LIPSCHITZ $p$-SUMMING OPERATORS

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Abstract. The notion of Lipschitz $p$-summing operator is introduced. A nonlinear Pietsch factorization theorem is proved for such operators, and it is shown that a Lipschitz $p$-summing operator that is linear is a $p$-summing operator in the usual sense.

1. Introduction

In this note we introduce a natural nonlinear version of a $p$-summing operator, which we call a Lipschitz $p$-summing operator. In section 2 we prove a nonlinear version of the Pietsch factorization theorem, show by example that the strong form of the Pietsch domination theorem is not true for Lipschitz $p$-summing operators, and make a few other remarks about these operators. In section 3 we “justify” our nomenclature by proving that for a linear operator, the Lipschitz $p$-summing norm is the same as the usual $p$-summing norm. Finally, in section 4 we raise some problems which we think are interesting.

2. Pietsch factorization

The Lipschitz $p$-summing ($1 \leq p < \infty$) norm, $\pi_{p}^{L}(T)$, of a (possibly nonlinear) mapping $T : X \to Y$ between metric spaces is the smallest constant $C$ so that for all $x_{i}, y_{i}$ in $X$ and all positive reals $a_{i}$,

$$\sum_{i} a_{i} \|Tx_{i} - Ty_{i}\|^{p} \leq C^{p} \sup_{f \in B_{X^{#}}} \sum_{i} a_{i} |f(x_{i}) - f(y_{i})|^{p}.$$ (2.1)

Here $B_{X^{#}}$ is the unit ball of $X^{#}$, the Lipschitz dual of $X$; i.e., $X^{#}$ is the space of all real valued Lipschitz functions under the (semi)-norm $\text{Lip}(\cdot)$; and $\|x - y\|$ is the distance from $x$ to $y$ in $Y$. We follow the usual convention of considering $X$ as a pointed metric space by designating a special point $0 \in X$ and identifying $X^{#}$ with the Lipschitz functions on $X$ that are zero at $0$. With this convention ($X^{#}, \text{Lip}(\cdot)$) is a Banach space and $B_{X^{#}}$ is a compact Hausdorff space in the topology of pointwise convergence on $X$.

Notice that the definition is the same if we restrict to $a_{i} = 1$. Indeed, by approximation it is enough to consider rational $a_{i}$ and thus, by clearing denominators, integer $a_{i}$. Then, given $a_{i}, x_{i},$ and $y_{i}$, consider the new collection of vectors in which

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the pair \((x_i, y_i)\) is repeated \(a_i\) times. (This observation was made with M. Mendel and G. Schechtman.)

It is clear that \(\pi^L_p\) has the ideal property; i.e., \(\pi^L_p(AB) \leq \Lip(A)\pi^L_p(T)\Lip(B)\)
whenever the compositions make sense. Also, if \(Y\) is a Banach space, the space of Lipschitz \(p\)-summing maps from any metric space into \(Y\) is a Banach space under
the norm \(\pi^L_p\).

If \(T\) is a linear operator, it is clear that \(\pi^L_p(T) \leq \pi_p(T)\), where \(\pi_p(\cdot)\) is the usual \(p\)-summing norm [5, p. 31]. In section 3 we prove that the reverse inequality is true.

We begin with a Pietsch factorization theorem for Lipschitz \(p\)-summing operators.

**Theorem 1.** The following are equivalent for a mapping \(T: X \to Y\) between metric spaces and \(C \geq 0\).

1. \(\pi^L_p(T) \leq C\).
2. There is a probability \(\mu\) on \(B_{X^#}\) such that
   \[
   ||Tx - Ty||^p \leq C^p \int_{B_{X^#}} |f(x) - f(y)|^p \ d\mu(f)
   \]
   \((\text{Pietsch domination}).\)
3. For some (or any) isometric embedding \(J\) of \(Y\) into a 1-injective space \(Z\),
   there is a factorization
   \[
   \begin{array}{ccc}
   L_{\infty}(\mu) & \overset{I_{\infty,p}}{\longrightarrow} & L_p(\mu) \\
   A \uparrow & & \downarrow B \\
   X & \overset{T}{\longrightarrow} & Y & \overset{J}{\longrightarrow} & Z
   \end{array}
   \]
   with \(\mu\) a probability and \(\Lip(A) \cdot \Lip(B) \leq C\) (Pietsch factorization).

