LIPSCHITZ \( p \)-SUMMING OPERATORS

JEFFREY D. FARMER AND WILLIAM B. JOHNSON

(Communicated by Marius Junge)

Abstract. The notion of Lipschitz \( p \)-summing operator is introduced. A nonlinear Pietsch factorization theorem is proved for such operators, and it is shown that a Lipschitz \( p \)-summing operator that is linear is a \( p \)-summing operator in the usual sense.

1. Introduction

In this note we introduce a natural nonlinear version of a \( p \)-summing operator, which we call a Lipschitz \( p \)-summing operator. In section 2 we prove a nonlinear version of the Pietsch factorization theorem, show by example that the strong form of the Pietsch domination theorem is not true for Lipschitz \( p \)-summing operators, and make a few other remarks about these operators. In section 3 we “justify” our nomenclature by proving that for a linear operator, the Lipschitz \( p \)-summing norm is the same as the usual \( p \)-summing norm. Finally, in section 4 we raise some problems which we think are interesting.

2. Pietsch factorization

The Lipschitz \( p \)-summing \((1 \leq p < \infty)\) norm, \( \pi^L_p(T) \), of a (possibly nonlinear) mapping \( T: X \to Y \) between metric spaces is the smallest constant \( C \) so that for all \((x_i), (y_i)\) in \( X \) and all positive reals \( a_i \),

\[
\sum a_i \|Tx_i - Ty_i\|^p \leq C^p \sup_{f \in B_{X\#}} \sum a_i |f(x_i) - f(y_i)|^p.
\]

Here \( B_{X\#} \) is the unit ball of \( X\# \), the Lipschitz dual of \( X \); i.e., \( X\# \) is the space of all real valued Lipschitz functions under the (semi)-norm \( \text{Lip}(\cdot) \); and \( \|x - y\| \) is the distance from \( x \) to \( y \) in \( Y \). We follow the usual convention of considering \( X \) as a pointed metric space by designating a special point \( 0 \in X \) and identifying \( X\# \) with the Lipschitz functions on \( X \) that are zero at \( 0 \). With this convention \((X\#, \text{Lip}(\cdot))\) is a Banach space and \( B_{X\#} \) is a compact Hausdorff space in the topology of pointwise convergence on \( X \).

Notice that the definition is the same if we restrict to \( a_i = 1 \). Indeed, by approximation it is enough to consider rational \( a_i \) and thus, by clearing denominators, integer \( a_i \). Then, given \( a_i, x_i, \) and \( y_i \), consider the new collection of vectors in which...
the pair \((x_i, y_i)\) is repeated \(a_i\) times. (This observation was made with M. Mendel and G. Schechtman.)

It is clear that \(\pi_p^L\) has the ideal property; i.e., \(\pi_p^L(ATB) \leq \text{Lip}(A)\pi_p^L(T)\text{Lip}(B)\) whenever the compositions make sense. Also, if \(Y\) is a Banach space, the space of Lipschitz \(p\)-summing maps from any metric space into \(Y\) is a Banach space under the norm \(\pi_p^L\).

If \(T\) is a linear operator, it is clear that \(\pi_p^L(T) \leq \pi_p(T)\), where \(\pi_p(\cdot)\) is the usual \(p\)-summing norm [5, p. 31]. In section 3 we prove that the reverse inequality is true.

We begin with a Pietsch factorization theorem for Lipschitz \(p\)-summing operators.

**Theorem 1.** The following are equivalent for a mapping \(T: X \rightarrow Y\) between metric spaces and \(C \geq 0\).

1. \(\pi_p^L(T) \leq C\).
2. There is a probability \(\mu\) on \(B_{X^\#}\) such that
   \[\|Tx - Ty\|^p \leq C^p \int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f)\]
   (Pietsch domination).
3. For some (or any) isometric embedding \(J\) of \(Y\) into a 1-injective space \(Z\), there is a factorization
   \[L_{\infty}(\mu) \xrightarrow{L_{\infty,p}} L_p(\mu) \xrightarrow{\downarrow B} X \xrightarrow{T} Y \xrightarrow{J} Z\]
   with \(\mu\) a probability and \(\text{Lip}(A) \cdot \text{Lip}(B) \leq C\) (Pietsch factorization).

