OPTIMAL LENGTH ESTIMATES FOR STABLE CMC SURFACES
IN 3-SPACE FORMS

LAURENT MAZET

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Abstract. In this paper, we study stable constant mean curvature $H$ surfaces in $\mathbb{R}^3$. We prove that, in such a surface, the distance from a point to the boundary is less than or equal to $\pi/(2H)$. This upper bound is optimal and is extended to stable constant mean curvature surfaces in space forms.

1. Introduction

A constant mean curvature (cmc) surface $\Sigma$ in a Riemannian 3-manifold $M^3$ is stable if its stability operator, $L = -\Delta - \text{Ric}(n, n) - |A|^2$, is nonnegative, where $\Delta$ is the Laplace operator on $\Sigma$, Ric is the Ricci tensor on $M^3$, $n$ is the normal along $\Sigma$ and $A$ is the second fundamental form on $\Sigma$. For minimal surfaces ($H = 0$), this characterization is only valid for two-sided surfaces, so in the following we restrict ourselves to such surfaces. The nonnegativity of the stability operator means that $\Sigma$ is a local minimizer of the area functional on surfaces with regard to the infinitesimal deformations fixing its boundary.

The stability hypothesis was studied by several authors and has many consequences (see [6] for an overview). For example, D. Fischer-Colbrie and R. Schoen [4] studied the case of complete stable minimal surfaces when $M^3$ has nonnegative scalar curvature. They obtain that the universal cover of $\Sigma$ is not conformally equivalent to the disk and, as a consequence, prove that the plane is the only complete stable minimal surface in $\mathbb{R}^3$. From this, R. Schoen [9] has derived a curvature estimate for stable cmc surfaces.

In [2], T. H. Colding and W. P. Minicozzi introduced new techniques and obtained area and curvature estimates for stable cmc surfaces. Afterward, these techniques were used by P. Castillon [1] to answer a question asked in [4] about the consequences of the positivity of certain elliptic operators. Recently, the same ideas have been used by J. Espinar and H. Rosenberg [3] to obtain similar results.

In [7], A. Ros and H. Rosenberg study constant mean curvature $H$ surfaces in $\mathbb{R}^3$ with $H \neq 0$. They prove a maximum principle at infinity. One of their tools is a length estimate for stable cmc surfaces. In fact, they prove that the intrinsic distance from a point $p$ in a stable cmc surface $\Sigma$ to the boundary of $\Sigma$ is less than $\pi/H$. H. Rosenberg [8] has generalized this result to any ambient 3-manifolds and large mean curvature. The aim of this paper is to improve the result of Ros...
2762 L. MAZET and Rosenberg. In fact, applying the ideas of [2], we prove that the distance is less than $\pi/(2H)$. This estimate is optimal since, for a hemisphere of radius $1/H$, the distance from the pole to the boundary is $\pi/(2H)$. Actually we prove that the hemisphere of radius $1/H$ is the only stable cmc $H$ surface where the distance $\pi/(2H)$ is reached. We can generalize this result to stable cmc $H$ surfaces in $M^3(\kappa)$, where $M^3(\kappa)$ is the 3-space form of sectional curvature $\kappa$. We prove that when $H^2 + \kappa > 0$ such an optimal estimate exists. In fact, it is already known that when $\kappa \leq 0$ and $H^2 + \kappa \leq 0$, there is no such estimate since there exist complete stable cmc $H$ surfaces. But, in some sense, our results are an extension of the fact that the planes (resp. the horospheres) are the only stable complete constant mean curvature $H$ surfaces in $\mathbb{R}^3$ (resp. $M^3(\kappa)$, $\kappa < 0$) when $H = 0$ (resp. $H^2 + \kappa = 0$).

