Abstract. We determine the structure of the first resonance variety of the cohomology ring of the group of automorphisms of a finitely generated free group which act by conjugation on a given basis.

1. Resonance of $P\Sigma_n$

Let $F_n$ be the free group generated by $x_1, \ldots, x_n$. The basis-conjugating automorphism group, or pure symmetric automorphism group, is the group $P\Sigma_n$ of all automorphisms of $F_n$ which send each generator $x_i$ to a conjugate of itself. Results of Dahm [4] and Goldsmith [7] imply that this group may also be realized as the “group of loops”, the group of motions of a collection of $n$ unknotted, unlinked oriented circles in 3-space, where each circle returns to its original position. McCool [9] found the following presentation for the basis-conjugating automorphism group:

$$P\Sigma_n = \langle \beta_{i,j}, 1 \leq i \neq j \leq n \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta_{j,k})] \rangle,$$

where $[u, v] = uvu^{-1}v^{-1}$ denotes the commutator, the indices in the relations are distinct, and the generators $\beta_{i,j}$ are the automorphisms of $F_n$ defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \neq j, \\ x_j^{-1}x_ix_j & \text{if } k = i. \end{cases}$$

The purpose of this paper is to determine the structure of the first resonance variety of the cohomology ring of this group.

Let $A = \bigoplus_{k=0}^\ell A^k$ be a finite-dimensional, graded, connected algebra over an algebraically closed field $k$ of characteristic 0. Since $a \cdot a = 0$ for each $a \in A^1$, multiplication by $a$ defines a cochain complex $(A, \delta_a)$:

$$A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} A^2 \xrightarrow{\delta_a} \cdots \xrightarrow{\delta_a} A^\ell,$$

where $\delta_a(x) = ax$. The resonance varieties of $A$ are the jumping loci for the cohomology of these complexes: $R^j_\delta(A) = \{ a \in A^1 \mid \dim_k H^j(A, \delta_a) \geq d \}$. As shown by Falk [6], these algebraic subvarieties of $A^1$ are isomorphism-type invariants of the algebra $A$. They have been the subject of considerable recent interest in the context of hyperplane arrangements and related areas; see, for instance, Dimca,
Papadima, and Suciu [11], and the references therein. We will focus on the first resonance variety \( R^1(\mathcal{G}) = \{ a \in A^1 \mid H^1(\mathcal{G}, \delta_a) \neq 0 \} \).

Since the relations in the presentation (1.1) of \( \mathcal{P}\Sigma_n \) are all commutators, the first homology group \( H_1(\mathcal{P}\Sigma_n; \mathbb{k}) \) is a vector space of dimension \( n(n-1) \) with basis \( \{ [\beta_{p,q}] \mid 1 \leq p \neq q \leq \ell \} \). Let \( \{ e_{p,q} \mid 1 \leq p \neq q \leq \ell \} \) be the dual basis of \( H^1(\mathcal{P}\Sigma_n; \mathbb{k}) \). Denote the first resonance variety of \( A = \mathcal{H}^*(\mathcal{P}\Sigma_n; \mathbb{k}) \) by \( R^1(\mathcal{P}\Sigma_n; \mathbb{k}) \).

**Theorem 1.1.** The first resonance variety of the cohomology ring \( \mathcal{H}^*(\mathcal{P}\Sigma_n; \mathbb{k}) \) of the basis-conjugating automorphism group is

\[
R^1(\mathcal{P}\Sigma_n; \mathbb{k}) = \bigcup_{1 \leq i < j \leq n} C_{i,j} \cup \bigcup_{1 \leq i < j < k \leq n} C_{i,j,k},
\]

where \( C_{i,j} = \text{span} \{ e_{i,j}, e_{j,i} \} \) and \( C_{i,j,k} = \text{span} \{ e_{j,i} - e_{k,i}, e_{i,j} - e_{k,j}, e_{i,k} - e_{j,k} \} \).

