COMPACT GRAPHS OVER A SPHERE
OF CONSTANT SECOND ORDER MEAN CURVATURE

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Abstract. The aim of this work is to show that a compact smooth star-shaped hypersurface \( \Sigma^n \) in the Euclidean sphere \( S^{n+1} \) whose second function of curvature \( S_2 \) is a positive constant must be a geodesic sphere \( S^n(\rho) \). This generalizes a result obtained by Jellett in 1853 for surfaces \( \Sigma^2 \) with constant mean curvature in the Euclidean space \( \mathbb{R}^3 \) as well as a recent result of the authors for this type of hypersurface in the Euclidean sphere \( S^{n+1} \) with constant mean curvature. In order to prove our theorem we shall present a formula for the operator \( L_r(g) = \text{div}(P_r \nabla g) \) associated with a new support function \( g \) defined over a hypersurface \( M^n \) in a Riemannian space form \( M^{n+1}_c \).

1. Introduction

This paper continues work of the authors concerning constant mean curvature \cite{5}, which was inspired by a paper due to Jellett \cite{11} published in the middle of the nineteenth century in which he proved that a smooth star-shaped constant mean curvature surface \( \Sigma^2 \subset \mathbb{R}^3 \) is a round sphere. On the other hand, if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is an oriented hypersurface and \( k_1, \ldots, k_n \) represent the principal curvatures of \( \Sigma^n \), we may consider similar problems related to the \( r^{th} \) elementary symmetric functions \( S_r \) given by \( S_r = \sum k_{i_1} \cdots k_{i_r}, \ r = 0, 1, \ldots, n \). For instance, Suss \cite{18} proved that a compact convex hypersurface \( \Sigma^n \) in the Euclidean space \( \mathbb{R}^{n+1} \) with some \( S_r \) constant must be a round sphere. The convexity condition was improved upon by Hsiung \cite{10}, who showed that a hypersurface \( \Sigma^n \subset \mathbb{R}^{n+1} \) whose classical support function has a well-defined sign with any symmetric function of the principal curvatures constant must also be a round sphere. Later, Ros proved in \cite{14} and \cite{15} that a round sphere is the unique compact embedded hypersurface in Euclidean space if any symmetric function \( S_r \) is constant. Subsequently, Montiel and Ros \cite{12} extended this result to any compact embedded hypersurface in the hyperbolic space \( \mathbb{H}^{n+1} \) as well as to one contained in an open hemisphere of the Euclidean sphere \( S^{n+1} \). On the other hand, it is well known that a product of spheres produces hypersurfaces in the Euclidean sphere \( S^{n+1} \) with \( S_r \) constant for any \( r = 1, \ldots, n \). Therefore, for hypersurfaces contained in the Euclidean sphere \( S^{n+1} \) we have a lot of examples with \( S_r \) constant which are not round spheres. However, returning
to Jellett’s idea, we may obtain a similar result in the Euclidean sphere $\mathbb{S}^{n+1}$ for compact smooth star-shaped hypersurfaces whose second function of curvature is positive while avoiding the condition that it be contained in an open hemisphere. More precisely, we shall prove the following theorem.

**Theorem 1.1.** Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a compact smooth star-shaped hypersurface with $S_2$ constant and positive. Then $\Sigma^n$ is totally umbilical.

2. Conformal fields

Given a vector field $V \in \chi(\overline{M})$ in a Riemannian manifold $\overline{M}$ and a $r$-covariant tensor field $\omega$, the Lie derivative of $\omega$ with respect to $V$ is defined by

$$(L_V \omega)(X_1, \ldots, X_r) = V(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^r \omega(X_1, \ldots, [V, X_i], \ldots, X_r).$$

For instance, if $\omega = \langle , \rangle$ stands for the Riemannian metric of $\overline{M}$, then

$$(L_V \langle , \rangle)(X, Y) = \langle \nabla_X Y, V \rangle + \langle X, \nabla_Y V \rangle.$$

We say that $V \in \chi(\overline{M})$ is a conformal vector field if there exists $\psi \in D(\overline{M})$, which is called the conformal factor of $V$, such that

$$L_V \langle , \rangle = 2\psi \langle , \rangle.$$

A useful conformal vector field in a space form $M^{n+1}_c$ is the position vector with origin at a fixed point $p_0 \in M^{n+1}_c$. It was introduced for non-Euclidean space forms by Heintze [9] according to the following: Let $d : M^{n+1}_c \to \mathbb{R}$ be the distance function defined by $d(\cdot) = \text{dist}(\cdot, p_0)$. The position vector field over $M^{n+1}_c$ is given by $V = s(d)\nabla d$ where $s(t)$ is the solution of the differential equation $y'' + cy = 0$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

