

**A VARIATION OF MULTIPLE L -VALUES
ARISING FROM THE SPECTRAL ZETA FUNCTION
OF THE NON-COMMUTATIVE HARMONIC OSCILLATOR**

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ABSTRACT. A variation of multiple L -values, which arises from the description of the special values of the spectral zeta function of the non-commutative harmonic oscillator, is introduced. In some special cases, we show that its generating function can be written in terms of the gamma functions. This result enables us to obtain explicit evaluations of them.

1. INTRODUCTION

The multiple zeta values

$$(1.1) \quad \zeta_k^\bullet(n_1, \dots, n_k) := \sum_{1 \leq i_1 < \dots < i_k} \frac{1}{i_1^{n_1} i_2^{n_2} \dots i_k^{n_k}}$$

are natural extensions of the Riemann zeta value $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$ introduced by Euler and have been of continuing interest to many mathematicians [18]. Recently, it has been shown by several authors that they appear in various fields in mathematics such as knot invariant theory, quantum group theory and mathematical physics (see, e.g., [11, 20]). This fact implies the richness of the theory of multiple zeta values and has encouraged recent studies of them. One of the main problems in studying multiple zeta values is to clarify the \mathbb{Q} -algebra structure of the space spanned by them, which is closely related to that of the category of mixed Tate motives. In fact, for this purpose, many results concerning relations among them and exact calculations of them are investigated. Furthermore, as a natural

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generalization, Arakawa and Kaneko [2] introduce two kinds of multiple L -values:

$$(1.2) \quad L_{\text{III}}(n_1, \dots, n_k; f_1, \dots, f_k) := \sum_{m_1 > \dots > m_k > 0} \frac{f_1(m_1 - m_2) \dots f_{k-1}(m_{k-1} - m_k) f_k(m_k)}{m_1^{n_1} m_2^{n_2} \dots m_k^{n_k}},$$

$$(1.3) \quad L_*(n_1, \dots, n_k; f_1, \dots, f_k) := \sum_{m_1 > \dots > m_k > 0} \frac{f_1(m_1) f_2(m_2) \dots f_k(m_k)}{m_1^{n_1} m_2^{n_2} \dots m_k^{n_k}},$$

where f_1, \dots, f_k are \mathbb{C} -valued periodic functions on \mathbb{Z} , and they also study their relations and exact evaluations.

In this paper, we study the following variation $S_k^{(N,M)}(n_1, \dots, n_k)$ ($N, M \in \mathbb{N}$) of the multiple L -values:

$$(1.4) \quad S_k^{(N,M)}(n_1, \dots, n_k) := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k} \varepsilon_{i_1 i_2 \dots i_k}^{(N)} \frac{\omega_M^{i_1 + i_2 + \dots + i_k}}{i_1^{n_1} i_2^{n_2} \dots i_k^{n_k}},$$

where ω_M is a primitive M th root of unity and

$$(1.5) \quad \varepsilon_{ij}^{(N)} := \begin{cases} 0 & i = j \not\equiv 0 \pmod{N} \\ 1 & \text{otherwise} \end{cases} = 1 - \delta_{ij} \left(1 - \frac{1}{N} \sum_{r=0}^{N-1} \omega_N^{ri} \right),$$

$$\varepsilon_{i_1 i_2 \dots i_k}^{(N)} := \prod_{j=1}^{k-1} \varepsilon_{i_j i_{j+1}}^{(N)}.$$

Here δ_{ij} is the Kronecker delta. For simplicity, we sometimes write

$$S_k^{(N)}(n_1, \dots, n_k) = S_k^{(N,N)}(n_1, \dots, n_k),$$

$$S_k^{(N,M)}(n) = S_k^{(N,M)}(n, \dots, n),$$

$$S_k^{(N)}(n) = S_k^{(N)}(n, \dots, n).$$

We note that $S_1^{(N,M)}(n) = Li_n(\omega_M)$, where $Li_n(z) := \sum_{i=1}^{\infty} z^i / i^n$ is the polylogarithm.

The aim of the paper is to establish generating function formulas for the series $S_k^{(N,M)}(n)$, and give an explicit evaluation of them in terms of Bernoulli numbers in the special case where $N = M = 2$ and n is even. It is quite remarkable that the values $S_k^{(2)}(n)$ can be fully computable; in fact, there are few examples of computable multiple L -values. In this sense, $S_k^{(N)}(n)$ seems to be a nice variant of the ordinary multiple L -values.

We will sometimes refer to $S_k^{(N,M)}(n_1, \dots, n_k)$ as a *partial multiple L -value* because it is indeed a partial sum of the “non-strict” multiple L -value

$$\sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k} \frac{\omega_M^{i_1 + i_2 + \dots + i_k}}{i_1^{n_1} i_2^{n_2} \dots i_k^{n_k}} = S_k^{(1,M)}(n_1, n_2, \dots, n_k).$$

In particular, $S_k^{(1)}(n_1, n_2, \dots, n_k)$ gives the non-strict multiple zeta value (see, e.g., [12]). It is also worth remarking that $\varepsilon_{ij}^{(N)} \rightarrow 1 - \delta_{ij}$ as $N \rightarrow \infty$ for fixed indices i, j , so that we may regard the (strict) multiple L -values (1.3) as “limiting cases” $S^{(\infty,M)}(n_1, n_2, \dots, n_k)$ of our series. We notice that our partial multiple L -value

$S_k^{(N,M)}(n_1, \dots, n_k)$ is a special case of *neither* the multiple L -values (1.2) nor (1.3) since $\varepsilon_{i_1 i_2 \dots i_k}^{(N)}$ does depend on *both* the differences $i_j - i_{j-1}$ of adjacent indices and the values of the indices i_1, \dots, i_k themselves. However, it is not difficult to see that $S_k^{(N,M)}(n_1, \dots, n_k)$ can be expressed as a \mathbb{Q} -linear combination of (1.2) (or (1.3)). Thus, for fixed N and M , it may be interesting to study the structure of the subalgebra spanned by all $S_k^{(N,M)}(n_1, n_2, \dots, n_k)$ in the \mathbb{Q} -algebra spanned by all multiple L -values $S_k^{(1,M)}(n_1, \dots, n_k)$. We leave these problems to future study.