This result reveals an interesting relationship between the resonance variety and the Bieri-Neumann-Strebel (BNS) invariant of the basis-conjugating automorphism group. For a finitely generated group \( G \), let \( \mathcal{C} \) be the Cayley graph corresponding to a finite generating set. Given an additive character \( \chi : \mathcal{G} \to \mathbb{R} \), let \( \mathcal{C}_+(\chi) \) be the full subgraph of \( \mathcal{C} \) on the vertex set \( \{ g \in \mathcal{G} \mid \chi(g) \geq 0 \} \). Then the (first) BNS invariant of \( G \) is the conical subset \( \Sigma(G) = \text{Hom}(\mathcal{G}, \mathbb{R}) \setminus \{ 0 \} \) where \( \mathcal{C}_+(\chi) \) is connected.

This invariant of \( G \) (which is independent of the choice of generating set) may be used to determine which subgroups above the commutator subgroup \( [G,G] \) are finitely generated; see [1]. The BNS invariant of the group \( G = \mathcal{P}\Sigma_n \) was determined by Orlandi-Korner [11]. Combining her result with the above theorem yields the following:

**Theorem 1.2.** The Bieri-Neumann-Strebel invariant of the basis-conjugating automorphism group is given by

\[
\Sigma(\mathcal{P}\Sigma_n) = H^1(\mathcal{P}\Sigma_n; \mathbb{R}) \setminus R^1(\mathcal{P}\Sigma_n, \mathbb{R}).
\]

This relationship between the resonance variety and the BNS invariant is known to hold for certain other groups, including right-angled Artin groups; see Meier and VanWyk [10] and Papadima andSuciu [12]. More recently, Papadima and Suciu have shown that the containment \( \Sigma(G) \subseteq H^1(G; \mathbb{R}) \setminus R^1(G; \mathbb{R}) \) holds for an arbitrary 1-formal group \( G \), where, as above, \( R^1(G; \mathbb{R}) \) denotes the first resonance variety of \( H^*(G; \mathbb{R}) \). See [13] for a detailed investigation of the relationship between BNS invariants and (co)homology jumping loci in a number of contexts.


In this section, we recall the structure of the cohomology ring of the basis-conjugating automorphism group and use it to prove Theorem [11]. The cohomology of \( \mathcal{P}\Sigma_n \) was computed by Jensen, McCammond, and Meier [8], resolving positively a conjecture of Brownstein and Lee [2].

**Theorem 2.1** (8). Let \( E_Z \) denote the exterior algebra over \( \mathbb{Z} \) generated by degree one elements \( e_{p,q}, 1 \leq p \neq q \leq n \), and let \( I_Z \) denote the two-sided ideal in \( E_Z \) generated by

\[
\begin{align*}
\eta_{i,j} &= e_{i,j}e_{j,i}, \quad 1 \leq i < j \leq n; \\
\tau^k_{i,j} &= (e_{k,i} - e_{j,i})(e_{k,j} - e_{i,j}), \quad 1 \leq k \leq n, \quad 1 \leq i < j \leq n, \quad k \notin \{i, j\}.
\end{align*}
\]

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Then the integral cohomology algebra of the basis-conjugating automorphism group $PS_n$ is isomorphic to the quotient of $E_2$ by $I_2$, $H^*(PS_n; \mathbb{Z}) \cong E_2/I_2$.

Remark 2.2. The above presentation of the cohomology ring $H^*(PS_n; \mathbb{Z})$ differs slightly from that given in [8, Thm. 6.7] but is easily seen to be equivalent. For instance, the relation in $H^*(PS_n; \mathbb{Z})$ arising from the generator $\tau_{i,j}^k$ of the ideal $I$ may be obtained by constructing an appropriate linear combination of the relations labeled 2 and 3 in [8, Thm. 6.7].

Let $k$ be a field of characteristic zero. From Theorem 2.1 and the Universal Coefficient Theorem, the cohomology algebra $H^*(PS_n; k)$ is isomorphic to $E_k/I_k$, where $E_k$ is the exterior algebra over $k$ generated by $e_{p,q}$, $1 \leq p \neq q \leq n$, and $I_k$ is the ideal in $E_k$ generated by the elements $\eta_{i,j}$ and $\tau_{i,j}^k$ above.

Recall from Section 1 that the first resonance variety of $A = H^*(PS_n; k)$ is $R^1(PS_n, k) = \{ a \in A \mid H^1(A, \delta_a) \neq 0 \}$.