On the other hand Greene and Wu [8] have shown an important property concerning the Hessian form of $d$ which states that

$$s(d)\langle D_X \nabla d, Y \rangle = s'(d)(\langle X, Y \rangle - \langle \nabla d, X \rangle \langle \nabla d, Y \rangle),$$

for any vector fields $X, Y \in \chi(M^{n+1}_c)$. Here and elsewhere $D$ stands for the Riemannian connection of $M^{n+1}_c$. From the last relation we conclude that $V$ is a conformal vector field with conformal factor $\psi = s'(d)$.

3. $L_r$ of a support function

In this section we compute the operator $L_r$ of a new support function defined on a hypersurface $M^n$ in a space form $M^{n+1}_c$. For generalized Robertson-Walker space a particular case appears in Alias and Colares [1]. Given a conformal vector field $V$ in $M^{n+1}_c$ and an isometric immersion $x : M^n \to M^{n+1}_c$, the support function is defined on $M$ by

$$g(p) = \langle V, N \rangle(x(p)),$$

where $N$ stands for a unit normal field to $M^n$. Whenever necessary we shall identify $p \in M^n$ with its image $x(p) \in M^{n+1}_c$.

We now consider on $M$ the second fundamental form $A$, the $r^{th}$ symmetric function of curvature $S_r$, the $r^{th}$ Newton tensor $P_r$ and the operator $L_r$. These Newton tensors are defined inductively according to $P_0 = I$ and, for $1 \leq r \leq n$, $P_r = S_rI - AP_{r-1}$, whereas the operator $L_r : D(M) \to D(M)$ is given by

$$L_r(f) = \text{tr} (P_r \circ Hess_f).$$
It is important to point out that for hypersurfaces in a space form \( M^r_{o+1} \) Rosenberg showed in [16] that each \( L_r \) takes a divergence form. More exactly, we have
\[
L_r(f) = \text{div}(P_r \nabla f).
\]
Moreover, under the above conditions we list some important properties concerning \( S_r \) and \( P_r \) that may be found in [13] and [16], whereas formulae for \( L_r X \) and \( L_r N \) can be found in Reilly [13] and Rosenberg [16], where \( X \) stands for the position vector while \( N \) is a unit normal vector field to \( M \).

**Theorem 3.1.** Let \( x : M^n \hookrightarrow M^r_{o+1} \) be an oriented isometric immersion with unit normal vector field \( N \). If \( V \) is a conformal vector field on \( M^r_{o+1} \) and \( g = \langle V, N \rangle \) represents the support function on \( M^n \), then
\[
L_r(g) = -(S_1 S_{r+1} - (r + 2) S_{r+2}) g - c(n - r) S_r g - (n - r) S_r N(\psi) - (r + 1) S_{r+1} \psi - \langle V, \nabla S_{r+1} \rangle,
\]
where \( \psi \), which we identify with \( \psi \circ x \), is the conformal factor of \( V \).

**Proof.** Given \( p \in x(M^n) \), let \( \{e_1(p), \ldots, e_n(p)\} \subset T_p M \) be an orthonormal basis which diagonalizes \( A \) at \( p \) and whose associated eigenvalues are \( \lambda_1, \ldots, \lambda_n \), respectively. Denote by \( \{e_1, \ldots, e_n\} \) the geodesic frame that extends the above basis to a neighborhood of \( p \) in \( x(M^n) \). We may do this on a neighborhood of \( p \) in \( M^r_{o+1} \) in such a way that \( D_N e_i(p) = 0 \). We now denote by \( (\cdot) \) the Riemannian metric of \( M^r_{o+1} \) and use the short notation \( \mathcal{L}_{i,N} = (\mathcal{L}_V(\cdot))(N, N) \), \( \mathcal{L}_{i,i} = (\mathcal{L}_V(\cdot))(e_i, e_i) \) and \( \mathcal{L}_{i,N} = (\mathcal{L}_V(\cdot))(e_i, N) \) for the Lie derivatives. First of all we notice that
\[
e_i e_i(g) = \langle D_{e_i} D_{e_i} N, V \rangle + 2 \langle D_{e_i} N, D_{e_i} V \rangle + \langle D_{e_i} D_{e_i} V, N \rangle.
\]