2. Definitions

On a constant mean curvature surface $\Sigma$ in a Riemannian 3-manifold $M^3$, the stability operator is defined by $L = -\Delta - \text{Ric}(n, n) - |A|^2$, where $\Delta$ is the Laplace operator on $\Sigma$, $\text{Ric}$ is the Ricci tensor on $M^3$, $n$ is the normal along $\Sigma$ and $A$ is the second fundamental form on $\Sigma$. When it is necessary, we will denote the stability operator by $L_f$ to refer to the immersion $f$ of $\Sigma$ in $M^3$.

The surface $\Sigma$ is called stable if the operator $L$ is nonnegative; i.e., for every compactly supported function $u$, we have

$$0 \leq \int_\Sigma uL(u)d\sigma = \int_\Sigma \|\nabla u\|^2 - (\text{Ric}(n, n) + |A|^2)u^2d\sigma.$$ 

We remark that this property is sometimes called strong stability since it means that the second derivatives of the area functional are nonnegative with respect to any compactly supported infinitesimal deformations $u$, whereas $\Sigma$ is critical for this functional only for compactly supported infinitesimal deformations with vanishing mean value, i.e. $\int_\Sigma ud\sigma = 0$.

In the following, on a cmc surface, the normal $n$ is always chosen such that $H$ is nonnegative.

We will denote by $d_\Sigma$ the intrinsic distance on $\Sigma$ and by $K$ the sectional curvature of the surface.

3. Results

The main result of this paper is the following theorem.

**Theorem 3.1.** Let $H$ be positive. Let $\Sigma$ be a stable constant mean curvature $H$ surface in $\mathbb{R}^3$. Then, for $p \in \Sigma$, we have

$$(3.1) \quad d_\Sigma(p, \partial \Sigma) \leq \frac{\pi}{2H}.$$ 

Moreover, if the equality is satisfied, $\Sigma$ is a hemisphere.

In $\mathbb{R}^3$, the stability operator can be written $L = -\Delta - 4H^2 + 2K$.

**Proof.** We denote by $R_0$ the distance $d_\Sigma(p, \partial \Sigma)$ and assume that $R_0 \geq \pi/(2H)$. If $R_0 < \pi/H$ we denote by $I$ the segment $[\pi/(2H), R_0]$; otherwise $I = [\pi/(2H), \pi/H]$. In fact, because of the work of Ros and Rosenberg [7], we already know that $R_0 \leq \pi/H$. Let $R$ be in $I$.

The surface $\Sigma$ has constant mean curvature $H$; thus its sectional curvature is less than $H^2$. So the exponential map $\exp_p$ is a local diffeomorphism on the disk.
$D(0, R) \subset T_p \Sigma$ of center 0 and radius $R$. On this disk, we consider the induced metric and the operator $L = -\Delta - 4H^2 + 2K$. The surface $\Sigma$ is stable, so there exists a positive function $g$ on $\Sigma$ such that $L(g) = 0$ (see Theorem 1 in [4]). On $D(0, R)$, the function $\tilde{g} = g \circ \exp_p$ is then positive and satisfies $L(\tilde{g}) = 0$ since $D(0, R)$ and $\Sigma$ are locally isometric. The operator $L$ is thus nonnegative on $D(0, R)$ [4].

For $r \in [0, R]$, we define $l(r)$ as the length of the circle $\{ v, |v| = r \} \subset D(0, R)$ and $K(r) = \int_{D(0, r)} Kd\sigma$. Since $D(0, R)$ and $\Sigma$ are locally isometric, the sectional curvature $K$ of $D(0, R)$ is less than $H^2$. Then

$$l(r) \geq \frac{2\pi}{H} \sin Hr.$$  

By Gauss-Bonnet, we have

$$K(r) = 2\pi - l'(r).$$  

Let us consider a function $\eta : [0, R] \to [0, 1]$ with $\eta(0) = 1$ and $\eta(R) = 0$. Let us write the nonnegativity of $L$ for the radial function $u = \eta(r)$:

$$0 \leq \int_0^R (\eta'(r))^2 l(r) dr - 4H^2 \int_0^R \eta^2(r) l(r) dr + 2 \int_0^R K(r) \eta^2(r) dr.$$  