Observe that $A^1 = H^1(PS_n; k)$ is a vector space of dimension $N = n(n - 1)$ over $k$. Elements of $A^1$ are of the form $a = \sum_{p \neq q} a_{p,q} e_{p,q}$, where $a_{p,q} \in k$.

(2.2) \[ C_{i,j} = \text{span} \{ e_{i,j}, e_{i,i} \} \{ a \in A^1 \mid a_{p,q} = 0 \text{ if } \{ p, q \} \neq \{ i, j \} \} \]

for $1 \leq i < j \leq n$ and that

(2.3) \[ C_{i,j,k} = \text{span} \{ e_{i,i} - e_{k,i}, e_{i,j} - e_{k,j}, e_{i,k} - e_{j,k} \} \]

for $1 \leq i < j < k \leq n$. Note that these are linear subspaces of $A^1 \cong k^N$ of dimensions 2 and 3 respectively. To prove Theorem 2.1, we must show that $R^1(PS_n, k)$ is equal to the union of these linear subspaces.

Proof of Theorem 2.1. In the case $n = 2$, the group $PS_2 = \langle \beta_{1,2}, \beta_{2,1} \rangle \cong F_2$ is a free group, and the theorem asserts that $R^1(PS_2; k) = C_{1,2} = H^1(PS_2; k)$, which is clear. So assume that $n \geq 3$.

Write $R = R^1(PS_n; k)$ and $C = \bigcup_{1 \leq i < j \leq n} C_{i,j} \bigcup \bigcup_{1 \leq i < j < k \leq n} C_{i,j,k}$. Observe that $0 \in C$ and $0 \in R$. So it is enough to show that $R \setminus \{ 0 \} = C \setminus \{ 0 \}$.

Write $E = E_k$ and $I = I_k$. For $a \in A^1 = E^1$, we have a short exact sequence of chain complexes $0 \to (I, \delta_a) \overset{i}{\to} (E, \delta_a) \overset{p}{\to} (A, \delta_a) \to 0$:

\[
\begin{array}{cccccc}
I^0 & \overset{\delta_a}{\to} & I^1 & \overset{\delta_a}{\to} & I^2 & \cdots \\
\downarrow i^0 & & \downarrow i^1 & & \downarrow i^2 & \\
E^0 & \overset{\delta_a}{\to} & E^1 & \overset{\delta_a}{\to} & E^2 & \cdots \\
\downarrow p^0 & & \downarrow p^1 & & \downarrow p^2 & \\
A^0 & \overset{\delta_a}{\to} & A^1 & \overset{\delta_a}{\to} & A^2 & \cdots \\
\end{array}
\]

where $i: I \to E$ is the inclusion, $p: E \to A$ the projection, and $\delta_a(x) = ax$. Note that since $I$ is generated in degree two, the maps $p^0: E^0 \to A^0$ and $p^1: E^1 \to A^1$ are identity maps. If $a \neq 0$, then the complex $(E, \delta_a)$ is acyclic. Consequently, the corresponding long exact cohomology sequence yields

\[ H^1(A, \delta_a) \cong H^2(I, \delta_a) = \ker(\delta_a: I^2 \to I^3) = \ker(i^3 \circ \delta_a: I^2 \to I^3). \]
Thus, \( a ∈ R \setminus \{0\} \) if and only if the map \( ψ_a := i^3 ◦ δ_a \) fails to inject.

Since the elements \( η_{i,j} \) and \( τ_{i,j}^k \) recorded in (2.4) generate the ideal \( I \) and are of degree two (and are linearly independent in \( E^2 \)), these elements form a basis for \( I^2 \).