We now explain the spectral-theoretic origin of our series $S_k^{(N,M)}(n_1, \dots, n_k)$. A system of differential equations defined by the operator

$$Q := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(x \frac{d}{dx} + \frac{1}{2} \right)$$

having two real parameters α, β is called the *non-commutative harmonic oscillator*. This system was first introduced and extensively studied by Parmeggiani and Wakayama [16, 17] (see also [15]). It is shown that when $\alpha, \beta > 0$ and $\alpha\beta > 1$, Q defines a positive, self-adjoint operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ which has only a discrete spectrum $(0 <) \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots (\nearrow +\infty)$, and the multiplicities of the eigenvalues are uniformly bounded. In order to describe the total behavior of the spectrum, Ichinose and Wakayama [6] studied the spectral zeta function $\zeta_Q(s) := \sum_{n=1}^{\infty} \lambda_n^{-s}$, which is absolutely convergent if $\Re(s) > 1$. This is analytically continued to the whole plane \mathbb{C} and gives a single-valued meromorphic function which has a simple pole at $s = 1$ and ‘trivial’ zeros at non-positive even integers. If $\alpha = \beta = 1/\sqrt{2}$, then Q is unitarily equivalent to a couple of the (ordinary) harmonic oscillators, from which it follows that $\zeta_Q(s) = 2(2^s - 1)\zeta(s)$. Thus one can regard $\zeta_Q(s)$ as a deformation of the Riemann zeta function $\zeta(s)$.

In describing the special values of the spectral zeta function $\zeta_Q(s)$, the integrals

$$J_m(n) = 2^m \int_0^1 \dots \int_0^1 \left(\frac{(1-x_1^4)(1-x_2^4) \dots (1-x_m^4)}{(1-x_1^2) \dots (1-x_m^2)} \right)^n \frac{dx_1 \dots dx_m}{1-x_1^2 \dots x_m^2} \quad (m = 2, 3, 4, \dots; n = 0, 1, 2, \dots)$$

and their generating functions $g_m(x) = \sum_{n=0}^{\infty} \binom{-1/2}{n} J_m(n) x^n$ play a very important role. In fact, Ichinose and Wakayama [7] calculated the first two special values $\zeta_Q(2)$ and $\zeta_Q(3)$ in terms of $g_2(x)$ and $g_3(x)$, respectively. The higher special values $\zeta_Q(m)$ ($m \geq 4$) are also expected to be expressed by $g_m(x)$ and their generalizations (see, e.g., [13, 9, 8]). In the case where $m = 2r$ is even, $J_{2r}(n)$ is explicitly given by

$$J_{2r}(n) = \sum_{p=0}^n (-1)^p \binom{-\frac{1}{2}}{p}^2 \binom{n}{p} \sum_{k=0}^{r-1} \zeta\left(2r - 2k, \frac{1}{2}\right) S_{k,p},$$

where $\zeta(s, x) := \sum_{n=0}^{\infty} (n+x)^{-s}$ is the Hurwitz zeta function and

$$S_{k,p} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq 2p} \varepsilon_{i_1 i_2 \dots i_k}^{(2)} \frac{(-1)^{i_1 + i_2 + \dots + i_k}}{i_1^2 i_2^2 \dots i_k^2}.$$

Now it is immediate to see that our series $S_k^{(N,M)}(n_1, \dots, n_k)$ is a natural generalization of $S_k^{(2)}(2) = \lim_{p \rightarrow \infty} S_{k,p}$ (we give the explicit formula of $S_k^{(2)}(2)$ in Example 3.4).

It is also worth remarking that another kind of generating function $w_2(t) = \sum_{n=0}^{\infty} J_2(n)t^n$ of $J_2(n)$ is regarded as a period integral for the universal family of the elliptic curves equipped with a rational point of order 4 and that it satisfies a Picard-Fuchs differential equation attached to this family of curves [10].

Conventions. We recall several basic conventions on partitions and symmetric functions (for further details, see [5]).

A partition is a weakly decreasing sequence of non-negative integers which has finitely many non-zero entries. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ ($\lambda_l \geq 1$), the sum $\lambda_1 + \dots + \lambda_l$ of entries in λ is denoted by $|\lambda|$ and the number l of non-zero entries in λ is denoted by $\ell(\lambda)$. We write $\lambda \vdash k$ to imply $|\lambda| = k$, and say λ is a partition of k . We denote by \emptyset the (only) partition of 0. To indicate a multiple of the same numbers in λ , we often write in an exponential form; Let $m_i = m_i(\lambda)$ be the number of i 's in λ . We call $m_i(\lambda)$ the multiplicity of i in λ . Then, we also write $\lambda = (k^{m_k}, \dots, 2^{m_2}, 1^{m_1})$ or $\lambda = 1^{m_1} 2^{m_2} \dots k^{m_k}$. For instance, $\lambda = (4, 2, 2, 1, 1, 1)$ is also written as $\lambda = (4, 2^2, 1^3) = 1^3 2^2 4^1$. When all the entries of λ are even, we call λ an even partition. For a given partition $\mu = (\mu_1, \dots, \mu_l)$ and a positive integer q , we define $q\mu = (q\mu_1, \dots, q\mu_l)$. We notice that $\{\lambda \vdash 2k \mid \lambda: \text{even}\} = \{2\mu \mid \mu \vdash k\}$. If a given pair of two partitions λ and μ satisfies that $\lambda_i - \mu_i = 0$ or 1 for any index i , then we say λ/μ is a vertical strip. For instance, $(4, 2, 2, 1, 1, 1)/(3, 2, 1, 1)$ is a vertical strip.