**Proof.** That (2) implies (3) is basically obvious: Let \(A: X \to L_{\infty}(\mu)\) be the natural isometric embedding composed with the formal identity from \(C(B_{X^#})\) into \(L_{\infty}(\mu)\).

Then (2) says that the Lipschitz norm of \(B\) restricted to \(I_{\infty,p}AX\) is bounded by \(C\),
which is just (3). (We have used implicitly the well known fact that every metric space embeds into \(\ell_{\infty}(\Gamma)\) for some set \(\Gamma\) and that, by the nonlinear Hahn-Banach theorem, \(\ell_{\infty}(\Gamma)\) is 1-injective. See Lemma 1.1 in [3].)

For (3) implies (1), use
\[
\pi^L_p(JT) = \pi^L_p(J)\pi^L_p(T) \leq \Lip(A)\pi^L_p(I_{\infty,p})\Lip(B) \leq \Lip(A)\pi_p(I_{\infty,p})\Lip(B.
\]

\[= \Lip(A)\Lip(B).\]

The proof of the main implication, that (1) implies (2), is like the proof of the (linear) Pietsch factorization theorem (see, e.g., [5] p. 44]). Suppose \(\pi^L_p(T) = 1\).
Let \(Q\) be the convex cone in \(C(B_{X^#})\) consisting of all positive linear combinations of functions of the form \(||Tx -Ty|| - C^p|f(x) - f(y)|^p\), as \(x\) and \(y\) range over \(X\). Condition (1) says that \(Q\) is disjoint from the positive cone \(P = \{ F \in C(B_{X^#}) \mid F(f) > 0 \ \forall f \in X^# \}\), which is an open convex subset of \(C(B_{X^#})\). Thus by the separation theorem and the Riesz representation theorem there is a finite signed Baire measure \(\mu\) on \(B_{X^#}\) and a real number \(c\) so that for all \(G \in Q\) and \(F \in P\),
\[
\int_{X^#} G \ d\mu \leq c < \int_{X^#} F \ d\mu.
\]
Since \(0 \in Q\) and all positive constants are in \(P\),
we see that \(c = 0\), and since \(\int_{X^#} \cdot \ d\mu\) is positive on the positive cone \(P\) of \(C(B_{X^#}),\)
the signed measure $\mu$ is a positive measure, which we can assume by rescaling is a probability measure. It is clear that the inequality in (2) is satisfied.

It is worth noting that the conditions in Theorem 1 are also equivalent to

(4) There is a probability $\mu$ on $K$, the closure in the topology of pointwise convergence on $X$ of the extreme points of $B_{X^*}$, such that

$$
\|Tx - Ty\|^p \leq C^p \int_K |f(x) - f(y)|^p d\mu(f).
$$

The proof that (1) implies (4) is the same as the proof that (1) implies (2) since the supremum on the right side of (2.1), the definition of the Lipschitz $p$-summing norm, is the same as

$$
\sup_{f \in K} \sum a_i |f(x_i) - f(y_i)|^p.
$$

One immediate consequence of Theorem 1 is that $\pi^p_L(T)$ is a monotonely decreasing function of $p$. Another consequence is that there is a version of Grothendieck’s theorem (that every linear operator from an $L_1$ space to a Hilbert space is 1-absolutely summing). In the category of metric spaces with Lipschitz mappings as morphisms, weighted trees play a role analogous to that of $L_1$ in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if $X$ is a finite weighted tree, $T: X \to Y$ is a Lipschitz mapping from $X$ into a metric space $Y$, and $Q: Z \to Y$ is a 1-Lipschitz quotient mapping in the sense of [2], [7], then for each $\varepsilon > 0$ there is a mapping $S: X \to Z$ so that $\text{Lip}(S) \leq \text{Lip}(T) + \varepsilon$ and $T = QS$. Letting $Y$ be a Hilbert space and $Z$ an $L_1$ space, we see from Grothendieck’s theorem and the ideal property of $\pi^1_L$ that if every finite subset of $X$ is a weighted tree (in particular, if $X$ is a tree or a metric tree; see [7]), then $\pi^1_L(T) \leq K_G \text{Lip}(T)$, where $K_G$ is Grothendieck’s constant. Here we use the obvious fact that $\pi^p_L(T: X \to Y)$ is the supremum of $\pi^p_L(T|_K)$ as $K$ ranges over finite subsets of $X$.