Because \(-D_{e_i} N(p) = A(e_i(p)) = \lambda_i e_i(p)\), we have \( \langle D_{e_i} N, D_{e_i} V \rangle(p) = -\frac{\lambda_i}{2} \mathcal{L}_{i,i}(p) \), which implies that
\[
e_i e_i(g)(p) = \langle D_{e_i} D_{e_i} N, V \rangle(p) + \langle D_{e_i} D_{e_i} V, N \rangle(p) + \lambda_i \mathcal{L}_{i,i}(p).
\]
We differentiate \( \langle D_{e_i} V, N \rangle + \langle D_N V, e_i \rangle = \mathcal{L}_{i,N} \) in the direction \( e_i \) to obtain the following identity:
\[
\langle D_{e_i} D_{e_i} V, N \rangle + \langle D_{e_i} V, D_{e_i} N \rangle + \langle D_{e_i} D_N V, e_i \rangle + \langle D_N V, D_{e_i} e_i \rangle = \lambda_i (\mathcal{L}_{i,N}).
\]

Using that \( \{e_1, \ldots, e_n\} \) is a geodesic frame, one gets \( (D_{e_i} e_j(p))^\top = 0 \), from which we have \( D_{e_i} e_i(p) = \lambda_i N(p) \). This yields
\[
\langle D_N V, D_{e_i} e_i \rangle(p) = \lambda_i \langle D_N V, N \rangle(p) = \lambda_i \frac{\mathcal{L}_{i,N}(p)}{2}.
\]
Therefore, we have at \( p \),
\[
\langle D_{e_i} D_{e_i} V, N \rangle = e_i (\mathcal{L}_{i,N}) + \frac{\lambda_i}{2} \mathcal{L}_{i,i} - \frac{\lambda_i}{2} \mathcal{L}_{i,N} - \langle D_{e_i} D_N V, e_i \rangle.
\]
On the other hand \([N, e_i](p) = D_N e_i(p) - D_{e_i} N(p) = -\lambda_i e_i(p), \) since \(D_N e_i(p) = 0.\) Thus we infer that
\[
\langle D_{[N,e_i]} V, e_i \rangle(p) = \lambda_i \langle D_{e_i} V, e_i \rangle(p) = \frac{\lambda_i}{2} \mathcal{L}_{i,i}(p).
\]

We now differentiate both sides of \(\langle D_{e_i} V, e_i \rangle = \frac{1}{2} \mathcal{L}_{i,i} \) with respect to \(N\) to obtain
\[
\langle D_N D_{e_i} V, e_i \rangle + \langle D_{e_i} V, D_N e_i \rangle = \frac{1}{2} N (\mathcal{L}_{i,i}).
\]

Then \(\langle D_N D_{e_i} V, e_i \rangle(p) = \frac{1}{2} N (\mathcal{L}_{i,i})(p).\) But the definition of curvature yields
\[
(3.3) \quad \langle R(N, e_i) V, e_i \rangle(p) = \langle D_{e_i} D_N V, e_i \rangle(p) - \frac{1}{2} N (\mathcal{L}_{i,i})(p) + \frac{\lambda_i}{2} \mathcal{L}_{i,i}(p).
\]

We now substitute (3.3) in (3.2) and then substitute the result in (3.1) to obtain
\[
\sum_{i=1}^n \lambda_i = \sum_{j=1}^n e_j e_i (g)(p) = -\langle R(N, e_i) V, e_i \rangle(p) + \langle D_{e_i} D_N N, V \rangle(p) + \langle e_i (\mathcal{L}_{j,N}) - \frac{1}{2} N (\mathcal{L}_{j,i})(p) - \frac{\lambda_i}{2} \mathcal{L}_{j,N,N}(p).
\]

Writing \(V = \sum_{j=1}^n v_j e_j + g N,\) where \(v_j = \langle V, e_j \rangle,\) we have
\[
\langle D_{e_i} D_{e_i} N, V \rangle = \langle D_{e_i} D_{e_i} N, \sum_{j=1}^n v_j e_j + g N \rangle
= \sum_{j=1}^n v_j \langle D_{e_i} D_{e_i} N, e_j \rangle + g \langle D_{e_i} D_{e_i} N, N \rangle.
\]