Let $f(n)$ be a function on \mathbb{N} and a_n a sequence. Then, for a partition λ and $q \in \mathbb{N}$, we put $f(q\lambda) := \prod_{j=1}^{\ell(\lambda)} f(q\lambda_j)$ and $a_{q\lambda} := \prod_{j=1}^{\ell(\lambda)} a_{q\lambda_j}$. For instance, $(q\lambda)! = \prod_{j=1}^{\ell(\lambda)} (q\lambda_j)!$.

Let x_1, x_2, \dots be (infinitely many) variables. For each positive integer r , we respectively denote by $e_r = e_r(x_1, x_2, \dots)$ and $h_r = h_r(x_1, x_2, \dots)$ the r -th elementary and r -th complete symmetric function defined by

$$e_r = \sum_{1 \leq i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}, \quad h_r = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \dots x_{i_r}.$$

We also put $e_0 = h_0 = 1$ for convenience. Moreover, for a partition λ , we put $e_\lambda = \prod_{i \geq 1} e_{\lambda_i}$ and $h_\lambda = \prod_{i \geq 1} h_{\lambda_i}$. The generating functions of e_r and h_r are given by

$$(1.6) \quad E(t) = \sum_{r=0}^{\infty} e_r t^r = \prod_{n=1}^{\infty} (1 + x_n t), \quad H(t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{n=1}^{\infty} (1 - x_n t)^{-1}.$$

2. GENERATING FUNCTIONS

In this section, we establish generating function formulas for the series $S_k^{(N,M)}(n)$. To achieve this, we first consider a decomposition of the non-strict multiple sum $S_k^{(N,M)}(n)$ into the sum of several strict multiple sums. Notice that each increasing sequence $1 \leq i_1 \leq i_2 \leq \dots \leq i_k$ of k positive integers uniquely determines a sequence $\mathbf{r} = (r_1, r_2, \dots, r_l)$, which we will refer to as the *multiplicity* of the sequence

(i_1, i_2, \dots, i_k) , through the condition

$$\underbrace{i_1 = \dots = i_{r_1}}_{r_1} < \underbrace{i_{r_1+1} = \dots = i_{r_1+r_2}}_{r_2} < i_{r_1+r_2+1} = \dots \\ = i_{r_1+\dots+r_{l-1}} < \underbrace{i_{r_1+\dots+r_{l-1}+1} = \dots = i_{r_1+\dots+r_l}}_{r_l}.$$

Obviously, \mathbf{r} is a permutation of a certain partition of k . We denote by $\tilde{S}^{(N,M)}(n; \mathbf{r})$ the partial sum of $S^{(N,M)}(n)$ whose running indices have multiplicity \mathbf{r} , i.e.

$$\tilde{S}^{(N,M)}(n; \mathbf{r}) = \sum_{j_1 < \dots < j_l} \varepsilon_{\underbrace{j_1, \dots, j_1}_{r_1}, \dots, \underbrace{j_l, \dots, j_l}_{r_l}}^{(N)} \frac{\omega_M^{r_1 j_1 + \dots + r_l j_l}}{j_1^{nr_1} \dots j_l^{nr_l}} = \sum_{\substack{j_1 < \dots < j_l \\ r_i > 1 \Rightarrow N | j_i}} \frac{\omega_M^{r_1 j_1 + \dots + r_l j_l}}{j_1^{nr_1} \dots j_l^{nr_l}}.$$

We also put

$$S^{(N,M)}(n; \emptyset) = 1, \quad S^{(N,M)}(n; \lambda) = \sum_{\mathbf{r} \in P(\lambda)} \tilde{S}^{(N,M)}(n; \mathbf{r}) \quad (\lambda \neq \emptyset),$$

where $P(\lambda)$ denotes the set consisting of the permutations of a partition λ . It is easy to see that

$$(2.1) \quad S_k^{(N,M)}(n) = \sum_{\lambda \vdash k} S^{(N,M)}(n; \lambda).$$

To study the series $S^{(N,M)}(n; \lambda)$, we here employ another function $R^{(N,M)}(n; \mu)$ defined by

$$R^{(N,M)}(n; \emptyset) := 1, \quad R^{(N,M)}(n; \mu) := S^{(N,M)}(n; \mu_{>1}) S^{(N,M)}(n; 1^{m_1(\mu)}) \quad (\mu \neq \emptyset).$$

Here $\mu_{>1}$ denotes the partition defined by $\mu_{>1} := 2^{m_2(\mu)} 3^{m_3(\mu)} \dots$. Fix a partition $\mu \vdash k$ and put $q = m_1(\mu)$, $p = \ell(\mu) - q$. We easily see that

$$R^{(N,M)}(n; \mu) = \sum_{\mathbf{r} \in P(\mu_{>1})} \tilde{R}_{p,q}^{(N,M)}(n; \mathbf{r}), \\ \tilde{R}_{p,q}^{(N,M)}(n; \mathbf{r}) = \sum_{\substack{s_1 < \dots < s_p \\ t_1 < \dots < t_q \\ N | s_i}} \frac{\omega_M^{r_1 s_1 + \dots + r_p s_p + t_1 + \dots + t_q}}{s_1^{r_1 n} \dots s_p^{r_p n} t_1^n \dots t_q^n}.$$

