l}{2^n}, \frac{l+1}{2^n})$, where $l \in \{0, 1, \dots, 2^n - 1\}$.

Theorem 2.1. *The L_p norm of \tilde{R}_n is attained on $S_{2^{n-1}}$, and the L_p norm of R_n is attained on S_{2^n} .*

Proof. We will prove the above for \tilde{R}_n ; for R_n the proof is the same. We have $\widetilde{Rad}_n \subset S_{2^{n-1}}$. Consider the projection

$$(2.2) \quad T_n = \tilde{R}_n / S_{2^{n-1}} : S_{2^{n-1}} \rightarrow \widetilde{Rad}_n.$$

$\|T_n\|_p \leq \|\tilde{R}_n\|_p$. To prove the inequality in the opposite direction, consider the norm-one projection $L_{n-1} : L_p[0, 1] \rightarrow S_{2^{n-1}}$ given by

$$(2.3) \quad L_{n-1}(f) = \sum_{l=0}^{2^{n-1}-1} 2^{n-1} \left(\int_{I_{l,2^{n-1}}} f(x) dx \right) \chi_{I_{l,2^{n-1}}}.$$

Furthermore, observe that $L_{n-1} / S_{2^{n-1}} = Id$, and since $r_i \in S_{2^{n-1}}$, we also have

$$(2.4) \quad r_i(f) = r_i(L_{n-1}f).$$

Taking a norming point for \tilde{R}_n , we have

$$(2.5) \quad \begin{aligned} \|\tilde{R}_n\|_p &= \|\tilde{R}_n(f)\|_p = \left\| \sum_{i=0}^{n-1} r_i(f)r_i \right\|_p = \left\| \sum_{i=0}^{n-1} r_i(L_{n-1}f)r_i \right\|_p \\ &= \|\tilde{R}_n(L_{n-1}f)\|_p = \|T_n(L_{n-1}f)\|_p \leq \|T_n\|_p \cdot \|L_{n-1}\|_p \leq \|T_n\|_p. \end{aligned}$$

Hence, $\|\tilde{R}_n\|_p = \|T_n\|_p$. □

The following is easy to observe.

Remark 2.2. $\|R_2\|_p = \|\tilde{R}_2\|_p = 1$.

Our goal is to prove

Theorem 2.3. *Consider $\tilde{R}_3 : L_p[0, 1] \rightarrow \{r_0, r_1, r_2\}$. Then*

$$(2.6) \quad \|\tilde{R}_3\|_p = \frac{(3^{p/q} + 1)^{1/p} (3^{q/p} + 1)^{1/q}}{4}.$$

Proof. Since $(\tilde{R}_n)^* = \tilde{R}_n$ and $\|(\tilde{R}_n)^*\| = \|\tilde{R}_n\|$, we can assume that $p > 2$. Put $w := r_1 r_2$. Then r_0, r_1, r_2, w is a basis for S_4 . Let $f = a_0 r_0 + a_1 r_1 + a_2 r_2 + a_3 w$. Then it is easy to see that finding the norm of R_3 is equivalent to finding the best constant in the following inequality:

$$(2.7) \quad \begin{aligned} &|a_o + a_1 + a_2|^p + |a_o + a_1 - a_2|^p + |a_o - a_1 + a_2|^p + |a_o - a_1 - a_2|^p \\ &\leq C(|a_o + a_1 + a_2 + a_3|^p + |a_o + a_1 - a_2 - a_3|^p \\ &\quad + |a_o - a_1 + a_2 - a_3|^p + |a_o - a_1 - a_2 + a_3|^p), \end{aligned}$$

where $C^{1/p} = \|R_3\|_p$. Making substitutions, $x_1 = a_0 + a_1 + a_2$, $x_2 = -a_0 - a_1 + a_2$, $x_3 = -a_0 + a_1 - a_2$, $x_4 = a_0 - a_1 - a_2$ and $d = a_3$, the above problem is reduced to

$$(2.8) \quad C = \max\left\{\sum_{i=1}^4 |x_i|^p : \sum_{i=1}^4 |x_i + d|^p = 1 \text{ and } \sum_{i=1}^4 x_i = 0\right\},$$

or equivalently,

$$(2.9) \quad \lambda = \frac{1}{C} = \min\left\{\sum_{i=1}^4 |x_i + d|^p : \sum_{i=1}^4 |x_i|^p = 1 \text{ and } \sum_{i=1}^4 x_i = 0\right\}.$$