Hence, following the ideas in [2] and using (3.3) and the boundary values of $\eta$, we have

$$\int_0^R (4H^2 \eta^2 - \eta'^2) l dr \leq 2 \left( [K(\eta^2(r))]_0^R - \int_0^R K(\eta^2(r))' dr \right)$$

$$= -2 \int_0^R K(\eta^2(r))' dr$$

$$= -2 \int_0^R (2\pi - l'(r))(\eta^2(r))' dr$$

$$= 4\pi + 2 \int_0^R (\eta^2(r))' l'(r) dr$$

$$= 4\pi + 2 \int_0^R (\eta^2(r))' l'(r) dr$$

Thus we obtain

$$\int_0^R \left( 4H^2 \eta^2 - \eta'^2 + 2(\eta^2)'' \right) l dr \leq 4\pi.$$

We shall apply this equation to the function $\eta(r) = \cos \frac{\pi r}{2R}$. In this case we have

$$\eta^2 = \frac{\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R},$$

$$(\eta^2)'' = -\frac{\pi^2}{2R^2} \left( \cos^2 \frac{\pi r}{2R} - \sin^2 \frac{\pi r}{2R} \right).$$

Thus

$$4H^2 \eta^2 - \eta'^2 + 2(\eta^2)'' = (4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R}.$$
As \( R \geq \frac{\pi}{2H} \), \( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \) is nonnegative and, by (3.2),
\[
\left(4H^2\eta^2 - \eta'^2 + 2(\eta^2)''\right) l \geq \left(4H^2 - \frac{\pi^2}{R^2}\cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2}\sin^2 \frac{\pi r}{2R}\right) \frac{2\pi}{H} \sin H r
\]
\[
\geq \frac{\pi}{H} \left(4H^2 - \frac{\pi^2}{4R^2}\sin H r + (4H^2 - \frac{7\pi^2}{4R^2}) \frac{1}{2} \left(\sin \frac{\pi r}{R}\eta - \sin \left(\frac{\pi}{R} - H r\right)\right)\right)
\]
Thus integrating in (3.3), we obtain (we recall that \( R < \pi/H \))
\[
4\pi \geq \frac{\pi}{H} \left(4H^2 - \frac{\pi^2}{4R^2}\right) \frac{1}{2} \left(1 - \cos HR\right)
\]
\[
+ \left(4H^2 - \frac{7\pi^2}{4R^2}\right) \frac{1}{2} \left(\frac{R}{\pi + HR} \left(1 - \cos (\pi + HR)\right) - \frac{R}{\pi - HR} \left(1 - \cos (\pi - HR)\right)\right).
\]
After some simplifications in the above expression, we obtain
\[
4\pi \geq \frac{\pi}{H} \left(-32H^2 R^4 + 24\pi^2 H^2 R^2 - \pi^4 - (10\pi^2 H^2 R^2 - \pi^4) \cos H R\right)
\]
Now, passing \( 4\pi \) on the right-hand side of the above inequality and simplifying by \( \pi \), we get
\[
F(R) := -\frac{(4H^2 R^2 - \pi^2)^2 - (10\pi^2 H^2 R^2 - \pi^4) \cos H R}{4H^2 R^2(\pi^2 - H^2 R^2)} \leq 0.
\]
If we write \( R = \pi/(2H) + x \), we compute the Taylor expansion of \( F \) and obtain
\[
F\left(\frac{\pi}{2H} + x\right) = 2Hx + o(x),
\]
which is positive if \( x > 0 \). Thus, if \( R_0 > \pi/(2H) \), we get a contradiction and the inequality (3.1) is proved.