Taking the second covariant derivative of \(\langle N, N \rangle\) in the direction \(e_i,\) we obtain \(\langle D_{e_i} D_{e_i} N, N \rangle(p) = -\lambda_i^2.\) We do the same for \(\langle N, e_j \rangle\) to obtain
\[
\langle D_{e_i} D_{e_i} N, e_j \rangle + 2 \langle D_{e_i} N, D_{e_i} e_j \rangle + \langle N, D_{e_i} D_{e_i} e_j \rangle = 0.
\]

Therefore
\[
(3.4) \quad \langle D_{e_i} D_{e_i} N, e_j \rangle(p) = -\langle D_{e_i} D_{e_i} e_j, N \rangle(p).
\]

Because \(\langle D_{e_i} e_j, N \rangle = \langle D_{e_i} e_i, N \rangle,\) we differentiate both sides in the direction \(e_i\) to find
\[
\langle D_{e_i} D_{e_i} e_j, N \rangle + \langle D_{e_i} e_j, D_{e_i} N \rangle = \langle D_{e_i} D_{e_i} e_i, N \rangle + \langle D_{e_i} e_i, D_{e_i} N \rangle.
\]

This equality, together with (3.4), allows us to write
\[
-\langle D_{e_i} D_{e_i} N, e_j \rangle(p) = \langle D_{e_i} D_{e_i} e_j, N \rangle(p) = \langle D_{e_j} D_{e_i} e_i, N \rangle(p).
\]

Again using the curvature tensor as well as \([e_i, e_j](p) = 0\) and to \(e_j \langle D_{e_j} e_i, N \rangle(p) = \langle D_{e_j} D_{e_j} e_i, N \rangle(p) + \langle D_{e_i} D_{e_j} e_i, N \rangle(p),\) we conclude that
\[
\langle D_{e_i} D_{e_i} N, e_j \rangle(p) = \langle R(e_i, e_j) e_i, e_j, N \rangle(p) - \langle D_{e_j} D_{e_i} e_i, N \rangle(p)
= \langle R(e_i, e_j) e_i, N \rangle(p) - e_j \langle D_{e_i} e_i, N \rangle(p).
\]
From this we arrive at

\[
\langle D_{e_i} D_{e_i} N, V \rangle (p) = \sum_{j=1}^{n} v_j (\langle R(e_i, e_j) e_i, N \rangle (p) - e_j \langle D_{e_i} e_i, N \rangle (p)) - g(p) \lambda_i^2 \\
= \sum_{j=1}^{n} v_j \langle R(e_i, e_j) e_i, N \rangle (p) - \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N \rangle (p) - g(p) \lambda_i^2 \\
= \langle R(e_i, \sum_{j=1}^{n} v_j e_j) e_i, N \rangle (p) - \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N \rangle (p) - g(p) \lambda_i^2 \\
= \langle R(e_i, V - gN) e_i, N \rangle (p) - \sum_{j=1}^{n} v_j E_j \langle D_{e_i} e_i, N \rangle (p) - g(p) \lambda_i^2 \\
= \langle R(e_i, V) e_i, N \rangle (p) - g(p) \langle R(e_i, N) e_i, N \rangle (p) - g(p) \lambda_i^2 \\
- \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N \rangle (p),
\]

which yields

\[
\langle D_{e_i} D_{e_i} N, V \rangle (p) - \langle R(e_i, V) e_i, N \rangle (p) = -g(p) \langle R(e_i, N) e_i, N \rangle (p) - g(p) \lambda_i^2 \\
- \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N \rangle (p).
\]

We now make use of the identity obtained for \(e_i e_i (g)\) to conclude that

\[
e_i e_i (g) (p) = e_i (L_{i,N}) (p) + \frac{1}{2} N (L_{i,i}) (p) - \frac{\lambda_i}{2} L_{i,N} (p) - g(p) \lambda_i^2 \\
- g(p) \langle R(e_i, N) e_i, N \rangle (p) - \sum_{i=1}^{n} \langle V, e_i \rangle \langle e_i (A(e_i), e_i) \rangle (p).
\]