We record the images of these basis elements under the map \( ψ_a \). For \( 1 ≤ i < j ≤ n \),

\[
(2.4)\quad ψ_a(η_{i,j}) = \sum_{\{p,q\} \neq \{i,j\}} a_{p,q} e_{p,q} η_{i,j} = \sum_{\{p,q\} \neq \{i,j\}} a_{p,q} e_{p,q} e_{i,j} e_{j,i}.
\]

For \( 1 ≤ k ≤ n, 1 ≤ i < j ≤ n, k \notin \{i,j\} \),

\[
ψ_a(τ_{i,j}^k) = (a_{j,i} + a_{k,i}) e_{j,i} e_{k,i} (e_{k,j} - e_{i,j}) - (a_{i,j} + a_{k,j}) e_{i,j} e_{k,j} (e_{k,i} - e_{i,j})
\]

\[
+ (a_{i,k} e_{i,k} + a_{j,k} e_{j,k}) τ_{i,j}^k + \sum_{\{p,q\} \notin \{i,j,k\}} a_{p,q} e_{p,q} τ_{i,j}^k.
\]

These calculations immediately yield the containment \( C \setminus \{0\} ⊆ R \setminus \{0\} \). If \( a ∈ C_{i,j} \), then \( a_{p,q} = 0 \) for \( \{p,q\} \neq \{i,j\} \). For such an \( a \), we have \( ψ_a(η_{i,j}) = 0 \) by (2.4), so \( C_{i,j} ⊂ R \). If \( 1 ≤ i < j < k ≤ n \) and \( a ∈ C_{i,j,k} \), then \( a_{j,i} + a_{k,i} = 0, a_{i,j} + a_{k,j} = 0, a_{k,i} + a_{i,k} = 0, \) and \( a_{p,q} = 0 \) for \( \{p,q\} \notin \{i,j,k\} \). In this instance,

\[
(2.5)\quad \psi_a(τ_{i,j}^k) = (a_{j,i} + a_{k,i}) e_{j,i} e_{k,i} (e_{k,j} - e_{i,j}) - (a_{i,j} + a_{k,j}) e_{i,j} e_{k,j} (e_{k,i} - e_{i,j})
\]

\[
+ (a_{i,k} e_{i,k} + a_{j,k} e_{j,k}) τ_{i,j}^k + \sum_{\{p,q\} \notin \{i,j,k\}} a_{p,q} e_{p,q} τ_{i,j}^k.
\]

Establishing the reverse containment, \( R \setminus \{0\} ⊆ C \setminus \{0\} \), is more involved. We will show that \( a \notin C \) implies that \( a \notin R \). If \( a \notin C \), then \( a \neq 0 \). So assume without loss that \( a_{2,1} \neq 0 \). Since \( a \notin C_{1,2} ⊂ C \), we must also have \( a_{p,q} \neq 0 \) for some \( \{p,q\} \neq \{1,2\} \). We will consider several cases depending on the relationship between the sets \( \{1,2\} \) and \( \{p,q\} \).

**Case 1.** \( \{1,2\} \cap \{p,q\} = \emptyset \)

Assume first that \( \{1,2\} \) and \( \{p,q\} \) are disjoint. Note that \( n ≥ 4 \) in this instance. Permuting indices if need be, we may assume that \( a ∈ H^1(PS_n; k) \) satisfies \( a_{2,1} \neq 0 \) and \( a_{3,4} \neq 0 \). We will show that this assumption implies that the map \( ψ_a: I^2 ⊂ E^3 \) injects; hence \( a \notin R \). Specifically, we will exhibit a subspace \( V ⊂ E^3 \) and a projection \( π: E^3 ⊂ V \) so that the composition \( π ◦ ψ_a: I^2 ⊂ V \) is an isomorphism.

Let \( V \) be the union of the sets

\[
\begin{align*}
&\{e_{1,2} e_{2,1} e_{3,4}, e_{2,1} e_{i,j} e_{j,i}, e_{i,j} e_{j,i} e_{i,j} | 1 ≤ i < j ≤ n, \{i,j\} \neq \{1,2\}\}, \\
&\{e_{3,4} e_{1,2} e_{k,1}, e_{3,4} e_{2,1} e_{1,k}, e_{3,4} e_{1,2} e_{1,k} | 3 ≤ k ≤ n\}, \\
&\{e_{2,1} e_{k,i} e_{k,j}, e_{2,1} e_{i,j} e_{j,k}, e_{2,1} e_{i,k} e_{k,j} | 2 ≤ j < k ≤ n \text{ or } 3 ≤ i < j < k ≤ n\}.
\end{align*}
\]