**Proof.** That (2) implies (3) is basically obvious: Let \(A: X \rightarrow L_{\infty}(\mu)\) be the natural isometric embedding composed with the formal identity from \(C(B_{X^\#})\) into \(L_{\infty}(\mu)\). Then (2) says that the Lipschitz norm of \(B\) restricted to \(I_{\infty,p}AX\) is bounded by \(C\), which is just (3). We have used implicitly the well-known fact that every metric space embeds into \(\ell_{\infty}(\Gamma)\) for some set \(\Gamma\) and that, by the nonlinear Hahn-Banach theorem, \(\ell_{\infty}(\Gamma)\) is 1-injective. See Lemma 1.1 in [3].

For (3) implies (1), use
\[
\pi_p^L(T) = \pi_p^L(JT) \leq \text{Lip}(A)\pi_p^L(I_{\infty,p})\text{Lip}(B) \leq \text{Lip}(A)\pi_p(I_{\infty,p})\text{Lip}(B)
= \text{Lip}(A)\text{Lip}(B).
\]

The proof of the main implication, that (1) implies (2), is like the proof of the (linear) Pietsch factorization theorem (see, e.g., [5, p. 44]). Suppose \(\pi_p^L(T) = 1\). Let \(Q\) be the convex cone in \(C(B_{X^\#})\) consisting of all positive linear combinations of functions of the form \(\|Tx - Ty\| - C\|f(x) - f(y)\|^p\), as \(x\) and \(y\) range over \(X\). Condition (1) says that \(Q\) is disjoint from the positive cone \(P = \{F \in C(B_{X^\#}) \mid F(f) > 0 \forall f \in X^\#\}\), which is an open convex subset of \(C(B_{X^\#})\). Thus by the separation theorem and the Riesz representation theorem there is a finite signed Baire measure \(\mu\) on \(B_{X^\#}\) and a real number \(c\) so that for all \(G \in Q\) and \(F \in P\), \(\int_{X^\#} G \, d\mu \leq c < \int_{X^\#} F \, d\mu\). Since \(0 \in Q\) and all positive constants are in \(P\), we see that \(c = 0\), and since \(\int_{X^\#} \cdot \, d\mu\) is positive on the positive cone \(P\) of \(C(B_{X^\#})\),
the signed measure $\mu$ is a positive measure, which we can assume by rescaling is a probability measure. It is clear that the inequality in (2) is satisfied.

It is worth noting that the conditions in Theorem 1 are also equivalent to

(4) There is a probability $\mu$ on $K$, the closure in the topology of pointwise convergence on $X$ of the extreme points of $B_{X^*}$, such that

$$\|Tx - Ty\|^p \leq C^p \int_K |f(x) - f(y)|^p \, d\mu(f).$$

The proof that (1) implies (4) is the same as the proof that (1) implies (2) since the supremum on the right side of (2.1), the definition of the Lipschitz $p$-summing norm, is the same as

$$\sup_{f \in K} \sum_{i} a_i |f(x_i) - f(y_i)|^p.$$

One immediate consequence of Theorem 1 is that $\pi^p_p(T)$ is a monotonely decreasing function of $p$. Another consequence is that there is a version of Grothendieck’s theorem (that every linear operator from an $L_1$ space to a Hilbert space is 1-absolutely summing). In the category of metric spaces with Lipschitz mappings as morphisms, weighted trees play a role analogous to that of $L_1$ in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if $X$ is a finite weighted tree, $T: X \to Y$ is a Lipschitz mapping from $X$ into a metric space $Y$, and $Q: Z \to Y$ is a 1-Lipschitz quotient mapping in the sense of [2, 7], then for each $\varepsilon > 0$ there is a mapping $S: X \to Z$ so that $\operatorname{Lip}(S) \leq \operatorname{Lip}(T) + \varepsilon$ and $T = QS$. Letting $Y$ be a Hilbert space and $Z$ an $L_1$ space, we see from Grothendieck’s theorem and the ideal property of $\pi^p_1$ that if every finite subset of $X$ is a weighted tree (in particular, if $X$ is a tree or a metric tree; see [7]), then $\pi^p_1(T) \leq K_G \operatorname{Lip}(T)$, where $K_G$ is Grothendieck’s constant. Here we use the obvious fact that $\pi^p_p(T: X \to Y)$ is the supremum of $\pi^p_p(T|_K)$ as $K$ ranges over finite subsets of $X$.