It is convenient to note that

\[
L_{r} (g) (p) = \text{tr} (P_r \circ Hess_g) (p) = \sum_{i=1}^{n} \langle P_r Hess_g (e_i), e_i \rangle (p) \\
= \sum_{i=1}^{n} \langle P_r \langle D_{e_i}, \nabla g \rangle, e_i \rangle (p) = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} \langle D_{e_i}, \nabla g, e_i \rangle (p).
\]

Hence, writing \(\nabla g = \sum_{i=1}^{n} g_j e_j\), we find

\[
D_{e_i} \nabla g (p) = \sum_{j=1}^{n} (g_j e_j + g_j D_{e_i} e_j) (p) = \sum_{j=1}^{n} g_j e_j (p),
\]

which gives us \(L_{r} (g) (p) = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} g_i (p)\).
Moreover, taking into account that the curvature of a space form $M^{n+1}$ is given by $R(X, Y) = c(X \wedge Y)$ and that $(L_X \langle \cdot, \cdot \rangle)(X, Y) = 2\psi(X, Y)$, we deduce that

\[ L_r(g)(p) = -\psi(p) \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} - N(\psi)(p) \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} - cg(p) \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} - g(p) \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i}\]

\[ - \sum_{i=1}^{n} \langle V, e_i \rangle (e_i \langle A(e_i), e_i \rangle)(p) \]

\[ = -(r+1)S_{r+1}\psi(p) - (n-r)S_rN(\psi)(p) - c(n-r)S_rg(p) \]

\[ - (S_1S_{r+1} - (r+2)S_{r+2})g(p) - I. \]

We now notice that

\[ tr \left(P, DX \right)(p) = \sum_{i=1}^{n} \langle (P_rDX)(e_i), e_i \rangle(p) \]

\[ = \sum_{i=1}^{n} \langle P_r(DX)(e_i), e_i \rangle(p) \]

\[ = \sum_{i=1}^{n} \langle DX(e_i), P_r(e_i) \rangle(p) \]

\[ = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} \langle DX(e_i) - A(DX(e_i), e_i) \rangle(p). \]

On the other hand, letting $X = e_i$ in the above equality and using $D_{e_i}e_i(p) = 0$, one gets

\[ \langle e_i, \nabla S_{r+1} \rangle(p) = tr \left(P, DX \right)(p) = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} \langle D_{e_i}A(e_i), e_i \rangle(p). \]

Finally we compute

\[ e_i \langle A(e_i), e_i \rangle(p) = \langle D_{e_i}A(e_i), e_i \rangle(p) + \langle A(e_i), D_{e_i}e_i \rangle(p) + \langle D_{e_i}A(e_i), e_i \rangle(p) \]

to deduce that

\[ I = \sum_{i=1}^{n} \langle V, e_i \rangle \langle e_i, \nabla S_{r+1} \rangle(p) = \langle V, \nabla S_{r+1} \rangle(p). \]

From the above one gets

\[ L_r(g) = -(S_1S_{r+1} - (r+2)S_{r+2})g - c(n-r)S_rg \]

\[ - (n-r)S_rN(\psi) - (r+1)S_{r+1}\psi - \langle V, \nabla S_{r+1} \rangle, \]

which concludes the proof of the theorem.

Now we combine Corollary 2.3 and formula (8.4) of Alias et al. [3] to derive the following lemma.

**Lemma 3.2.** Let $x : M^n \hookrightarrow M^{n+1}$ be an oriented isometric immersion with unit normal vector field $N$. If $V$ is a conformal vector field on $M^{n+1}$ with conformal factor $\psi$ and $g = \langle V, N \rangle$ represents the support function on $M^n$, then

\[ div \left(P \langle V^T \rangle \right) = (n-r)S_r\psi + (r+1)S_{r+1}g. \]

Putting together Theorem [3.1] and Lemma [3.2] we derive the next corollary.
Corollary 3.3. Let $x : M^n \rightarrow M^{n+1}_r$ be an oriented isometric immersion with $S_{r+1}$ constant. If $V = s(d)\nabla d$, $g = \langle V, N \rangle$ and $W = (n-r-1)P_v \nabla g + (r+1)P_{r+1}(V^T)$, then
\[
\text{div}(W) = -n(r+2) \left( \frac{n}{r+2} \right) (H_1 H_{r+1} - H_{r+2}) g,
\]
where $N$ is a unit normal vector field to $M^n$ and $d$ is the distance on $M^{n+1}_r$ to a fixed point $p_0$.