There is a bijection between \( V \) and the set of generators of \( I^2 \) given by

\[
\begin{align*}
&η_{1,2} ↦ e_{1,2} e_{2,1} e_{3,4}, \quad η_{i,j} ↦ e_{2,1} e_{i,j} e_{j,i}, \quad \{i,j\} \neq \{1,2\}, \\
&τ_{i,2}^{k} ↦ e_{3,4} e_{1,2} e_{k,1}, \quad τ_{1,k}^{i} ↦ e_{3,4} e_{2,1} e_{1,k}, \quad τ_{2,k}^{i} ↦ e_{3,4} e_{1,2} e_{1,k} (3 ≤ k ≤ n), \\
&τ_{i,j}^{k} ↦ e_{2,1} e_{k,i} e_{k,j}, \quad τ_{i,k}^{j} ↦ e_{2,1} e_{i,j} e_{j,k}, \quad τ_{j,k}^{i} ↦ e_{2,1} e_{i,k} e_{k,j} (i ≤ 2 < j < k ≤ n).
\end{align*}
\]

In particular, the monomials in \( V \) are distinct. Hence, \( V \) is a linearly independent set in \( E^3 \) of cardinality \( |V| = \dim_k I^2 = \binom{n}{3}(n-1) \). Let \( V = \text{span} V ⊂ E^3 \).

Define \( π: E^3 ⊂ V \) on basis elements by

\[
π(e_{a,b} e_{c,d} e_{p,q}) = \begin{cases} e_{a,b} e_{c,d} e_{p,q} & \text{if } e_{a,b} e_{c,d} e_{p,q} ∈ V, \\
0 & \text{if } e_{a,b} e_{c,d} e_{p,q} \notin V. \end{cases}
\]
Then, a calculation using (2.3) and (2.5) reveals that $\pi \circ \psi_a : I^2 \rightarrow V$ is an isomorphism. For instance, ordering the bases of $I^2$ and $V$ appropriately, one can check that the matrix $M$ of $\pi \circ \psi_a$ has determinant $\det M = \psi_{m,2,1} \psi_{3,4} \neq 0$, where $m_{3,4} = 3n - 5$ and $m_{2,1} = (\frac{n}{2})(n - 1) - m_{3,4}$. Thus, if $a \notin C$ satisfies $a_{2,1} \neq 0$ and $a_{p,q} \neq 0$ for some $p, q$ with $\{1, 2\} \cap \{p, q\} = \emptyset$, then $a \notin R$.

Case 2. $\{1, 2\} \cap \{p, q\} \neq \emptyset$

Now assume that $a \notin C$, $a_{2,1} \neq 0$, and $a_{r,s} = 0$ for all $r, s$ with $\{1, 2\} \cap \{r, s\} = \emptyset$. Since $a \notin C_{1,2} \subset C$, we must have $a_{p,q} \neq 0$ for some $p, q$ with $\{|p, q\} \cap \{1, 2\} \neq 1$. Permuting indices if necessary, we may assume that $3 \in \{p, q\}$.

In the case $n = 3$, since $a \notin C_{1,2,3} \subset C$, one of the sums $a_{2,1} + a_{3,1}, a_{1,2} + a_{3,2}, a_{1,3} + a_{2,3}$ must be nonzero; see (2.3). In this instance, ordering bases appropriately, the map $\psi_a : I^2 \rightarrow E^3$ has matrix