The strong form of the Pietsch domination theorem says that if $X$ is a subspace of $C(K)$ for some compact Hausdorff space $K$, and if $T$ is a $p$-summing linear operator with domain $X$, then there is a probability measure $\mu$ on $K$ so that for all $x \in X$, $\|Tx\|^p \leq \pi^p_p(T)^p \int_K |x(t)|^p \, d\mu(t)$. It is easy to see that there is not a nonlinear version of this result. Let $D_n$ be the discrete metric space with $n$ points so that the distance between any two distinct points is one. We can embed $D_n$ into $C(\{-1,1\}^n)$ in two essentially different ways. First, if $D_n = \{x_1, \ldots, x_n\}$, let $f(x_k) = \frac{1}{2} r_k$, where $r_k$ is the projection onto the $k$th coordinate. The image of this set under the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \mu)$ with $\mu$ the uniform probability on $\{-1,1\}^n$ is a discrete set with the $p$-th power of all distances one-half. This shows that the identity on $D_n$ has Lipschitz $p$-summing norm at most two. Secondly, let $g(k)$, $1 \leq k \leq n$, be disjointly supported unit vectors in $C(\{-1,1\}^n)$. Then for any probability measure $\nu$ on $\{-1,1\}^n$, the injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \nu)$ shrinks the distance between some pair of the $g(k)$’s to at most $(2/n)^{1/p}$.

Incidentally, $\pi^p_p(I_{D_n})$ tends to $2^{p\frac{1}{p}}$ as $n \to \infty$ and can be computed exactly. To see this, note that the extreme points, $K_n$, of $B_{D^n}$ are of the form $\pm \chi_A$ with $A$ a nonempty subset of $D_n \sim \{0\}$. This can be calculated directly or deduced from Theorem 1 in [6]. We calculate $\pi^p_p(I_{D_n})$ in the (easier) case that $n$ is even. Define
a probability \( \mu \) on \( K_n \) by letting \( \mu \) be the uniform measure on \( J_{n/2} := \{ \chi_A : |A| = n/2, A \subset D_n \sim \{0\} \} \) (so that \( \mu(e) = 0 \) for elements \( e \) of \( K_n \sim J_{n/2} \)). Then for each pair of distinct points \( x \) and \( y \) in \( D_n \), \( \int_{K_n} |f(x) - f(y)|^p \, d\mu(f) = \frac{n}{2^n} \), so that \( \pi_p^\mu(I_{D_n}) \leq (2 - \frac{2}{n})^\frac{1}{p} \). To see that \( \mu \) is a Pietsch measure for \( I_{D_n} \), let \( \nu \) be any Pietsch probability for \( I_{D_n} \) on \( K_n \). We can clearly assume that \( \nu \) is supported on the positive elements in \( K_n \). By averaging \( \nu \) against the permutations of \( D_n \) which fix 0, which is a group of isometries on \( D_n \), we get another Pietsch probability for \( I_{D_n} \) (which we continue to denote by \( \nu \)) so that if we condition \( \nu \) on \( J_k := \{ \chi_A : |A| = k, A \subset D_n \sim \{0\} \}, 1 \leq k \leq n - 1 \), the resulting probability \( \nu_k \) on \( J_k \) is the uniform probability. A trivial calculation shows that for \( x, y \) in \( D_n \sim \{0\} \), \( \int_{J_k} |f(x) - f(y)|^p \, d\nu_k(f) \leq \frac{n}{2^n} \). This proves that \( \mu \) is a Pietsch measure for \( I_{D_n} \) and hence \( \pi_p^\mu(I_{D_n}) = (2 - \frac{2}{n})^\frac{1}{p} \).

Our final comment on Lipschitz 1-summing operators is that the concept has appeared in the literature even if the definition is new. In [4], Bourgain proved that every \( n \)-point metric space can be embedded into a Hilbert space with distortion at most \( C \log n \), where \( C \) is an absolute constant. In fact, he really proved the much stronger result that \( \pi_1^\mu(I_X) \leq C \log n \) if \( I_X \) is the identity mapping on an \( n \)-point space \( X \) by making use of a special embedding of \( X \) into a space \( C(K_X) \) with \( K_X \) a finite metric space and constructing a probability on \( K_X \). Moreover, Bourgain’s construction has occasionally been used in the computer science literature. The strong form of Bourgain’s theorem is also used in [8] to prove an inequality that is valid for all metric spaces.

3. Linear operators

In this section we show that the Lipschitz \( p \)-summing norm of a linear operator is the same as its \( p \)-summing norm. This justifies that the notion of Lipschitz \( p \)-summing operator is really a generalization of the concept of linear \( p \)-summing operator.

**Theorem 2.** Let \( u \) be a bounded linear operator from \( X \) into \( Y \) and \( 1 \leq p < \infty \). Then \( \pi_p^u(u) = \pi_p(u) \).