We will write the above in the quotient form

$$(2.10) \quad \lambda = \frac{1}{C} = \min\left\{\frac{\sum_{i=1}^4 |x_i + d|^p}{\sum_{i=1}^4 |x_i|^p} : \sum_{i=1}^4 x_i = 0\right\} = \min\left\{\frac{\sum_{i=1}^4 |x_i + 1|^p}{\sum_{i=1}^4 |x_i|^p} : \sum_{i=1}^4 x_i = 0\right\}.$$

Assume the above minimum is attained at $x = (x_1, x_2, x_3, x_4)$, where $\sum_{i=1}^4 x_i = 0$. Observe that $x \neq 0$, because if $x \rightarrow 0$, then $\frac{\sum_{i=1}^4 |x_i + 1|^p}{\sum_{i=1}^4 |x_i|^p} \rightarrow \infty$. Take any $v = (v_1, v_2, v_3, v_4)$ where $\sum_{i=1}^4 v_i = 0$ and consider the function

$$(2.11) \quad g(\epsilon) = \frac{\sum_{i=1}^4 |x_i + \epsilon v_i + 1|^p}{\sum_{i=1}^4 |x_i + \epsilon v_i|^p}.$$

Since $p > 2$ and $x = (x_1, x_2, x_3, x_4) \neq 0$, g is a well-defined C^1 function around 0. The function g has a minimum at 0. As a result, $g'(0) = 0$ and $g''(0) \geq 0$. From now on we will use the notation

$$(2.12) \quad \langle x \rangle^p = |x|^p \text{sign}(x).$$

Observe that $g = \frac{T}{S}$; therefore $gS = T$ and $g'S + gS' = T'$. Using $g(0) = \lambda$ we obtain

$$(2.13) \quad g'(0) = \frac{p \sum_{i=1}^4 v_i (\langle x_i + 1 \rangle^{p-1} - \lambda \langle x_i \rangle^{p-1})}{\sum_{i=1}^4 |x_i|^p} = 0.$$

Taking a second derivative we get $g''S + 2g'S' + gS'' = T''$. Using $g(0) = \lambda$ and $g'(0) = 0$ we get

$$(2.14) \quad g''(0) = \frac{p(p-1) \sum_{i=1}^4 v_i^2 (|x_i + 1|^{p-2} - \lambda |x_i|^{p-2})}{\sum_{i=1}^4 |x_i|^p} \geq 0.$$

Consider the following function:

$$(2.15) \quad h(t) = \langle t + 1 \rangle^{p-1} - \lambda \langle t \rangle^{p-1}.$$

Putting $v = (1, -1, 0, 0)$, $v = (1, 0, -1, 0)$ and $v = (1, 0, 0, -1)$ in (2.13) gives us that for all i, j ,

$$(2.16) \quad h(x_i) = h(x_j) = A.$$

Observe that $\lambda < 1$. Taking the derivative of the function h , we can easily see that it has only two extreme points, a maximum at $t_0 = -\frac{1}{1-\lambda^{1/p-2}}$ and a minimum at $t_1 = -\frac{1}{1+\lambda^{1/p-2}}$. As a result, $h(t) = A$ can have a maximum of 3 different solutions. Therefore only 3 numbers among the x_i can be different. We will prove that, in fact, only 2 numbers among the x_i can be different.