\[
M_3 = \begin{pmatrix}
a_{3,2} & 0 & 0 & 0 & -a_{1,2} - a_{3,2} & 0 \\
a_{3,1} & 0 & 0 & 0 & -a_{1,2} - a_{3,1} & 0 \\
a_{2,3} & 0 & a_{1,2} & -a_{2,3} & a_{2,1} & 0 \\
a_{1,3} & 0 & 0 & a_{1,2} & a_{1,3} & a_{2,1} \\
0 & a_{3,2} & 0 & -a_{3,2} & a_{1,3} & -a_{3,1} \\
0 & -a_{3,2} & 0 & a_{1,3} + a_{2,3} & 0 & 0 \\
0 & -a_{2,1} & 0 & a_{2,1} + a_{3,1} & 0 & 0 \\
0 & a_{1,2} & 0 & -a_{1,2} & -a_{1,3} & a_{3,1} \\
0 & 0 & -a_{3,1} & a_{3,2} & a_{2,3} & a_{3,1} \\
0 & 0 & a_{2,1} & a_{3,2} & a_{2,3} & a_{3,1} \\
0 & a_{1,3} & 0 & 0 & -a_{1,3} - a_{2,3} & 0 \\
0 & a_{1,2} & 0 & 0 & -a_{1,2} - a_{3,2} & 0 \\
0 & 0 & 0 & a_{3,2} & -a_{1,3} & -a_{2,1} \\
0 & 0 & 0 & a_{2,1} + a_{3,1} & 0 & 0 \\
0 & 0 & 0 & a_{1,3} + a_{2,3} & 0 & 0 \\
0 & 0 & 0 & a_{2,1} + a_{3,1} & 0 & 0 \\
0 & 0 & 0 & 0 & a_{1,2} + a_{3,2} & 0 \\
0 & 0 & 0 & 0 & a_{1,3} + a_{2,3} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{1,2} + a_{3,2} \\
0 & 0 & 0 & 0 & 0 & a_{1,3} + a_{2,3} \\
\end{pmatrix}
\]

Using the assumptions on $a_{2,1}$, the sums $a_{2,1} + a_{3,1}, a_{1,2} + a_{3,2}, a_{1,3} + a_{2,3}$, and $a_{p,q} \in \{a_{1,3}, a_{2,3}, a_{3,1}, a_{3,2}\}$, one can check that the matrix $M_3$ has maximal rank. Hence, if $a \notin C$, then $\psi_a : I^2 \rightarrow E^3$ injects in the case $n = 3$.

For general $n$, the assumption that $a \notin C_{1,2,3}$ implies that the set

$$\{a_{2,1} + a_{3,1}, a_{1,2} + a_{3,2}, a_{1,3} + a_{2,3}\} \cup \{a_{r,s} \mid \{r, s\} \not\subseteq \{1, 2, 3\}\}$$

contains a nonzero element. Recall that, by Case 1, we may assume that $a_{r,s} = 0$ for all $r, s$ with $\{1, 2\} \cap \{r, s\} = \emptyset$. If $a_{r,s} = 0$ for all $\{r, s\} \not\subseteq \{1, 2, 3\}$, let $W$ be the subspace of $E^3$ spanned by the union of the sets

$$\{e_{i_1,j_1}e_{i_2,j_2}e_{i_3,j_3} \mid 1 \leq i_k, j_k \leq 3, i_k \neq j_k\},$$

$$\{e_{1,2}e_{2,1}e_{p,q} \cup \{e_{2,1}e_{i,j}e_{j,i} \mid 1 \leq i < j \leq n, \{i,j\} \not\subseteq \{1, 2\}\},$$

$$\{e_{2,1}e_{k,i}e_{k,j} + e_{2,1}e_{j,i}e_{k,k} + e_{2,1}e_{i,k}e_{k,j} \mid i \leq 2 < j < k \leq n \text{ or } 3 \leq i < j < k \leq n\},$$

$$\{e_{p,q}e_{k,1}e_{k,2}, e_{p,q}e_{k,1}e_{2,1}, e_{p,q}e_{2,1}e_{k,2}, e_{p,q}e_{k,1}e_{2,1}, e_{p,q}e_{k,2}e_{1,k}, e_{p,q}e_{k,2}e_{2,1}, e_{p,q}e_{k,2}e_{1,k} \mid 3 \leq k \leq n\}.$$
Define \( \pi: E^3 \to W \) on basis elements as before. Ordering bases appropriately, one can use (2.4) and (2.5) to find a submatrix \( M \) of the matrix of \( \pi \circ \psi_n: I^2 \to W \) of the form

\[
M = \begin{pmatrix} U & \ast \\ 0 & M_3 \end{pmatrix},
\]

where \( M_3 \) is given by (2.6) and \( U \) is upper triangular, with diagonal entries \( a_{2,1} \neq 0 \) and \( a_{p,q} \neq 0 \). (The choice of \( U \) depends on which \( a_{p,q} \in \{a_{1,3}, a_{2,3}, a_{3,1}, a_{3,2} \} \) is nonzero.) Hence, the matrix \( M \) has maximal rank. It follows that \( \psi_n: I^2 \to E^3 \) injects.