In the sum $\tilde{R}_{p,q}^{(N,M)}(n; \mathbf{r})$ for each $\mathbf{r} \in P(\mu_{>1})$, several of the running indices t_1, \dots, t_q may coincide with certain s_1, \dots, s_p . To describe the situation, we introduce the following map: Put

$$I(p, q) = \left\{ (\boldsymbol{\tau}, \boldsymbol{\varepsilon}) = (\tau_0, \tau_1, \dots, \tau_p, \varepsilon_1, \dots, \varepsilon_p) \mid \tau_i \in \mathbb{Z}_{\geq 0}, \varepsilon_i \in \{0, 1\}, \sum_{i=0}^p \tau_i + \sum_{i=1}^p \varepsilon_i = q \right\}.$$

For each element $(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) \in P(\mu_{>1}) \times I(p, q)$, we associate a new sequence $\pi_\mu(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}))$ by

$$\pi_\mu(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) = (1^{\tau_0}, r_1 + \varepsilon_1, 1^{\tau_1}, r_2 + \varepsilon_2, \dots, r_p + \varepsilon_p, 1^{\tau_p}).$$

Notice that there exists a partition $\lambda \vdash k$ such that $\lambda/\mu_{>1}$ is a vertical strip and $\pi_\mu(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) \in P(\lambda)$. Namely, the correspondence π_μ defines a map $\pi_\mu : P(\mu_{>1}) \times I(p, q) \rightarrow \prod_{\lambda/\mu_{>1}: \text{vertical strip}}^{\lambda \vdash k} P(\lambda)$. Thus it follows that

$$\sum_{\mathbf{r} \in P(\mu_{>1})} \tilde{R}_{p,q}^{(N,M)}(n; \mathbf{r}) = \sum_{\lambda/\mu_{>1}: \text{vertical strip}}^{\lambda \vdash k} \sum_{\mathbf{r} \in P(\lambda)} |\pi_\mu^{-1}(\mathbf{r})| \tilde{S}^{(N,M)}(n; \mathbf{r}).$$

Since each $|\pi_\mu^{-1}(\mathbf{r})|$ depends *only* on λ , we obtain

$$R^{(N,M)}(n; \mu) = \sum_{\lambda/\mu_{>1}: \text{vertical strip}}^{\lambda \vdash k} |\pi_\mu^{-1}(\lambda)| S^{(N,M)}(n; \lambda).$$

Next, we calculate $|\pi_\mu^{-1}(\lambda)|$. For each $a > 2$, we assume that $\lambda_{i_{a1}} = \dots = \lambda_{i_{a,d(a)}} = a$, where $d(a) = m_a(\lambda)$. Let us count the number of elements $(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}))$ in $I(p, q)$ such that $\pi_\mu(\mathbf{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) = \lambda$. Notice that $\boldsymbol{\tau}$ is uniquely determined by the assumption. If $r_{i_{aj}} + \varepsilon_{i_{aj}} = \lambda_{i_{aj}} = a$, then it is possible that $(r_{i_{aj}}, \varepsilon_{i_{aj}}) = (a, 0)$ or $(a-1, 1)$, and there are exactly $\binom{m_a(\lambda)}{m_a(\lambda; \mu)}$ ways of choosing i_{aj} such that $(r_{i_{aj}}, \varepsilon_{i_{aj}}) = (a, 0)$, where $m_i(\lambda; \mu) = |\{j \mid \lambda_j = \mu_j = i\}|$. (Remark that $m_2(\lambda; \mu) = m_2(\lambda)$.) Thus we have $|\pi_\mu^{-1}(\lambda)| = \prod_{i>2} \binom{m_a(\lambda)}{m_a(\lambda; \mu)}$. If μ is an *even* partition and $\mu/\lambda_{>1}$ is a vertical strip, then $m_i(\lambda; \mu) = m_i(\lambda)$ (if i is even) or 0 (if i is odd) by definition, and hence $|\pi_\mu^{-1}(\lambda)| = 1$. Consequently, we get the following lemma.

Lemma 2.1. *For each $\mu \vdash k$, it follows that*

$$(2.2) \quad R^{(N,M)}(n; \mu) = \sum_{\lambda/\mu_{>1}: \text{vertical strip}}^{\lambda \vdash k} \prod_{i>2} \binom{m_i(\lambda)}{m_i(\lambda; \mu)} S^{(N,M)}(n; \lambda),$$

where $m_i(\lambda; \mu) = |\{j \mid \lambda_j = \mu_j = i\}|$. In particular, if μ is even, then

$$(2.3) \quad R^{(N,M)}(n; \mu) = \sum_{\lambda/\mu_{>1}: \text{vertical strip}}^{\lambda \vdash k} S^{(N,M)}(n; \lambda).$$

□

Lemma 2.2. *For any $\lambda \vdash k$, there uniquely exists $\mu \vdash k$ such that $\mu_{>1}$ is even and $\lambda/\mu_{>1}$ is a vertical strip.*