The strong form of the Pietsch domination theorem says that if $X$ is a subspace of $C(K)$ for some compact Hausdorff space $K$, and if $T$ is a $p$-summing linear operator with domain $X$, then there is a probability measure $\mu$ on $K$ so that for all $x \in X$, $\|Tx\|^p \leq \pi^p(T)^p \int_K |x(t)|^p d\mu(t)$. It is easy to see that there is not a nonlinear version of this result. Let $D_n$ be the discrete metric space with $n$ points so that the distance between any two distinct points is one. We can embed $D_n$ into $C(\{-1,1\}^n)$ in two essentially different ways. First, if $D_n = \{x_1, \ldots, x_n\}$, let $f(x_k) = \frac{1}{2}r_k$, where $r_k$ is the projection onto the $k$th coordinate. The image of this set under the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \mu)$ with $\mu$ the uniform probability on $\{-1,1\}^n$ is a discrete set with the $p$-th power of all distances one-half. This shows that the identity on $D_n$ has Lipschitz $p$-summing norm at most two. Secondly, let $g(k), 1 \leq k \leq n$, be disjointly supported unit vectors in $C(\{-1,1\}^n)$. Then for any probability measure $\nu$ on $\{-1,1\}^n$, the injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \nu)$ shrinks the distance between some pair of the $g(k)$’s to at most $(2/n)^{1/p}$.

Incidentally, $\pi^p_L(I_{D_n})$ tends to $2\pi^p$ as $n \to \infty$ and can be computed exactly. To see this, note that the extreme points, $K_n$, of $B_{D_n^2}$ are of the form $\pm \chi_A$ with $A$ a nonempty subset of $D_n \sim \{0\}$. This can be calculated directly or deduced from Theorem 1 in [6]. We calculate $\pi^p_L(I_{D_n})$ in the (easier) case that $n$ is even. Define
a probability $\mu$ on $K_n$ by letting $\mu$ be the uniform measure on $J_{n/2} := \{\chi_A: |A| = n/2, A \subset D_n \sim \{0\}\}$ (so that $\mu(e) = 0$ for elements $e$ of $K_n \sim J_{n/2}$). Then for each pair of distinct points $x$ and $y$ in $D_n$, $\int_{K_n} |f(x) - f(y)|^p \, d\mu(f) = \frac{n}{2(n-1)}$, so that $\pi_p^L(I_{D_n}) \leq (2 - \frac{2}{n})^\frac{1}{p}$. To see that $\mu$ is a Pietsch measure for $I_{D_n}$, let $\nu$ be any Pietsch probability for $I_{D_n}$ on $K_n$. We can clearly assume that $\nu$ is supported on the positive elements in $K_n$. By averaging $\nu$ against the permutations of $D_n$ which fix 0, which is a group of isometries on $D_n$, we get another Pietsch probability for $I_{D_n}$ (which we continue to denote by $\nu$) so that if we condition $\nu$ on $J_k := \{\chi_A: |A| = k, A \subset D_n \sim \{0\}\}$, $1 \leq k \leq n - 1$, the resulting probability $\nu_k$ on $J_k$ is the uniform probability. A trivial calculation shows that for $x, y$ in $D_n \sim \{0\}$, $\int_{J_k} |f(x) - f(y)|^p \, d\nu_k(f) \leq \frac{n}{2(n-1)}$. This proves that $\mu$ is a Pietsch measure for $I_{D_n}$ and hence $\pi_p^L(I_{D_n}) = (2 - \frac{2}{n})^\frac{1}{p}$.

Our final comment on Lipschitz 1-summing operators is that the concept has appeared in the literature even if the definition is new. In [4], Bourgain proved that every $n$-point metric space can be embedded into a Hilbert space with distortion at most $C \log n$, where $C$ is an absolute constant. In fact, he really proved the much stronger result that $\pi_1^L(I_X) \leq C \log n$ if $I_X$ is the identity mapping on an $n$-point space $X$ by making use of a special embedding of $X$ into a space $C(K_X)$ with $K_X$ a finite metric space and constructing a probability on $K_X$. Moreover, Bourgain's construction has occasionally been used in the computer science literature. The strong form of Bourgain's theorem is also used in [5] to prove an inequality that is valid for all metric spaces.

3. Linear operators

In this section we show that the Lipschitz $p$-summing norm of a linear operator is the same as its $p$-summing norm. This justifies that the notion of Lipschitz $p$-summing operator is really a generalization of the concept of linear $p$-summing operator.