Now if \( R_0 = \pi/(2H) \), we have in fact equality all along the computation, so \( l(r) = (2\pi/H) \sin H r \) and \( K(r) = 2\pi - l'(r) = 2\pi(1 - \cos H r) \). But we also know that the sectional curvature is less than \( H^2 \); thus \( K(r) \leq H^2 \int_0^r l(u) du = 2\pi(1 - \cos H r) \). Since this inequality is in fact an equality, the sectional curvature is in fact \( H^2 \) at every point. Thus the principal curvatures of a point in \( \Sigma \) are \( H \) and \( H \); i.e. there are only umbilical points. Hence \( \Sigma \) is a piece of a sphere of radius \( 1/H \) and, since \( d\Sigma(p, \partial \Sigma) = \frac{\pi}{H} \), it contains the hemisphere of pole \( p \). A hemisphere cannot be strictly contained in a stable subdomain of the sphere, so \( \Sigma \) is a hemisphere. \( \Box \)

With this result we have an important corollary.

**Corollary 3.2.** Let \( H \geq 0 \) and \( \kappa \in \mathbb{R} \) such that \( H^2 + \kappa > 0 \). Let \( \Sigma \) be a stable constant mean curvature \( H \) surface in \( \mathbb{M}^3(\kappa) \). Then for \( p \in \Sigma \), we have
\[
d\Sigma(p, \partial \Sigma) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}.
\]
Moreover, if the equality is satisfied, \( \Sigma \) is a geodesical hemisphere of \( \mathbb{M}^3(\kappa) \).

The proof is based on the Lawson correspondence between constant mean curvature surfaces in space forms (see [5]).

**Proof.** First, the case \( \kappa = 0 \) is Theorem 3.1.

Let \( \Pi : \widetilde{\Sigma} \to \Sigma \) be the universal cover of \( \Sigma \). We then have a constant mean curvature immersion of \( \Sigma \) in \( \mathbb{M}^3(\kappa) \). Let \( L = -\Delta - 2\kappa - |A|^2 \) be the stability operator on \( \Sigma \). \( \Sigma \) is stable, so there exists a positive function \( g \) on \( \Sigma \) such that \( L(g) = -\Delta g - (2\kappa + |A|^2)g = 0 \). Thus the function \( \tilde{g} = g \circ \Pi \) is a positive function.
on $\tilde{\Sigma}$ satisfying $L(\tilde{g}) = 0$. Hence $\tilde{\Sigma}$ is stable. Let $I$ and $S$ be respectively the first fundamental form and the shape operator on $\tilde{\Sigma}$. They satisfy the Gauss and Codazzi equations for $M^3(\kappa)$.

We define $S' = S + (-H + \sqrt{H^2 + \kappa})\text{id}$ on $\tilde{\Sigma}$. Then $I$ and $S'$ satisfy the Gauss and Codazzi equations for $M^3(0) = \mathbb{R}^3$ (see [5]). Hence there exists an immersion $f$ of $\tilde{\Sigma}$ in $\mathbb{R}^3$ with first fundamental form $I$ and shape operator $S'$ (we notice that the induced metric is the same). Its mean curvature is then $H + (-H + \sqrt{H^2 + \kappa}) = \sqrt{H^2 + \kappa}$; i.e. the immersion has constant mean curvature. The stability operator is

$$L_f = -\Delta - \|S'\|^2$$

$$= -\Delta - (\|S\|^2 + 4H(-H + \sqrt{H^2 + \kappa}) + 2(-H + \sqrt{H^2 + \kappa})^2)$$

$$= -\Delta - (\|S\|^2 + 2\kappa)$$

$$= L.$$

Hence the surface $f(\tilde{\Sigma})$ is stable. So, from Theorem 3.1, we have

$$d_{\Sigma}(p, \partial \Sigma) = d_{\tilde{\Sigma}}(\tilde{p}, \partial \tilde{\Sigma}) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}},$$

where $\Pi(\tilde{p}) = p$.

The equality case comes from the equality case in Theorem 3.1 and since the Lawson correspondence sends spheres into spheres. □

REFERENCES