Assume for a contradiction that 3 numbers among the x_i are different. Since $\sum_{i=1}^4 x_i = 0$ at least one of $x_i > 0$. Thus, $h(t) = A$ has to have a positive solution. Observe now that $h(-1) = \lambda < 1 = h(0)$; therefore all negative solutions to $h(t) = A$ have to be less than -1 . As a result, all the solutions of $h(t) = A$ have to be of the form

$$(2.17) \quad -\alpha, -\beta, \gamma, \quad \text{where } \alpha > \beta > 1, \gamma > 0.$$

Observe that, until now, we have only used the first derivative of g . The above conditions guarantee us that g is actually a C^∞ function. There could be only one $-\beta$ among the x_i (otherwise, since $h'(-\beta) < 0, g''(0) < 0$, a contradiction to (2.14)). The possible choices for x_1, x_2, x_3, x_4 are now $-\alpha, -\alpha, -\beta, \gamma$ or $-\alpha, -\beta, \gamma, \gamma$. But since $\sum_{i=1}^4 x_i = 0$ in the first case, we have $\gamma = 2\alpha + \beta$, and secondly $\gamma = (\alpha + \beta)/2$. In both cases

$$(2.18) \quad \gamma \geq \beta.$$

Using (2.17), the equation $h(\gamma) = h(-\beta)$ gives

$$(2.19) \quad \frac{(\gamma + 1)^{p-1} + (\beta - 1)^{p-1}}{\gamma^{p-1} + \beta^{p-1}} = \lambda < 1.$$

On the other hand, since $\gamma \geq \beta$,

$$(2.20) \quad (\gamma + 1)^{p-1} - \gamma^{p-1} \geq (p - 1)\gamma^{p-2} \geq (p - 1)\beta^{p-2} \geq \beta^{p-1} - (\beta - 1)^{p-1}.$$

As a result,

$$(2.21) \quad \frac{(\gamma + 1)^{p-1} + (\beta - 1)^{p-1}}{\gamma^{p-1} + \beta^{p-1}} \geq 1,$$

contrary to (2.19). Therefore we can have only 2 different numbers among the x_i , giving the possibilities $-a, -a, b, b$ and $-a, b, b, b$ ($a > 1, b > 0$). The first one is, in fact, not possible since it would lead to $b = a$, a contradiction. Hence x_1, x_2, x_3, x_4 has to be $-3b, b, b, b$ ($b > 1/3$). Plugging it into (2.10) and using $s = 1/b$ we get

$$(2.22) \quad \lambda = \frac{1}{C} = \min_{b > 1/3} \left\{ \frac{3|b + 1|^p + |3b - 1|^p}{3|b|^p + 3^p|b|^p} \right\} = \frac{1}{3 + 3^p} \min_{0 < s < 3} \{3(1 + s)^p + (3 - s)^p\}.$$

Standard calculations yield that the minimum is attained for $s = \frac{3 - 3^{q/p}}{1 + 3^{q/p}}$ and

$$(2.23) \quad C = \frac{(3^{p/q} + 1)(3^{q/p} + 1)^{p/q}}{4^p}.$$

□

Theorem 2.4. $\|R_3\|_p = \|\tilde{R}_3\|_p$.

Proof. Using the convexity of a function $f(x) = |x|^p$ we have

$$(2.24) \quad \begin{aligned} \frac{|a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7|^p + |-a_0 + a_1 + a_2 - a_3 + a_4 - a_5 - a_6 + a_7|^p}{2} &\geq |a_1 + a_2 + a_4 + a_7|^p, \\ \frac{|a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7|^p + |-a_0 + a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7|^p}{2} &\geq |a_1 + a_2 - a_4 - a_7|^p, \\ \frac{|a_0 + a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7|^p + |-a_0 + a_1 - a_2 + a_3 + a_4 - a_5 + a_6 - a_7|^p}{2} &\geq |a_1 - a_2 + a_4 - a_7|^p, \\ \frac{|a_0 + a_1 - a_2 - a_3 - a_4 - a_5 + a_6 + a_7|^p + |-a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7|^p}{2} &\geq |a_1 - a_2 - a_4 + a_7|^p. \end{aligned}$$

Using (1.2) and Theorem 2.3 we get

$$(2.25) \quad \|a_1 r_1 + a_2 r_2 + a_4 r_4\|_p = 2\|a_1 r_0 + a_2 r_1 + a_3 r_2\|_p \leq 2\|\tilde{R}_3\|_p \|a_1 r_0 + a_2 r_1 + a_3 r_2 + a_7 w\|_p,$$

and applying equations (2.24), we get

$$(2.26) \quad \|R_3\|_p \leq \|\tilde{R}_3\|_p.$$

But since the above inequalities will become equalities when $a_3 = a_5 = a_6 = 0$, we will have equality. \square