Proof. It is immediate to see that

$$\mu = 1^{m_1(\lambda)+m_3(\lambda)+m_5(\lambda)+\dots} 2^{m_2(\lambda)+m_3(\lambda)} 4^{m_4(\lambda)+m_5(\lambda)} 6^{m_6(\lambda)+m_7(\lambda)} \dots \vdash k$$

is the unique partition which satisfies all the desired conditions. □

By Lemmas 2.1 and 2.2, we readily obtain the

Lemma 2.3. *Let $U_d^{(N,M)}(n) := \sum_{\mu \vdash d} S^{(N,M)}(n; 2\mu)$. Then it follows that*

$$(2.4) \quad \begin{aligned} S_k^{(N,M)}(n) &= \sum_{\lambda \vdash k} S^{(N,M)}(n; \lambda) = \sum_{\substack{\mu \vdash k \\ \mu_{>1}: \text{even}}} R^{(N,M)}(n; \mu) \\ &= \sum_{0 \leq 2d \leq k} S^{(N,M)}(n; 1^{k-2d}) U_d^{(N,M)}(n). \end{aligned}$$

□

We next study the generating function of $S_k^{(N,M)}(n)$. For this purpose, the following formula, which is obtained by the canonical product expression of the gamma function, is useful.

Lemma 2.4. For $a_i, b_i \in \mathbb{C}$ satisfying $\sum_{i=1}^l a_i = \sum_{i=1}^l b_i$, the equality

$$(2.5) \quad \prod_{m=k}^{\infty} \prod_{j=1}^l \frac{m+a_j}{m+b_j} = \prod_{j=1}^l \frac{\Gamma(k+b_j)}{\Gamma(k+a_j)}$$

holds for any integer k . □

Lemma 2.5. The generating function of $U_d^{(N,M)}(n)$ is given by

$$(2.6) \quad \mathcal{H}^{(N,M)}(n; x) := \sum_{d=0}^{\infty} U_d^{(N,M)}(n) x^{2nd} = \prod_{k=1}^M \prod_{j=0}^{2n-1} \frac{\Gamma\left(\frac{1}{M}\left(k - \frac{1}{N}\omega_{2n}^j \omega_{Mn}^{kN} x\right)\right)}{\Gamma\left(\frac{k}{M}\right)}.$$

Proof. We notice that

$$U_d^{(N,M)}(n) = \sum_{\mu \vdash d} S^{(N)}(n; 2\mu) = h_d\left(\frac{\omega_M^{2N}}{N^{2n}}, \frac{\omega_M^{4N}}{(2N)^{2n}}, \frac{\omega_M^{6N}}{(3N)^{2n}}, \dots\right)$$

since the complete symmetric function h_d is the sum of all monomials of degree d . Therefore, by specializing $x_m = \omega_M^{2mN}/(Nm)^{2n}$ and $t = x^{2n}$ in the generating function $H(t)$ in (1.6), we obtain

$$\begin{aligned} \mathcal{H}^{(N,M)}(n; x) &= \prod_{m=1}^{\infty} \left(1 - \frac{\omega_M^{2mN}}{(Nm)^{2n}} x^{2n}\right)^{-1} = \prod_{m=0}^{\infty} \prod_{k=1}^M \left\{1 - \left(\frac{\omega_{Mn}^{kN} x}{N(Mm+k)}\right)^{2n}\right\}^{-1} \\ &= \prod_{m=0}^{\infty} \prod_{k=1}^M \prod_{j=0}^{2n-1} \left(1 - \omega_{2n}^j \frac{\omega_{Mn}^{kN} x}{N(Mm+k)}\right)^{-1} \\ &= \prod_{m=0}^{\infty} \prod_{k=1}^M \prod_{j=0}^{2n-1} \frac{m + \frac{k}{M} - \frac{\omega_{2n}^j \omega_{Mn}^{kN} x}{MN}}{m + \frac{k}{M}}. \end{aligned}$$

Applying Lemma 2.4 to the equation above, we have (2.6). □

Lemma 2.6. The generating function of $S^{(N,M)}(n; 1^r)$ is given by

$$(2.7) \quad \mathcal{E}^{(M)}(n; x) := \sum_{r=0}^{\infty} S^{(N,M)}(n; 1^r) x^{nr} = \prod_{k=1}^M \prod_{j=0}^{n-1} \frac{\Gamma\left(\frac{k}{M}\right)}{\Gamma\left(\frac{1}{M}\left(k - \omega_{2n}^{2j-1} \omega_{Mn}^k x\right)\right)}.$$

Proof. We notice that

$$S^{(N)}(n; 1^r) = e_r\left(\frac{\omega_M}{1^n}, \frac{\omega_M^2}{2^n}, \frac{\omega_M^3}{3^n}, \dots\right).$$

Hence, if we specialize $x_m = \omega_M^m/m^n$ and set $t = x^n$ in the generating function $E(t)$ in (1.6), then we obtain the lemma by a similar calculation as in the case of $\mathcal{H}^{(N,M)}(n; x)$. □

Now, we obtain the following.

Theorem 2.7. *The generating function of $S_k^{(N,M)}(n)$ is given by*
 (2.8)

$$\mathcal{S}^{(N,M)}(n; x) := \sum_{k=0}^{\infty} S_k^{(N)}(n)x^{nk} = \prod_{k=1}^M \frac{\prod_{j=0}^{2n-1} \Gamma\left(\frac{1}{M}\left(k - \frac{1}{N}\omega_{2n}^j \omega_{Mn}^{kN} x\right)\right)}{\Gamma\left(\frac{k}{M}\right)^n \prod_{j=0}^{n-1} \Gamma\left(\frac{1}{M}\left(k - \omega_{2n}^{2j-1} \omega_{Mn}^k x\right)\right)}.$$

Proof. It is clear that $\mathcal{S}^{(N,M)}(n; x) = \mathcal{H}^{(N,M)}(n; x)\mathcal{E}^{(M)}(n; x)$ from equation (2.4). Hence one immediately obtains the formula (2.8) from (2.6) and (2.7). \square