**Theorem 2.** Let $u$ be a bounded linear operator from $X$ into $Y$ and $1 \leq p < \infty$. Then $\pi_p^L(u) = \pi_p(u)$.

**Proof.** Note that we can assume, without loss of generality, that $\dim Y \leq \dim X = N < \infty$. Indeed, it is clear from the definition that $\pi_p^L(u)$ is the supremum of $\pi_p^L(u_E)$ as $E$ ranges over finite dimensional subspaces of $X$ and similarly for $\pi_p^L(u)$. That we can assume $\dim Y \leq \dim X$ is clear from the linearity of $u$.

Since $\dim Y \leq N$, there is an embedding $J$ of $Y$ into $\ell_\infty^m$ with $m \leq (\frac{3}{2})^N$ so that $||J|| = 1$ and $||J^{-1}|| \leq 1 + \varepsilon$. We then get the following nonlinear Pietsch factorization:

$$
\xymatrix{ L_\infty(\mu) \ar[r]^{i_{\infty,p}} & L_p(\mu) \ar[d]_{\beta} \\
X \ar[r]^u & Y \ar[r]^J & \ell_\infty^m }
$$

where $\text{Lip}(\alpha) = 1$, $\text{Lip}(\beta) \leq \pi_p^L(Ju) \leq \pi_p^L(u)$. We can also assume, without loss of generality, that the probability $\mu$ is a separable measure.

We now use some nonlinear theory that can be found in the book [3].

(1) The mapping $\alpha$ is weak* differentiable almost everywhere. This means that for (Lebesgue) almost every $x_0$ in $X$, there is a linear operator
$D_{x_0}^w(\alpha) : X \to L_\infty(\mu)$ such that for all $f \in L_1(\mu)$ and for every $y \in X$,

$$\lim_{t \to 0} \frac{\alpha(x_0 + ty) - \alpha(x_0)}{t} = (D_{x_0}^w(\alpha)(y), f).$$

(2) The operator $i_{\infty,p,\alpha}$ is differentiable almost everywhere. This means that

for almost every $x_0 \in X$, there is a linear operator $D_{x_0}(i_{\infty,p,\alpha}) : X \to L_p(\mu)$ such that

$$\sup_{\|f\| \leq 1} \left\| \frac{i_{\infty,p,\alpha}(x_0 + ty) - i_{\infty,p,\alpha}(x_0)}{t} - D_{x_0}(i_{\infty,p,\alpha})(y) \right\|_p \to 0 \text{ as } t \to 0.$$

When $1 < p < \infty$, statement (2) follows from the reflexivity of $L_p$ (see [3 Corollary 5.12 & Proposition 6.1]). For $p = 1$, just use (2) for $p = 2$ and compose with $i_{2,1}$.

The mapping $i_{\infty,p}$ is weak* to weak continuous, so $D_{x_0}(i_{\infty,p,\alpha}) = i_{\infty,p}D_{x_0}^w(\alpha)$ whenever both derivatives exist. Since they both exist almost everywhere, by making several translations we can assume without loss of generality that this equation is true for $x_0 = 0$ and also that $\alpha(0) = 0$.

Next we show that in the factorization diagram the nonlinear map $\alpha$ can be replaced by the linear operator $D_0^w(\alpha)$ by constructing a mapping $\tilde{\beta} : L_p(\mu) \to \ell_\infty^m$ so that $\beta i_{\infty,p}D_0^w(\alpha) = Ju$ and $\text{Lip}(\beta) \leq \text{Lip}(\tilde{\beta})$. To do this, define $\beta_n : L_p(\mu) \to \ell_\infty^m$ by $\beta_n(y) := n\beta(y/n)$ and note that $\text{Lip}(\beta_n) = \text{Lip}(\beta)$. We have for each $x \in X$,

$$\|Ju(x) - \beta_n i_{\infty,p}D_0^w(\alpha)(x)\| = \|\beta_n ni_{\infty,p}\alpha(x/n) - \beta_n D_0(i_{\infty,p,\alpha})(x)\| \leq \text{Lip}(\beta)\|ni_{\infty,p}\alpha(x/n) - D_0(i_{\infty,p,\alpha})(x)\||,$$

which tends to zero as $n \to \infty$. For $\tilde{\beta}$ we can take any cluster point of $\beta_n$ in the space of functions from $L_p(\mu)$ into $\ell_\infty^m$; such points exist because $\beta_n$ is uniformly Lipschitz and $\beta_n(0) = 0$.