From the proof of the above Theorem 2.3 we can deduce that

Corollary 2.5. *For any $a_0, a_1, a_2, a_3 \in \mathbb{R}$,*

$$(2.27) \quad \begin{aligned} & |a_0 + a_1 + a_2|^p + |a_0 + a_1 - a_2|^p + |a_0 - a_1 + a_2|^p + |a_0 - a_1 - a_2|^p \\ & \leq C_p (|a_0 + a_1 + a_2 + a_3|^p + |a_0 + a_1 - a_2 - a_3|^p \\ & \quad + |a_0 - a_1 + a_2 - a_3|^p + |a_0 - a_1 - a_2 + a_3|^p), \end{aligned}$$

where $C_p = \frac{(3^{p/q} + 1)(3^{q/p} + 1)^{p/q}}{4^p}$ is the best possible constant.

There are several inequalities known that contain the left part of the above inequality. The following inequality was used to obtain important results on sums of independent random variables.

Theorem 2.6 ([15], Lemma 5). *For any $a_0, a_1, a_2 \in \mathbb{R}$,*

$$(2.28) \quad \begin{aligned} & |a_0 + a_1 + a_2|^p + |a_0 + a_1 - a_2|^p + |a_0 - a_1 + a_2|^p + |a_0 - a_1 - a_2|^p \\ & \leq \frac{(|a_0 + a_1|^p + |a_0 - a_1|^p)(|a_0 + a_2|^p + |a_0 - a_2|^p)}{|a_0|^p}. \end{aligned}$$

The left side of (2.27) can be viewed as

$$(2.29) \quad 4 * \text{Average}\{|\epsilon_0 a_0 + \epsilon_1 a_1 + \epsilon_2 a_2|^p : \epsilon_i = \pm 1\};$$

therefore it is related to Khintchine-type Inequalities. The following inequality can be deduced from it.

Theorem 2.7 ([9], Proposition 2.1; [13], Corollary 3). *For any $a_0, a_1, a_2 \in \mathbb{R}$ and $p > 2$,*

$$(2.30) \quad \begin{aligned} & |a_0 + a_1 + a_2|^p + |a_0 + a_1 - a_2|^p + |a_0 - a_1 + a_2|^p + |a_0 - a_1 - a_2|^p \\ & \leq (3^{p/q} + 1)(|a_0|^p + |a_1|^p + |a_2|^p). \end{aligned}$$

The above results are also connected to the famous Clarkson Inequalities (see [19, 12] for generalizations).

Theorem 2.8 (Clarkson Inequality [8]). *For any $a_0, a_1 \in \mathbb{R}$ and $p > 2$,*

$$(2.31) \quad |a_0 + a_1|^p + |a_0 - a_1|^p \leq 2^{p/q}(|a_0|^p + |a_1|^p).$$

We will now compute the relative projection constants of $\ker(1, \dots, 1)$ in ℓ_p^n . We will need the following.

Theorem 2.9. *The following holds:*

$$(2.32) \quad \min\left\{\frac{\sum_{i=1}^n |a_i + d|^p}{\sum_{i=1}^n |a_i|^p} : \sum_{i=1}^n a_i = 0\right\} = \frac{n^p}{((n-1)^{p/q} + 1)((n-1)^{q/p} + 1)^{p/q}}.$$

Proof. The case $n = 2$ is trivial, and the case $n = 4$ follows from Theorem 2.3. If $n = 3$, the proof goes the same way as in Theorem 2.3, since $x_1 + x_2 + x_3 = 0$ gives $\alpha + \beta + \gamma = 0$ and then $\gamma = \alpha + \beta > \beta$ and the rest follows from the same reasoning as in the $n = 4$ case. The general case can be handled in a similar way as in $n = 4$. The proof becomes extremely tedious and depends on the consideration of many cases. The author does not know a satisfactory solution in the general case. \square

The following is an easy consequence of the above.