If $M \mid N$, then, using the Gauss-Legendre formula of the gamma function, we have the following reduced formulas:

$$(2.9) \quad \mathcal{H}^{(N,M)}(n; x) = \prod_{j=0}^{2n-1} \Gamma\left(1 - \frac{\omega_{2n}^j x}{N}\right),$$

$$(2.10) \quad \mathcal{S}^{(N,M)}(n; x) = \prod_{k=1}^M \frac{\Gamma\left(\frac{k}{M}\right)^n \prod_{j=0}^{2n-1} \Gamma\left(1 - \frac{\omega_{2n}^j x}{N}\right)}{\prod_{j=0}^{n-1} \Gamma\left(\frac{1}{M}\left(k - \omega_{2n}^{2j-1} \omega_{Mn}^k x\right)\right)}.$$

Notice that $\mathcal{E}^{(M)}(n; x)$ depends *only* on M .

3. PARTIAL ALTERNATING MULTIPLE ZETA VALUES

In this section, we concentrate on the special case where $N = M = 2$. From the definition, the sums $S_k(n) := S_k^{(2,2)}(n)$ in this case may be called *partial alternating multiple zeta values*. From (2.10), we have

$$\begin{aligned} \mathcal{S}(n; x) := \mathcal{S}^{(2)}(n; x) &= \frac{\Gamma\left(\frac{1}{2}\right)^n \prod_{j=0}^{2n-1} \Gamma\left(1 - \frac{x}{2}\omega_{2n}^j\right)}{\prod_{j=0}^{n-1} \Gamma\left(\frac{1}{2} - \frac{x}{2}\omega_n^j\right) \Gamma\left(1 - \frac{x}{2}\omega_n^j \omega_{2n}\right)} \\ &= \frac{\Gamma\left(\frac{1}{2}\right)^n \prod_{j=0}^{n-1} \Gamma\left(1 - \frac{x}{2}\omega_n^j\right)}{\prod_{j=0}^{n-1} \Gamma\left(\frac{1}{2} - \frac{x}{2}\omega_n^j\right)}. \end{aligned}$$

Furthermore, using the duplication formula $\Gamma(2a)\Gamma(1/2) = 2^{2a-1}\Gamma(a)\Gamma(1/2+a)$ with $a = -x\omega_n^j/2$ and the equation $\sum_{j=0}^{n-1} \omega_n^j = \delta_{n,1}$, we see that

$$(3.1) \quad \mathcal{S}(n; x) = \frac{\Gamma\left(\frac{1}{2}\right)^n \prod_{j=0}^{n-1} \Gamma\left(1 - \frac{x}{2}\omega_n^j\right)}{\prod_{j=0}^{n-1} \Gamma(-x\omega_n^j)\Gamma\left(\frac{1}{2}\right) 2^{x\omega_n^j+1} \Gamma\left(-\frac{x}{2}\omega_n^j\right)^{-1}} = 2^{-x\delta_{n,1}} \prod_{j=0}^{n-1} \frac{\Gamma\left(1 - \frac{x}{2}\omega_n^j\right)^2}{\Gamma\left(1 - x\omega_n^j\right)}.$$

For $m \geq 0$, define the sequence $\{A^\bullet(m)\}_{m \geq 0}$ by $A^\bullet(0) := 1, A^\bullet(1) := 0$ and

$$A^\bullet(m) := \sum_{a=1}^{m-1} \zeta_a^\bullet(\underbrace{1, \dots, 1}_{a-1}, m-a+1) \quad (m \geq 2).$$

Namely, $A^\bullet(m)$ ($m \geq 2$) denotes the sum of multiple zeta values of weight m and height 1. It is known that $A^\bullet(m)$ can be expressed as a polynomial in $\zeta(2), \zeta(3), \dots$

and $\zeta(m)$ with rational coefficients (see [14]). For example, we have $A^\bullet(3) = \zeta(3) + \zeta_2^\bullet(1, 2) = 2\zeta(3)$ since $\zeta_2^\bullet(1, 2) = \zeta(3)$, which is due to Euler. Furthermore, we put

$$A_n^\bullet(m) := \sum_{\substack{m_1, \dots, m_n \geq 0 \\ m_1 + \dots + m_n = m}} A^\bullet(m_1) \cdots A^\bullet(m_n), \quad Z_n(k) := \sum_{\substack{\mu \vdash k \\ \mu_{\ell(\mu)} > \delta_{n,1}}} \frac{\nu(n\mu)}{z_\mu} \zeta(n\mu),$$

where $\nu(x) := 2^{1-x} - 1$ and $z_\mu := \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!$. Note that $A_1^\bullet(m) = A^\bullet(m)$. Then, we get the following expressions of the values $S_k(n) = S_k^{(2)}(n)$.