Summarizing, we see that we have a factorization

$$L_\infty(\mu) \xrightarrow{i_{\infty,p}} L_p(\mu) \xrightarrow{\beta} \ell_\infty^m$$

with $\tilde{\alpha}$ linear, $\|\tilde{\alpha}\| \leq \text{Lip}(\alpha)$, and $\text{Lip}(\tilde{\beta}) \leq \text{Lip}(\beta)$.

The final step involves replacing $\tilde{\beta}$ with a linear operator. Since the restriction $\beta$ of $\tilde{\beta}$ to the linear subspace $i_{\infty,p}\tilde{\alpha}[X]$ is linear and $\ell_\infty^m$ is reflexive, this follows from [3 Theorem 7.2], which is proved by a simple invariant means argument. Alternatively, one can use the injectivity of $\ell_\infty^m$ to extend $\beta$ to $L_p(\mu)$.

### 4. Open Problems and Concluding Remarks

**Problem 1.** Is there a composition formula for Lipschitz $p$-summing operators? That is, do we have $\pi^T_p(TS) \leq \pi^T_p(T)\pi^T_p(S)$, when $\frac{1}{p} \leq \frac{1}{q} + \frac{1}{r} \text{ and } T \text{ is a linear operator}?$

Say that a Lipschitz mapping $T : X \to Y$ is Lipschitz $p$-integral if it satisfies a factorization diagram as in condition (3) of Theorem 1 except with $J$ being the canonical isometry from $Y$ into $(Y^\#)^*$. We then define the Lipschitz $p$-integral norm $L_p^T(T)$ of $T$ to be the infimum of $\text{Lip}(\alpha) \cdot \text{Lip}(B)$, the infimum being taken over all such factorizations. When $T$ is a linear operator, this is the same as the usual $p$-integral norm of $T$. Indeed, in this case one can use for $J$ the canonical
isometry from $Y$ into $Y^{**}$ because $Y^{**}$ is norm one complemented in $(Y^*)^*$. Then the proof that $I_p(T) \leq I^L_p(T)$ is identical to the proof of Theorem 2.

**Problem 2.** Is every Lipschitz $2$-summing operator Lipschitz $2$-integral?

In the case where the target space $Y$ is a Hilbert space, problem 2 has an affirmative answer by Kirszbraun’s theorem [3, p. 18]. If $Y$ has K. Ball’s Markov cotype 2 property [1], it follows from Ball’s work that the answer is still positive, although his result does not yield that $I^L_p(T)$ and $\pi_p(T)$ are equal. It is worth mentioning that the work of Naor, Peres, Schramm, and Sheffield [4] combines with Ball’s result to yield that for $2 \leq p < \infty$, every Lipschitz $p$-summing operator into $L_r$, $1 < r \leq 2$, is Lipschitz $p$-integral.

We mentioned in section 2 that $\Pi^n_p(X,Y)$, the class of Lipschitz $p$-summing operators from $X$ into $Y$, is a Banach space under the norm $\pi_p^n(\cdot)$ when $Y$ is a Banach space.

**Problem 3.** When $Y$ is a Banach space and $X$ is finite, what is the dual of $\Pi^n_p(X,Y)$?

In section 2 we noted that there is a version of Grothendieck’s theorem that is true in the nonlinear setting. Are there other versions? In particular, we ask the following:

**Problem 4.** Is every Lipschitz mapping from an $L_1$ space to a Hilbert space Lipschitz $1$-summing? Is every Lipschitz mapping from a $C(K)$ space to a Hilbert space Lipschitz $2$-summing?

It is elementary that for a linear operator $T : X \to Y$, $\pi_p(T)$ is the supremum of $\pi_p(TS)$ as $S$ ranges over all operators from $\ell_p'$ into $X$ of norm at most one. This leads us to ask:

**Problem 5.** If $T : X \to Y$ is Lipschitz, is $\pi_p^L(T)$ the supremum of $\pi_p^L(TS)$ as $S$ ranges over all mappings from finite subsets of $\ell_p'$ into $X$ having Lipschitz constant at most one?

Since all finite metric spaces embed isometrically into $\ell_\infty$, the answer to problem 5 is yes for $p = 1$.

Of course, all of the above problems are special cases of the general

**Problem 6.** What results about $p$-summing operators have analogues for Lipschitz $p$-summing operators?

*Added in proof.* Problem 3 has been solved by J. A. Chávez Domínguez (unpublished).

**References**


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