Theorem 2.10. *Consider $P = Id - \frac{1}{n}1 \otimes 1 : \ell_p^n \rightarrow \ker 1$. Then for $n = 2, 3, 4$ and $1 \leq p \leq \infty$,*

$$(2.33) \quad \|P\|_p = \frac{((n-1)^{p/q} + 1)^{1/p}((n-1)^{q/p} + 1)^{1/q}}{n}.$$

Moreover, for $1 < p < \infty, p \neq 2$, if $x \in S(\ell_p^n)$ is a norming point for P , then $x = (x_1, \dots, x_n) = \pm(u_{\pi(1)}, \dots, u_{\pi(n)})$, where π is any permutation and

$$(2.34) \quad u = \frac{1}{((n-1)+(n-1)^q)^{1/p}}(-(n-1)^{q/p}, 1, \dots, 1).$$

Proof. We have

$$(2.35) \quad Px = x - \frac{1}{n}\left(\sum_{i=1}^n x_i\right)(1, \dots, 1).$$

Taking $a_i = x_i - \frac{1}{n}(\sum_{i=1}^n x_i)$ and $d = \frac{1}{n}(\sum_{i=1}^n x_i)$ and applying Theorem 2.9, we get the result. \square

The above projection is minimal [16]. Interestingly, because of the form of the norming points of P , we can see that it is not a minimal projection with respect to the numerical radius (see [1] for details). As a result of the above theorem, we automatically get

Theorem 2.11. *For $n = 2, 3, 4$ and $1 \leq p \leq \infty$, the relative projection constant is*

$$(2.36) \quad \lambda(\ker(1, 1, \dots, 1), \ell_p^n) = \frac{((n-1)^{p/q} + 1)^{1/p}((n-1)^{q/p} + 1)^{1/q}}{n}.$$

We will conclude this article by stating two conjectures that relate to the geometry of the ℓ_p^n unit ball.

Conjecture 2.12. *Assume $n \geq 3$ and $1 < p < \infty, p \neq 2$. Let S be a unit sphere in ℓ_p^n and let S_d be the shift of S by the vector (d, \dots, d) . Consider the ring $R = S \cap \ker(1, \dots, 1)$. Now we can shrink or enlarge S_d in such a way that it will be tangent to R and R will be outside of S_d . Then the points where R and $c \cdot S_d$ meet have exactly $n - 1$ coordinates equal. As a result we have exactly $2n$ points of tangency of $c \cdot S_d$ and R .*

The above conjecture is equivalent to showing that

$$(2.37) \quad \min\left\{\sum_{i=1}^n |x_i + d|^p : \sum_{i=1}^n |x_i|^p = 1 \text{ and } \sum_{i=1}^n x_i = 0\right\}$$

is attained at points that have exactly $n - 1$ coordinates equal. This is also equivalent to Theorem 2.9 (we gave a satisfactory proof of it for $n = 3, 4$).

Conjecture 2.13. Assume $n \geq 3$ and $1 < p < \infty, p \neq 2$. Let S be a unit sphere in ℓ_p^n and let S_d be the shift of S by the vector (d, \dots, d) . Consider the ring $R = S \cap \ker(1, \dots, 1)$. Now we can shrink or enlarge S_d in such a way that it will be tangent to R and R will be inside of S_d . Then the coordinates of the points where R and $c \cdot S_d$ meet are such that $n - 2$ of them are equal and also the remaining 2 are equal to each other. As a result we have exactly $n(n - 1)$ points of tangency of $c \cdot S_d$ and R .

The above conjecture is equivalent to showing that

$$(2.38) \quad \max \left\{ \sum_{i=1}^n |x_i + d|^p : \sum_{i=1}^n |x_i|^p = 1 \text{ and } \sum_{i=1}^n x_i = 0 \right\}$$

is attained at points in which the coordinates are such that $n - 2$ of them are equal and also the remaining 2 are equal to each other.

The dilemma in (2.37) and (2.38) is that both functions have several local minima and maxima (for instance, numerical calculations show that (2.38) has a local maximum with three different numbers among x_1, \dots, x_n), so one has to be very careful when trying to find any numerical solutions to the above.

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