Finally, consider the case where \( a \notin C_1 \cup C_2 \cup C_3 \). \( a_{2,1} \neq 0 \) and \( a_{p,q} \neq 0 \) for some \( a_{p,q} \in \{a_{1,3}, a_{2,3}, a_{3,1}, a_{3,2} \} \), and \( a_r,s \neq 0 \) for some \( \{r,s\} \notin \{1,2,3\} \). Since we may assume by Case 1 that \( a_r,s = 0 \) if \( \{1,2\} \cap \{r,s\} = \emptyset \), we have \( a_r,s \neq 0 \) for some \( r,s \) with \( r \in \{1,2\} \) and \( 4 \leq s \leq n \). In this instance, let \( W \) be the subspace of \( E^3 \) spanned by the union of the sets

\[
\{e_{1,2}e_{1,1}e_{p,q} \} \cup \{e_{2,1}e_{i,j}e_{j,i} \mid 1 \leq i < j \leq n, \{i,j\} \neq \{1,2\} \},
\{e_{2,1}e_{k,i}e_{j,k}, e_{2,1}e_{i,j}e_{k,k}, e_{2,1}e_{i,k}e_{k,j} \mid 1 \leq 2 < j < k \leq n \text{ or } 3 \leq i < j < k \leq n \},
\{e_{r,s}e_{3,2}, e_{r,s}e_{3,3} \} \cup \{e_{r,s}e_{3,3} \} \cup \{e_{r,s}e_{2,3} \} \cup \{e_{r,s}e_{2,3}e_{3,1} \},
\{e_{p,q}e_{k,1}e_{k,2}, e_{p,q}e_{k,2}e_{k,1}, e_{p,q}e_{1,k}e_{k,2} \mid 4 \leq k \leq n \}.
\]

Defining \( \pi: E^3 \to W \) on basis elements as above, a calculation using (2.4) and (2.5) shows that \( \pi \circ \psi_n: I^2 \to W \) is an isomorphism. For instance, ordering bases appropriately, one can check that the matrix \( M \) of \( \pi \circ \psi_n \) has determinant \( \det M = a_{2,1}a_{3,2}a_{p,q}a_{r,s} \neq 0 \), where \( m_{p,q} = 3n - 8 \) and \( m_{2,1} = \binom{n}{2}(n-1) - m_{p,q} \). Hence, \( \psi_n \) injects in this final case.

Thus, for any \( a \notin C \), the map \( \psi_n: I^2 \to E^3 \) injects, and \( a \notin R \). This completes the proof of Theorem 2.1.

\[\square\]

**Remark 2.3**. It follows from Theorem 2.1 that the integral cohomology groups of \( PS_n \) are torsion free, with Betti numbers \( b_k(PS_n) = \text{rank } H^k(PS_n; \mathbb{Z}) \) given by the coefficients of the Poincaré polynomial \( p(PS_n, t) = \sum_{k \geq 0} b_k(PS_n; \mathbb{Z}) t^k = (1+nt)^{n-1}; \) see [8, §6]. Thus the cohomology groups cannot distinguish \( PS_n \) from a direct product \( F_n \times \cdots \times F_n \) of \( n-1 \) free groups of rank \( n \). For \( n = 2 \), this is to be expected, since \( PS_2 \cong F_2 \).

For \( n \geq 3 \), the groups \( PS_n \) and \( F_n \times \cdots \times F_n \) are not isomorphic and are, in fact, distinguished by their cohomology rings. By Theorem 1.1 the irreducible components of \( R^1(PS_n; k) \) are two- and three-dimensional. On the other hand, the results of [3] or [12] may be used to show that the irreducible components of the first resonance variety of \( H^*(F_n \times \cdots \times F_n; k) \) are all \( n \)-dimensional.

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**References**


Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

E-mail address: cohen@math.lsu.edu
URL: http://www.math.lsu.edu/~cohen/