Proof. First we notice that $V = s(d)\nabla d$ is a conformal vector field with factor $\psi = s'(d)$. Moreover, $N(\psi) = -cg$. On the other hand, since $S_{r+1}$ is constant, Theorem 3.1 yields
\[
(n-r-1)L_r(g) = -(n-r-1)(r+1)S_{r+1}s'(d) - (n-r-1)(S_1 S_{r+1} - (r+2)S_{r+2}) g.
\]

We now use Lemma 3.2 to find
\[
(r+1)\text{div} \left( P_{r+1}(V^T) \right) = (n-r-1)(r+1)S_{r+1}s'(d) + (r+1)(r+2)S_{r+2} g.
\]

Therefore, using $L_r(g) = \text{div} (P_v \nabla g)$ and $W = (n-r-1)P_v \nabla g + (r+1)P_{r+1}(V^T)$, we infer that
\[
\text{div}(W) = -(n-r-1)S_1 S_{r+1} g + n(r+2)S_{r+2} g.
\]

In order to deduce the desired result it is enough to notice that $S_r = \binom{n}{r} H_r$ and
\[
(n-r-1)n \binom{n}{r+1} = n(r+2) \binom{n}{r+2}.
\]

\[
\square
\]

4. Compact radial graphs

By a compact smooth radial graph $\Sigma^n \subset \mathbb{R}^{n+1}$ we mean a differentiable graph whose domain is a full Euclidean sphere $\mathbb{S}^n(r)$ of radius $r > 0$. In order to construct such a graph we fix a point $p_0 \in \mathbb{R}^{n+1}$ called the origin, which coincides with the center of the sphere, and for each direction $v \in T_{p_0} \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ we consider a point $p(v) \in \Sigma^n$ that corresponds to the end point of a nontrivial geodesic segment on $\mathbb{R}^{n+1}$ starting from $p_0$ in the direction of $v$. We also call this type of graph a smooth star-shaped hypersurface of the Euclidean space $\mathbb{R}^{n+1}$.

A similar construction can be done for a compact smooth radial graph $\Sigma^n \subset \mathbb{S}^{n+1}$. In fact, we fix a point $p_0 \in \mathbb{S}^{n+1}$ and for each direction $v \in T_{p_0} \mathbb{S}^{n+1}$ we consider a point $p(v) \in \Sigma^n$ that corresponds to the end point of a nontrivial geodesic segment on $\mathbb{S}^{n+1}$ starting from $p_0$ in the direction of $v$.

Now we consider a compact smooth radial graph $\Sigma^n \subset \mathbb{R}^{n+1}$ whose domain is a Euclidean sphere $\mathbb{S}^n(r)$ of radius $r > 0$. Next, we introduce local coordinates $u = (u_1, \ldots, u_n)$ and let $X(u)$ and $Y(u)$ be parametrizations of $\mathbb{S}^n(r)$ and $\Sigma^n$ respectively. If $\rho(u) = |Y(u)| > 0$, then $Y = \rho X$.

Let $f : \Sigma^n \rightarrow \mathbb{R}$ be the function defined by $f(Y) = \langle Y, N_Y \rangle$, where $N_Y$ is a unit vector field normal to $\Sigma^n$. Letting $\rho_i = h_i$, we have $Y_i = \rho X_i + \rho_i X$, from which we obtain
\[
\langle \rho X, \frac{Y_1 \wedge \ldots \wedge Y_n}{|Y_1 \wedge \ldots \wedge Y_n|} \rangle = \langle \rho X, \frac{(\rho X_1) \wedge \ldots \wedge (\rho X_n)}{|Y_1 \wedge \ldots \wedge Y_n|} \rangle.
\]
Hence we deduce that
\[ f(Y) = \langle \rho X, Y_1 \wedge \ldots \wedge Y_n \rangle = \rho^{n+1} \frac{|X_1 \wedge \ldots \wedge X_n|}{|Y_1 \wedge \ldots \wedge Y_n|} \langle X, N \rangle \]
\[ = -\frac{\rho^{n+1}}{r} \frac{|X_1 \wedge \ldots \wedge X_n|}{|Y_1 \wedge \ldots \wedge Y_n|} \langle X, X \rangle = -r \rho^{n+1} \frac{|X_1 \wedge \ldots \wedge X_n|}{|Y_1 \wedge \ldots \wedge Y_n|} < 0. \]

Here we are considering \( N = -\frac{1}{r} X \) as the unit normal vector field to \( S^n(r) \) in such a way that its mean curvature is positive.