Theorem 3.1. (i) *If $n = 1$, then*

$$\begin{aligned} (3.2) \quad S_k(1) &= \sum_{m=0}^k \frac{(-\log 2)^{k-m}}{(k-m)! 2^m} A^\bullet(m) \\ &= \sum_{m=0}^k \frac{(-\log 2)^{k-m}}{(k-m)!} Z_1(m) \in \mathbb{Q}[\log 2, \zeta(2), \zeta(3), \dots, \zeta(k)]. \end{aligned}$$

(ii) *If $n \geq 2$, then*

$$(3.3) \quad S_k(n) = \frac{1}{2^{nk}} A_n^\bullet(nk) = Z_n(k) \in \mathbb{Q}[\zeta(n), \zeta(2n), \dots, \zeta(kn)].$$

Proof. From the generating function (3.1), it is sufficient to show that

$$(3.4) \quad \prod_{j=0}^{n-1} \frac{\Gamma(1 - \frac{x}{2}\omega_n^j)^2}{\Gamma(1 - x\omega_n^j)} = \sum_{m=0}^{\infty} A_n^\bullet(nm) \left(\frac{x}{2}\right)^{nm} = \sum_{m=0}^{\infty} Z_n(m) x^{nm}.$$

To prove this, we recall the identity (see [1, 4])

$$\begin{aligned} (3.5) \quad \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} &= 1 - \sum_{a,b=1}^{\infty} \zeta_a^\bullet(\underbrace{1, \dots, 1}_{a-1}, b+1) X^a Y^b \\ &= \exp\left(\sum_{m=2}^{\infty} \frac{X^m + Y^m - (X+Y)^m}{m} \zeta(m)\right). \end{aligned}$$

Putting $X = Y = x\omega_n^j/2$ and writing $a + b = m$ in the middle term in (3.5), we have

$$\frac{\Gamma(1 - \frac{x}{2}\omega_n^j)^2}{\Gamma(1 - x\omega_n^j)} = \sum_{m=0}^{\infty} A^\bullet(m) \left(\frac{x\omega_n^j}{2}\right)^m = \exp\left(\sum_{m=2}^{\infty} \frac{\nu(m)}{m} \zeta(m) (\omega_n^j x)^m\right).$$

Then, taking the product $\prod_{j=0}^{n-1}$ of this equation, one sees that

$$(3.6) \quad \prod_{j=0}^{n-1} \frac{\Gamma(1 - \frac{x}{2}\omega_n^j)^2}{\Gamma(1 - x\omega_n^j)} = \sum_{m=0}^{\infty} A_n^\bullet(m) \left(\frac{x\omega_n^j}{2}\right)^m = \exp\left(\sum_{\substack{m=1 \\ nm \geq 2}}^{\infty} \frac{\nu(nm)}{m} \zeta(nm) x^{nm}\right)$$

because $\sum_{j=0}^{n-1} \omega_n^{jm} = n$ if $n \mid m$ and 0 otherwise. Here, the rightmost side of (3.6) can be written as

$$\begin{aligned} & \prod_{\substack{m=1 \\ nm \geq 2}}^{\infty} \exp\left(\frac{\nu(nm)}{m} \zeta(nm)x^{nm}\right) \\ &= \prod_{\substack{m=1 \\ nm \geq 2}}^{\infty} \sum_{l_m=0}^{\infty} \frac{1}{l_m!} \left(\frac{\nu(nm)}{m} \zeta(nm)x^{nm}\right)^{l_m} \\ &= \begin{cases} \sum_{l_2, l_3, \dots=0}^{\infty} \frac{\nu(2)^{l_2} \nu(3)^{l_3} \dots}{(2^{l_2} 3^{l_3} \dots)(l_2! l_3! \dots)} (\zeta(2)^{l_1} \zeta(3)^{l_3} \dots) x^{2l_2+3l_3+\dots} & (n=1) \\ \sum_{l_1, l_2, \dots=0}^{\infty} \frac{\nu(n)^{l_1} \nu(2n)^{l_2} \dots}{(1^{l_1} 2^{l_2} \dots)(l_1! l_2! \dots)} (\zeta(n)^{l_1} \zeta(2n)^{l_2} \dots) x^{l_1+2l_2+\dots} & (n \geq 2) \end{cases} \\ &= \sum_{m=0}^{\infty} \left\{ \sum_{\substack{\mu \vdash m \\ \mu \ell(\mu) > \delta_{n,1}}} \frac{\nu(n\mu)}{z_{\mu}} \zeta(n\mu) \right\} x^{nm} = \sum_{m=0}^{\infty} Z_n(m)x^{nm}. \end{aligned}$$

Note that from the second equality in (3.6), this shows that $A_n^{\bullet}(m) = 0$ if $n \nmid m$. Therefore, one can actually obtain the equations (3.4). This completes the proof of the theorem. \square

Example 3.2. We have

$$\begin{aligned} S_1(1) &= -\log 2, & S_2(1) &= \frac{(\log 2)^2}{2} - \frac{\zeta(2)}{4}, \\ S_3(1) &= -\frac{(\log 2)^3}{6} + \frac{\log 2}{4} \zeta(2) - \frac{1}{4} \zeta(3), \end{aligned}$$

and

$$\begin{aligned} S_1(3) &= -\frac{3}{4} \zeta(3), & S_2(3) &= -\frac{31}{64} \zeta(6) + \frac{9}{32} \zeta(3)^2, \\ S_3(3) &= -\frac{255}{768} \zeta(9) + \frac{93}{128} \zeta(6) \zeta(3) - \frac{27}{384} \zeta(3)^3. \end{aligned}$$

If one further assumes that n is even, then one can obtain the following various expressions.