**Proof.** Note that we can assume, without loss of generality, that dim \( Y \leq \text{dim} \, X = N < \infty \). Indeed, it is clear from the definition that \( \pi_p^u(u) \) is the supremum of \( \pi_p^u(u_E) \) as \( E \) ranges over finite dimensional subspaces of \( X \) and similarly for \( \pi_p^u(u) \). That we can assume dim \( Y \leq \text{dim} \, X \) is clear from the linearity of \( u \).

Since dim \( Y \leq N \), there is an embedding \( J \) of \( Y \) into \( \ell^m_\infty \) with \( m \leq (\frac{3}{2})^N \) so that \( \|J\| = 1 \) and \( \|J^{-1}\| \leq 1 + \varepsilon \). We then get the following nonlinear Pietsch factorization:

\[
\begin{align*}
L_\infty(\mu) & \xrightarrow{\alpha} L_p(\mu) \\
X & \xrightarrow{u} Y & J & \xrightarrow{} \ell^m_\infty
\end{align*}
\]

where Lip(\( \alpha \)) = 1, Lip(\( \beta \)) \leq \( \pi_p^u(Ju) \leq \pi_p^u(u) \). We can also assume, without loss of generality, that the probability \( \mu \) is a separable measure.

We now use some nonlinear theory that can be found in the book [3].

1. The mapping \( \alpha \) is weak* differentiable almost everywhere. This means that for (Lebesgue) almost every \( x_0 \) in \( X \), there is a linear operator
\[
D_{x_0}^\alpha(\ell^m_p) : X \to L_\infty(\mu) \text{ such that for all } f \in L_1(\mu) \text{ and for every } y \in X,
\lim_{t \to 0} \left\langle \frac{\alpha(x_0 + ty) - \alpha(x_0)}{t}, f \right\rangle = (D_{x_0}^\alpha(\ell^m_p)(y), f).
\]

(2) The operator \( i_{\infty,p} \alpha \) is differentiable almost everywhere. This means that
for almost every \( x_0 \) in \( X \), there is a linear operator \( D_{x_0}(i_{\infty,p} \alpha) : X \to L_\mu(\mu) \)
such that
\[
\sup_{\|y\| \leq 1} \left\| i_{\infty,p} \alpha(x_0 + ty) - i_{\infty,p} \alpha(x_0) - D_{x_0}(i_{\infty,p} \alpha)(y) \right\|_p \to 0 \quad \text{as } t \to 0.
\]

When \( 1 < p < \infty \), statement (2) follows from the reflexivity of \( L_p \) (see [3, Corollary 5.12 & Proposition 6.1]). For \( p = 1 \), just use (2) for \( p = 2 \) and compose with \( i_{2,1} \).

The mapping \( i_{\infty,p} \) is weak* to weak continuous, so \( D_{x_0}(i_{\infty,p} \alpha) = i_{\infty,p} D_{x_0}^\alpha(\alpha) \)
whenever both derivatives exist. Since they both exist almost everywhere, by making several translations we can assume without loss of generality that this equation is true for \( x_0 = 0 \) and also that \( \alpha(0) = 0 \).

Next we show that in the factorization diagram the nonlinear map \( \alpha \) can be replaced by the linear operator \( D_0^\alpha(\alpha) \) by constructing a mapping \( \beta : L_\mu(\mu) \to \ell^\infty_\mu \)
so that \( \beta i_{\infty,p} D_0^\alpha(\alpha) = Ju \) and \( \text{Lip}(\beta) \leq \text{Lip}(\beta) \). To do this, define \( \beta_n : L_\mu(\mu) \to \ell^m_n \) by \( \beta_n(x) := n\beta(\frac{x}{n}) \) and note that \( \text{Lip}(\beta_n) = \text{Lip}(\beta) \). We have for each \( x \) in \( X \),
\[
\|Ju(x) - \beta_n i_{\infty,p} D_0^\alpha(\alpha)(x)\| = \|\beta_n n i_{\infty,p} \alpha(x/n) - \beta_n D_0(i_{\infty,p} \alpha)(x)\| \\
\leq \text{Lip}(\beta) \|n i_{\infty,p} \alpha(x/n) - D_0(i_{\infty,p} \alpha)(x)\|,
\]
which tends to zero as \( n \to \infty \). For \( \beta \) we can take any cluster point of \( \beta_n \) in the space of functions from \( L_\mu(\mu) \) into \( \ell^m_\infty \); such points exist because \( \beta_n \) is uniformly Lipschitz and \( \beta_n(0) = 0 \).