Finally, given a compact smooth radial graph \( \Sigma \subset S^{n+1} \) as above, we consider the stereographic projection \( \pi : S^{n+1}\setminus\{p_0\} \to \mathbb{R}^{n+1} \) and let \( V_{g_{n+1}} \) be the position vector field with basis point \( p_0 \) in \( S^{n+1} \). Hence we have the next lemma.

**Lemma 4.1.** Under the above conditions the function \( g = \langle V_{g_{n+1}}, N \rangle \) has a well-defined sign.

**Proof.** Let \( X \) and \( Y \) be parametrizations of \( S^n(r) \) and \( S^n \) respectively. Then \( \pi(X) \) is a parametrization of a sphere on \( \mathbb{R}^{n+1} \) while \( \pi(Y) \) is a parametrization of a smooth radial graph over \( \pi(X) \) on \( \mathbb{R}^{n+1} \).

Let \( g_1, g_2 : \pi(Y) \to \mathbb{R} \) be continuous functions such that either \( g_1 > 0 \) or \( g_1 < 0 \), \( d\pi(V_{g_{n+1}}) = g_1 V_{g_{n+1}} \) and \( d\pi(N) = g_2 N \). If \( e^\phi \) stands for the conformal factor of \( \pi \), then we obtain
\[ e^{2\phi} \langle V_{g_{n+1}}, N \rangle = \langle d\pi(V_{g_{n+1}}), d\pi(N) \rangle = \langle g_1 V_{g_{n+1}}, g_2 N \rangle > 0 \text{ (or } < 0 \text{),} \]
from which we derive the desired result. \( \square \)

5. **Proof of Theorem 1.1**

**Proof.** Letting \( r = 1 \) in Corollary 3.3 and integrating \( \text{div} W \) over \( \Sigma \), we obtain
\[ \int_{\Sigma} (H_1 H_2 - H_3) g d\Sigma = 0. \tag{5.1} \]

On the other hand, it is well known that for immersions \( M^n \hookrightarrow M^{n+1}_c \),
\[ H_r^2 \geq H_{r-1} H_{r+1}, \quad \forall \ r \in \{1, \ldots, n-1\}, \tag{5.2} \]
with equality occurring if and only if \( M \) is totally umbilical; see [6] or [1] for more details.

Taking into account that \( S^2 = |A|^2 + 2S_2 \), we see that \( H_1^2 > 0 \). Hence we may choose the orientation for \( \Sigma \) in such a way that \( H_1 > 0 \). Since \( H_0 = 1 \) we obtain \( H_1^2 \geq H_2^2 > 0 \). Now letting \( r = 2 \) on (5.2) we have \( -H_1 H_3 \geq -H_2^2 \), from which we obtain
\[ H_1 (H_1 H_2 - H_3) \geq H_2 (H_1^2 - H_2) \geq 0. \tag{5.3} \]

Since \( g \) has a well-defined sign, we make use of equations (5.1) and (5.3) to arrive at \( H_1 H_2 - H_3 = 0 \). Thus taking this into account in inequality (5.3), we find that \( H_2^2 = H_2 = 0 \). Hence we deduce that \( x(M^n) \) is totally umbilical, which finishes the proof of the theorem. \( \square \)

For a compact smooth radial graph \( \Sigma \subset S^{n+1} \) of non-null constant mean curvature, we proved in [5] that \( \Sigma \) is a round sphere. However, we avoided minimal graphs. In fact, this is easily extended to the minimal case according to the next theorem.
Theorem 5.1. Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a compact smooth star-shaped minimal hypersurface. Then $\Sigma^n$ is totally geodesic.

Proof. Referring to Lemma 3.2, we have $\text{div} \left( P_1(V_T) \right) = 2S_2 g$. This implies that

$$\int_{\Sigma} S_2 g d\Sigma = 0. \tag{1}$$

Since $2S_2 = -|A|^2 \leq 0$ and $g$ does not change sign, we get $S_1 = S_2 = 0$. Therefore, we deduce that $|A|^2 = 0$, which yields that $\Sigma^n$ is totally geodesic. \hfill \Box

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