Theorem 3.3. *It follows that*

$$(3.7) \quad S_k(2n) = (-\pi^2)^{nk} \sum_{\substack{m_1, \dots, m_n \geq 0 \\ m_1 + \dots + m_n = nk}} \omega_n^{m_1+2m_2+\dots+nm_n} \frac{B_{2m_1}}{(2m_1)!} \dots \frac{B_{2m_n}}{(2m_n)!}$$

$$(3.8) \quad = (-\pi^2)^{nk} \sum_{\substack{\lambda \vdash nk \\ \ell(\lambda) \leq n}} \langle p_n \circ h_k, m_{\lambda} \rangle \frac{B_{2\lambda}}{(2\lambda)!}$$

$$(3.9) \quad = (-\pi^2)^{nk} \sum_{\mu \vdash k} \frac{\tilde{\nu}(2n\mu)}{z_{\mu}} \frac{B_{2n\mu}}{(2n\mu)!},$$

where $\tilde{\nu}(x) := 2^{x-1} - 1$, p_n is the n -th power-sum symmetric function, m_{λ} is the monomial symmetric function for λ , \circ is the plethysm, and $\langle \cdot, \cdot \rangle$ is the standard

scalar product in the ring of symmetric functions defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ with $\delta_{\lambda\mu}$ being the Kronecker delta (see [5] for details).

Proof. If we apply the reflection formula for the gamma function in (3.1), then we have

$$\begin{aligned} \mathcal{S}(2n; x) &= \prod_{j=1}^n \frac{\Gamma\left(1 - \frac{\omega_{2n}^j x}{2}\right)^2 \Gamma\left(1 + \frac{\omega_{2n}^j x}{2}\right)^2}{\Gamma\left(1 - \omega_{2n}^j x\right) \Gamma\left(1 + \omega_{2n}^j x\right)} \\ (3.10) \quad &= \prod_{j=1}^n \frac{\pi x \omega_{2n}^j}{2} \cot \frac{\omega_{2n}^j \pi x}{2} = \prod_{j=1}^n \sum_{m=0}^{\infty} \frac{(-\omega_n^j)^m B_{2m} \pi^{2m}}{(2m)!} x^{2m}, \end{aligned}$$

from which we immediately obtain (3.7). Next, it readily follows from (3.7) that

$$S_k^{(2n)} = (-\pi^2)^{nk} \sum_{\substack{\lambda \vdash nk \\ \ell(\lambda) \leq n}} m_\lambda(1, \omega_n, \dots, \omega_n^{n-1}, 0, \dots) \frac{B_{2\lambda}}{(2\lambda)!}.$$

Thus we should calculate $m_\lambda(1, \omega_n, \dots, \omega_n^{n-1}, 0, \dots)$. Let us recall the expansion formula (see, e.g., [5])

$$(3.11) \quad \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} h_\lambda(x) m_\lambda(y).$$

If we set $y_j = \omega_n^{j-1}$ for $j = 1, 2, \dots, n$ and $y_j = 0$ for $j > n$ in (3.11), then we have

$$\begin{aligned} &\sum_{\ell(\lambda) \leq n} h_\lambda(x) m_\lambda(1, \omega_n, \dots, \omega_n^{n-1}, 0, \dots) \\ (3.12) \quad &= \prod_{i \geq 1} \frac{1}{1 - x_i^n} = \sum_{k=0}^{\infty} h_k(x_1^n, x_2^n, \dots) = \sum_{k=0}^{\infty} p_n \circ h_k. \end{aligned}$$

By taking the terms of homogeneous degree nk in (3.12), we have

$$(3.13) \quad \sum_{\substack{\lambda \vdash nk \\ \ell(\lambda) \leq n}} h_\lambda m_\lambda(1, \omega_n, \dots, \omega_n^{n-1}, 0, \dots) = p_n \circ h_k$$

for each k . Hence we get $m_\lambda(1, \omega_n, \dots, \omega_n^{n-1}, 0, \dots) = \langle p_n \circ h_k, m_\lambda \rangle$, which readily implies (3.8). The equation (3.9) follows immediately from (3.3) together with the classical result $\zeta(2m) = (-1)^{m-1} 2^{2m-1} B_{2m} \pi^{2m} / (2m)!$ due to Euler. This completes the proof. \square

Example 3.4. From the equation (3.7), we have

$$S_k(2) = \frac{(-1)^k B_{2k}}{(2k)!} \pi^{2k} = -\frac{\zeta(2k)}{2^{2k-1}}, \quad S_k(4) = \left\{ \sum_{m=0}^{2k} (-1)^m \frac{B_{2m} B_{4k-2m}}{(2m)!(4k-2m)!} \right\} \pi^{4k}.$$

See [19] for a similar discussion on the multiple Dirichlet L -values.

Remark 3.5. It is remarkable that $S_k(2) = S_k^{(2)}(2)$ can be reduced as above. We recall that $S_k^{(2)}(2)$ is closely related to the special value $\zeta_Q(2)$ of the spectral zeta function. Can one explain the simplicity (or “exact solvability”) of $S_k^{(2)}(2)$ by, for instance, the existence of the Picard-Fuchs differential equation for $w_2(t)$?

Remark 3.6. Let us give an example of the partial alternating *double* zeta value with *distinct* indices:

$$\begin{aligned} S_2^{(2)}(1, 2k) &= (k+1)S_1^{(2)}(2k+1) + 2(1-2^{-2k})\zeta(2k)\log 2 \\ &\quad - \sum_{p=1}^{k-1} S_1^{(2)}(2p+1)\zeta(2k-2p), \\ S_2^{(2)}(2k, 1) &= -kS_1^{(2)}(2k+1) - \zeta(2k)\log 2 \\ &\quad + \sum_{p=1}^{k-1} S_1^{(2)}(2p+1)\zeta(2k-2p). \end{aligned}$$

Notice that $S_1^{(2)}(n) = (2^{1-n} - 1)\zeta(n)$ for $n \geq 2$. This is regarded as an analogue of Euler's formula $\zeta_2^*(1, 2k) = k\zeta(2k+1) - \frac{1}{2} \sum_{p=2}^{2k-1} \zeta(p)\zeta(2k-p+1)$. See also [3] for related calculations.

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