Summarizing, we see that we have a factorization
\[
\begin{array}{c}
L_\infty(\mu) \\
\beta \uparrow \\
X \\
\downarrow u \\
Y \\
\downarrow J \\
\ell^\infty_\mu
\end{array}
\]
with \( \beta \) linear, \( \|\beta\| \leq \text{Lip}(\alpha) \), and \( \text{Lip}(\beta) \leq \text{Lip}(\beta) \).

The final step involves replacing \( \beta \) with a linear operator. Since the restriction \( \overline{\beta} \) of \( \beta \) to the linear subspace \( i_{\infty,p} \alpha[X] \) is linear and \( \ell^\infty_\mu \) is reflexive, this follows from [3, Theorem 7.2], which is proved by a simple invariant means argument. Alternatively, one can use the injectivity of \( \ell^m_n \) to extend \( \overline{\beta} \) to \( L_p(\mu) \). \( \square \)

4. Open problems and concluding remarks

**Problem 1.** Is there a composition formula for Lipschitz \( p \)-summing operators? That is, do we have \( \pi^L_p(TS) \leq \pi^L_p(T) \pi^L_p(S) \), when \( \frac{1}{p} \leq \frac{1}{q} + \frac{1}{s} \) and \( 1? \)

Say that a Lipschitz mapping \( T : X \to Y \) is Lipschitz \( p \)-integral if it satisfies a factorization diagram as in condition (3) of Theorem 1 except with \( J \) being the canonical isometry from \( Y \) into \( (Y^\#)^* \). We then define the Lipschitz \( p \)-integral
\[
\ell^p_p(T) \text{ of } T \text{ to be the infimum of } \text{Lip}(A) \cdot \text{Lip}(B), \text{ the infimum being taken over all such factorizations. When } T \text{ is a linear operator, this is the same as the usual } p \text{-integral norm of } T.
\]
Indeed, in this case one can use for \( J \) the canonical
isometry from $Y$ into $Y^{**}$ because $Y^{**}$ is norm one complemented in $(Y^#)^*$. Then
the proof that $I_p(T) \leq I_p^L(T)$ is identical to the proof of Theorem 2.

**Problem 2.** Is every Lipschitz 2-summing operator Lipschitz 2-integral?

In the case where the target space $Y$ is a Hilbert space, problem 2 has an affirmative answer by Kirszbraun’s theorem [3, p. 18]. If $Y$ has K. Ball’s Markov cotype 2 property [1], it follows from Ball’s work that the answer is still positive, although his result does not yield that $I_p^L(T)$ and $\pi_p^L(T)$ are equal. It is worth mentioning that the work of Naor, Peres, Schramm, and Sheffield [4] combines with Ball’s result to yield that for $2 \leq p < \infty$, every Lipschitz $p$-summing operator into $L_r$, $1 < r \leq 2$, is Lipschitz $p$-integral.

We mentioned in section 2 that $\Pi_p^L(X, Y)$, the class of Lipschitz $p$-summing operators from $X$ into $Y$, is a Banach space under the norm $\pi_p^L(\cdot)$ when $Y$ is a Banach space.

**Problem 3.** When $Y$ is a Banach space and $X$ is finite, what is the dual of $\Pi_p^L(X, Y)$?

In section 2, we noted that there is a version of Grothendieck’s theorem that is true in the nonlinear setting. Are there other versions? In particular, we ask the following:

**Problem 4.** Is every Lipschitz mapping from an $L_1$ space to a Hilbert space Lipschitz 1-summing? Is every Lipschitz mapping from a $C(K)$ space to a Hilbert space Lipschitz 2-summing?

It is elementary that for a linear operator $T: X \rightarrow Y$, $\pi_p(T)$ is the supremum of $\pi_p(TS)$ as $S$ ranges over all operators from $\ell_p'$ into $X$ of norm at most one. This leads us to ask:

**Problem 5.** If $T: X \rightarrow Y$ is Lipschitz, is $\pi_p^L(T)$ the supremum of $\pi_p^L(TS)$ as $S$ ranges over all mappings from finite subsets of $\ell_p'$ into $X$ having Lipschitz constant at most one?

Since all finite metric spaces embed isometrically into $\ell_\infty$, the answer to problem 5 is yes for $p = 1$.

Of course, all of the above problems are special cases of the general

**Problem 6.** What results about $p$-summing operators have analogues for Lipschitz $p$-summing operators?

*Added in proof.* Problem 3 has been solved by J. A. Chávez Domínguez (unpublished).

**References**


Department of Mathematics, University of Denver, Denver, Colorado 80208
E-mail address: jdfarmer89@hotmail.com

Department Mathematics, Texas A&M University, College Station, Texas 77843
E-mail address: johnson@math.tamu.edu